# DRESSED SKELETON EXPANSION AND THE COUPLING SCALE AMBIGUITY PROBLEM 

Hung Jung Lu

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# DRESSED SKELETON EXPANSION AND THE COUPLING SCALE AMBIGUITY PROBLEM 

Hung Jung Lu, Ph.D.<br>Stanford University, 1992


#### Abstract

Perturbative expansions in quantum field theories are usually expressed in powers of a coupling constant. In principle, the infinite sum of the expansion series is independent of the renormalization scale of the coupling constant. In practice, there is a remnant dependence of the truncated series on the renormalization scale. This scale ambiguity can severely restrict the predictive power of theoretical calculations.

The dressed skeleton expansion is developed as a calculational method which avoids the coupling scale ambiguity problem. In this method, physical quantities are expressed as functional expansions in terms of a coupling vertex function. The arguments of the vertex function are given by the physical momenta of each process. These physical momenta effectively replace the unspecified renormalization scale and eliminate the ambiguity problem.

This method is applied to various field theoretical models and its main features and limitations are explored. For quantum chromodynamics, an expression for the running coupling constant of the three-gluon vertex is obtained. The effective coupling scale of this vertex is shown to be essentially given by $\mu^{2} \sim Q_{\min }^{2} Q_{\operatorname{med}}^{2} / Q_{\max }^{2}$, where $Q_{\text {min }}^{2}, Q_{\text {med }}^{2}$ and $Q_{\text {max }}^{2}$ are respectively the smallest, the next-to-smallest and the largest scale among the three gluon virtualities. This functional form suggests that the three-gluon vertex becomes non-perturbative at asymmetric momentum configurations. Implications for four-jet physics is discussed.


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To Everyone in the World

## Help me! <br> I don't want to become a human...



## CHAPTER 1:

INTRODUCTION

The progress in the study of elementary particles has been spectacular. We now have a comprehensive theory of particle interactions that describes satisfactorily all non-gravitational physics. Often neglected in the popular literature is the fact that, from the modern point of view, the fundamental objects under study are not particles but fields. In the current interpretation, all particles correspond to excitations of diverse quantum fields. This picture is very different from the classical conception of particles as tiny billiard balls. As an example, electrons and their antiparticles, positrons, are to be interpreted as quantized excitations of an "electron field", much like the situation depicted in Fig. 1.1.


Fig. 1.1 Interpretation of electrons and positrons as different quantized excitations of an electron field.

There has been steady and extraordinary progress in the understanding of quantum field theories. Our physical world seems to be described by a special class of quantum field theories named gauge theories, where the interactions are dictated by an invariance principle. Strong, weak and electromagnetic interactions are all gauge interactions. The success of the Standard Model, which emcompases these
three interactions in a single Lagrangian, has become one of the major achievements in the history of science.

Theoretical predictions in quantum field theories frequently rely on perturbative methods. Perturbative expansions offer a fruitful approach to calculations in quantum field theories. For instance, theoretical prediction for the electron magnetic anomaly from perturbative calculations gives

$$
\begin{equation*}
a^{\text {th }}=\left(\frac{g-2}{2}\right)^{\text {th }}=1159652140(5.3)(4.1)(27.1) \times 10^{-12} \tag{1.1}
\end{equation*}
$$

which is in near perfect agreement with the experimental measurement

$$
\begin{equation*}
a^{\exp }=\left(\frac{g-2}{2}\right)^{\exp }=1159652188.4(4.3) \times 10^{-12} \tag{1.2}
\end{equation*}
$$

(See Ref. [1] for details on the theoretical and experimental development in the calculation and measurement of this quantity, as well as the sources of the various errors quoted in the expressions given above.)

Despite this and other successful predictions, perturbative expansions in quantum field theories are not free of difficulties. The "Scale Ambiguity Problem" $[2,3,4]$ is one of them, which we shall explain shortly.

The scale ambiguity problem has its roots in the renormalization procedure of quantum field theories. To understand this, we should first clarify what scale we are referring to and where it comes from. It is well-known that we encounter ultraviolet divergences in a particular class of quantum field theories; that is, the analytical expressions computed from these theories are riddled with mathematical infinities. A "renormalization" operation is needed in order to extract physically meaningful results. All the gauge theories of the standard model fall into this class
of theories. During the stage of renormalization, we inevitably introduce a spurious momentum scale which we call the renormalization scale. All physical predictions should therefore be independent of this scale. This unfortunately is not the case in practice.

More concretely, consider the power series expansion of a physical quantity $R$ in terms of a coupling constant $\alpha$. As a consequence of the renormalization procedure, a dependence on a spurious scale $\mu$ shows up both in the coupling constant $\alpha(\mu)$ and in the expansion coefficients $r_{i}(\mu)$ :

$$
\begin{equation*}
R=r_{0} \alpha^{p}(\mu)+r_{1}(\mu) \alpha^{p+1}(\mu)+r_{2}(\mu) \alpha^{p+2}(\mu)+\ldots . \tag{1.3}
\end{equation*}
$$

(Here $p$ denotes the lowest power in the coupling constant. The lowest order coefficient $r_{0}$ does not depend on the renormalization scale $\mu$ since it comes from tree-level Feynman diagrams where no renormalization is involved. In a theory with running masses, the mass dependence is included in the coefficients $r_{i}(\mu)$.)

These two kinds of dependence on the renormalization scale will in principle conspire to cancel each other, rendering the overall result $R$ independent of $\mu$. In reality, only a few terms in the expansion series can be computed, and the truncated series

$$
\begin{equation*}
R_{N}=r_{0} \alpha^{p}(\mu)+r_{1}(\mu) \alpha^{p+1}(\mu)+\ldots+r_{N}(\mu) \alpha^{p+N}(\mu) \tag{1.4}
\end{equation*}
$$

carries a residual dependence on the renormalization scale $\mu$. Different choices of this scale will therefore lead to different theoretical predictions. The arbitrariness in the choice of the scale $\mu$ is known as the coupling scale ambiguity problem.

Let us see a concrete example of the choice of coupling scale, taken from recent analyses of hadronic jet events performed by the four collaborations (OPAL, DELPHI, L3 and ALEPH) at LEP, the European electron-positron storage ring in CERN.

1) The OPAL collaboration, in a study of jet production rates and a test of QCD on the $Z^{0}$ resonance [5], has concluded that a value of

$$
\begin{equation*}
\mu^{2}=0.001-0.003 M_{Z}^{2} \tag{1.5}
\end{equation*}
$$

is appropriate.
2) The DELPHI collaboration, in a comparison of jet production rates on the $Z^{0}$ resonance to perturbative QCD [6], has employed both

$$
\begin{equation*}
\mu^{2}=M_{Z}^{2} \quad \text { and } \quad \mu^{2}=0.001 M_{Z}^{2}, \tag{1.6}
\end{equation*}
$$

but favored the last value to fit the four-jet cross section.
3) The L3 group, in determination of $\alpha_{s}$ from jet multiplicities [7] has chosen

$$
\begin{equation*}
\mu^{2}=0.08 M_{Z}^{2} \tag{1.7}
\end{equation*}
$$

to fit their data. This scale is motivated from the typical squared momentum $y_{\text {cut }} s$ transferred to hard gluons.
4) The ALEPH collaboration, in its measurement of the strong coupling constant $\alpha_{s}$ from global event-shape variables of hadronic $Z$ decays $[8]$, has used

$$
\begin{equation*}
\mu^{2}=0.25 M_{Z}^{2} \tag{1.8}
\end{equation*}
$$

They estimate the error coming from scale ambiguity to be around $8 \%$.

As can be seen from these cases, there is in general no consensus on the choice of coupling scales, even for seemingly identical experiments.

The bothersome aspect of the scale ambiguity is that our theory does not provide a "clean" prediction. Unlike traditional perturbation theories, in which our calculation gives us a definite number, quantum field theories requires us to first choose a scale in order to obtain a numerical answer. The choice of an appropriate scale seems to involve more art than science.

As mentioned previously, the scale ambiguity problem has its roots in the renormalization procedure. We can imagine a renormalization procedure where, instead of introducing an extraneous scale $\mu$, we employ directly the physical momenta involved in each problem as the renormalization scale. A theory so renormalized will then be free of scale ambiguity. This is exactly the approach taken in the Dressed Skeleton Expansion to resolve the scale ambiguity problem.

In spite of its horror-inspiring name, the skeleton expansion forms part of the standard techniques in field theoretical analysis. F. J. Dyson [9] apparently is responsible for the introduction of this technique and for its naming. The dressed skeleton expansion is an adaptation of the traditional skeleton expansion with the following two ingredients:

1) All scattering amplitudes are expanded as skeleton graphs in terms of a renormalized vertex function.
2) This vertex function is computed from a multi-momentum renormalization group equation.

The purpose of this thesis is to explore the main features and limitations of the dressed skeleton expansion as a perturbative calculational tool in quantum field theories.

The following is a glossary of the content of this thesis.
Chapter 2 explains and discusses the problem of scale ambiguity, and surveys the standard approaches in handling this problem.

Chapter 3 introduces the method of dressed skeleton expansion, and points out its potential as a perturbative calculational method without scale ambiguity.

Chapter 4 is devoted to the application of the dressed skeleton method to simple field theory models. As a first example, the two-particle scattering amplitude in $\phi_{6}^{3}$ theory is computed, and the dressed skeleton result is found to be compatible with other scale-setting methods. Then the dressed skeleton method is applied to field theory models in 1+1-dimension to illustrate its various features. In particular, it is shown to give the exact answer in leading $1 / N$ Gross-Neveu model and to absorb renormalons at loop level. Its extension to theories involving more complicated vertices is discussed.

Although the skeleton expansion is a relatively straightforward technique in simple field theories, it is not as simple to apply to gauge field theories, since the naïve skeleton graphs in these theories are not gauge-invariant. Nonetheless, some interesting lowest-order results can still be obtained. In Chapter 5 the multimomentum renormalization group equation is applied to gauge-invariant QCD verticcs. In particular, we obtain an expression for the effective coupling constant of the three-gluon vertex.

Finally, Chapter 6 summarizes the discussions and conclusions.

CHAPTER 2:
THE COUPLING SCALE AMBIGUITY PROBLEM

The scale-scheme ambiguity problem is present in all field theories, but its relevance was not fully appreciated until Quantum Chromodynamics (QCD) became the accepted theory of strong interactions. Contributing to this misappreciation is the fact that its elder sibling Quantum Electrodynamics (QED) has both a substantially smaller coupling constant and a convenient renormalization scale for low-energy phenomena. The scheme-scale ambiguity problem presently remains one of the major obstacles impeding precise QCD predictions, and a deeper understanding in this area is urgently needed.

As explained in Chapter 1, the scale ambiguity problem has its origin in the renormalization process. Hence it is appropriate here to give a description of renormalization theory here; this will also clarify the concept of physical coupling constant.

Consider a quantum field theory with a single coupling constant. Let us assume that the theory is renormalizable. A physical quantity $R$ in this theory (e.g., a particular scattering amplitude or decay rate) can be expanded as a power series in the bare coupling constant $\alpha_{o}$ of the theory

$$
\begin{equation*}
R=r_{0} \alpha_{o}^{p}+r_{1} \alpha_{o}^{p+1}+r_{2} \alpha_{o}^{p+2}+\ldots, \tag{2.1}
\end{equation*}
$$

where $r_{0}$ is known as the tree-level term, $r_{1}$ the one-loop correction, $r_{2}$ the twoloop correction, etc., and $p$ is the power of coupling constant associated to the tree-level term. It is well-known that all the coefficients in the power series beyond the tree level suffer ultraviolet divergence; that is, their Feynman integrals are not finite. Thus the series in Eq. (2.1) as it stands is ill-defined. Since $R$ is a physical quantity, we expect its series to represent a finite result. We must conclude then that the bare coupling constant $\alpha_{o}$ itself is ill-defined. That is, the infinities from
the coefficients $r_{i}$ and from the bare coupling $\alpha_{o}$ must somehow cancel each other. If instead of $\alpha_{o}$ we use a physical quantity to expand the power series (2.1), we would expect the new expansion coefficients to be finite.

The idea of renormalization is the following. Take another physical quantity $S$

$$
\begin{equation*}
S=s_{0} \alpha_{o}^{q}+s_{1} \alpha_{o}^{q+1}+s_{2} \alpha_{o}^{q+2}+\ldots \tag{2.2}
\end{equation*}
$$

As the previous quantity $R$, all the coefficients in this series beyond the tree level are infinite. However, if we define a "physical coupling-constant in the $S$-scheme" $\alpha_{S}$ (also known as the "renormalized coupling-constant in the $S$-scheme") by

$$
\begin{equation*}
S \equiv s_{0} \alpha_{S}^{q} \tag{2.3}
\end{equation*}
$$

and express the series of $R$ in Eq. (2.1) as a power series in $\alpha_{S}$, that is

$$
\begin{equation*}
R=r_{0}^{\prime} \alpha_{S}^{p}+r_{1}^{\prime} \alpha_{S}^{p+1}+r_{2}^{\prime} \alpha_{S}^{p+2}+\ldots \tag{2.4}
\end{equation*}
$$

where

$$
\begin{align*}
r_{0}^{\prime} & =r_{0} \\
r_{1}^{\prime} & =r_{1}-\frac{p}{q} \frac{s_{1}}{s_{0}} r_{0}, \quad \text { etc. } \tag{2.5}
\end{align*}
$$

then the new series for $R$ will have finite coefficients. For instance, in the expression for the coefficient $r_{1}^{\prime}$, the infinities coming from the coefficient $r_{1}$ and $s_{1}$ will conspire to cancel each other, yielding a finite result for $r_{1}^{\prime}$. The process of expressing a physical quantity $R$ in terms of a physical coupling $\alpha_{S}$ is known as the renormalization procedure.

We see that the underlying theory actually does not provide us a direct prediction for the quantities $R$ and $S$; rather, it only allows us to relate these two quantities. We see also that it makes no sense to measure the bare coupling constant, since it disappears after the renormalization procedure.

The name "physical coupling constant" for $\alpha_{S}$ can be somewhat misleading. Strictly speaking, $\alpha_{S}$ is not really a. coupling, since it is not the quantity that appears in the original Lagrangian. Also, $\alpha_{S}$ is not really a constant, because in general it contains a dependence on the renormalization scale.

Renormalization then simply expresscs a physical quantity $R$ in terms of another physical quantity $\alpha_{S}$ (or equivalently, $S$ ). Wc arc allowed to choose any physical quantity $S$ to define our physical coupling constant; different choices of $S$ lead to different definitions of $\alpha_{S}$. Also, all physical processes depend on one or more scales. The scale $\mu$ that characterizes the overall scale dependence of $\alpha_{S}$ is named the renormalization scale. Thus,

$$
\begin{equation*}
\alpha_{S}=\alpha_{S}(\mu) \tag{2.6}
\end{equation*}
$$

and we are allowed to choose any value for $\mu$. The value of $R$ in principle should be independent of our choice of $S$ and $\mu$, which is known as the renormalization scheme and renormalization scale invariance. However, due to the truncation of the expansion series, the finite series

$$
\begin{equation*}
R_{N}=r_{0}^{\prime} \alpha_{S}^{p}(\mu)+r_{1}^{\prime}(\mu) \alpha_{S}^{p+1}(\mu)+\ldots+r_{N}^{\prime}(\mu) \alpha_{S}^{p+N}(\mu) \tag{2.7}
\end{equation*}
$$

becomes dependent on the choice of scheme and scale. This is the scheme and scale ambiguity problem.

The freedom in choosing physical constants has not threatened perturbative calculations in QED (Quantum Electrodynamics) for two reasons:

1. Due to the Ward-Takahashi identity, the photon vacuum polarization defines a natural coupling constant $\alpha_{\mathrm{em}}\left(Q^{2}\right)$ in QED (known as the on-shell scheme coupling constant), where $Q^{2}$ is the squared momentum transfer for the photon.
2. At zero momentum transfer squared, $\alpha_{\mathrm{em}}\left(Q^{2}\right)$ approaches a small and fixed value (called the fine-structure constant; see Fig. 2.1.)

$$
\begin{equation*}
\alpha=\alpha_{\mathrm{em}}(0)=1 / 137.03598 \ldots \tag{2.8}
\end{equation*}
$$

Because of the naturalness of the on-shell scheme and the small value of the fine-structure constant, the problem of scale-scheme ambiguity has been virtually neglected in QED. Adding to this is the fact that traditional tests of QED are performed in the low-energy region (energies within a few orders of magnitude of the mass of the electron). As a consequence, the uncertainty in scheme and scale has little impact on the numerical results.

In QCD the situation is quite different. The discovery of the asymptotic freedom [10] (i.e., the fact that color interactions become weak at short distances,) enables us to use perturbative methods at high-energy limits. However, unlike QED, QCD apparently has no natural scheme associated to gluons. The scale dependence of the strong coupling constant $\alpha_{s}$ also has its peculiar features. At - low energies, $\alpha_{s}$ becomes large. Perturbative methods eventually cease to be valid at low enough energies (around 1 GeV ). At high energies, $\alpha_{s}$ becomes small due to asymptotic freedom, and $\alpha_{s}$ tends to zero as the energy goes to infinity. Unlike
the case of QED, where there is a preferred value of the coupling constant at zero momentum scale, in the case of QCD we do not have any special values for the coupling constant in the perturbative region. See Fig. 2.1.


Fig. 2.1 Energy dependence of the QED and QCD effective coupling constants.

The value of the strong coupling constant in the currently accessible energy range is rather large. For instance, the strong coupling constant in the popular $\overline{\mathrm{MS}}$ scheme [11] has the values [12] (assuming five light-quark flavors)

$$
\begin{array}{ll}
\alpha_{\overline{\mathrm{MS}}}(\mu)=0.148 \pm 0.018 & \text { for } \mu=34 \mathrm{GeV}, \\
\alpha_{\overline{\mathrm{MS}}}(\mu)=0.115 \pm 0.008 & \text { for } \mu=91.17 \mathrm{GeV} . \tag{2.9}
\end{array}
$$

As a consequence, the uncertainty from the choice of coupling scheme and scale affects theoretical predictions appreciably, and the study of these issues becomes unavoidable.

The scheme and scale ambiguity problems are not particular to QED or QCD. They are also present in all other quantum field theories. However, as we shall
argue here, the scheme ambiguity is conceptually less severe since it can be reduced to a scale ambiguity. The freedom to select various renormalization schemes is actually no more than the freedom to adopt 'meter' or 'foot' as the basic unit of length. As long as a scheme is well defined, we can always express the result in a particular scheme. Notice that in the process of translating results from one scheme to another-namely, replacing one coupling constant by another-we inevitably re-encounter the problem of scale setting. More precisely, two coupling constants $\alpha_{1}(\mu)$ and $\alpha_{2}(\mu)$ of different schemes are related by an equation

$$
\begin{equation*}
\alpha_{1}\left(\mu_{1}\right)=\alpha_{2}\left(\mu_{2}\right)+C_{1}\left(\mu_{1} / \mu_{2}\right) \alpha_{2}^{2}\left(\mu_{2}\right)+C_{2}\left(\mu_{1} / \mu_{2}\right) \alpha_{2}^{3}\left(\mu_{2}\right)+\ldots \tag{2.10}
\end{equation*}
$$

Given a finite number of terms in this series, we must choose an appropriate value of $\mu_{2}$ for each value of $\mu_{1}$. That is, we again run into a scale ambiguity problem. The scale ambiguity is thus a somewhat more fundamental problem than the corresponding scheme ambiguity problem, in the sense that if we know how to choose the "best" scale in all cases, then we can translate our result freely from one scheme to another.

Several methods have been proposed to solve the coupling scale ambiguity. Among them we shall mention:

1. Fastest Apparent Convergence (FAC) $[3,4]$ :

According to FAC, we should choose the coupling scale that makes the series look most convergent. Operationally we will define this method as setting the contribution of the second order term (i.e., next to tree level) to be zero. That
is, if

$$
\begin{equation*}
R_{N}=r_{0} \alpha^{p}(\mu)+r_{1}(\mu) \alpha^{p+1}(\mu)+\ldots+r_{N}(\mu) \alpha^{p+N}(\mu) \tag{2.11}
\end{equation*}
$$

then the scale $\mu$ should be chosen as the solution of

$$
\begin{equation*}
r_{1}(\mu)=0 . \tag{2.12}
\end{equation*}
$$

2. Principle of Minimal Sensitivity (PMS) [3]:

We define this method here as the choice of the coupling scale at the stationary point of the truncated series:

$$
\begin{equation*}
\left.\frac{d R_{N}}{d \mu}\right|_{\mu}=0 \tag{2.13}
\end{equation*}
$$

The full PMS method also requires the choice of a renormalization scheme. Beyond two-loop order, this method proposes the optimization of scheme parameters in addition to the coupling scale. (The scheme parameters can be defined as the $\beta$-function coefficients in each scheme. See Ref. [3] for detail.)
3. Brodsky-Lepage-Mackenzie Method (BLM) [2]:

This method is inspired by QED. The philosophy is to absorb all fermionic vacuum polarization effects into the running coupling constant. In 1-loop order
massless QCD, it is operationally equivalent to the condition of a vanishing coefficient of the $n_{f}$ (number of light fermions) term. Therefore BLM results are formally invariant under the change of number of light flavors:

$$
\begin{equation*}
\frac{\partial R}{\partial n_{f}}\left[\alpha(\mu), n_{f}\right]=0 \tag{2.14}
\end{equation*}
$$

Extension of the BLM method based on the fermion-number criterion has recently been studied by Grunberg and Kataev [13]. (See also Surguladze and Samuel [14] for a recent application of the flavor-independence criterion to the next-to-leading coefficient in the total hadronic cross section in $e^{+} e^{-}$annihilation and in the $\tau$ hadronic decay rate.) IIowever, in the next Chapter we will show that BLM's method in QED effectively corresponds to the dressed photon expansion; thus, the general dressed skeleton expansion can be considered as another extension of the BLM method.
4. Renormalization Scheme Invariant Calculation (RSI) $[4,15]$ :

This is yet another point of view on the subject. Given a physical quantity, we can define an effective coupling (or effective charge) associated to it (which we shall call the R -scheme coupling constant):

$$
\begin{align*}
R & =r_{0} \alpha^{p}(\mu)+r_{1}(\mu) \alpha^{p+1}(\mu)+\ldots \\
& \equiv r_{0} \alpha_{R}^{s} \tag{2.15}
\end{align*}
$$

If $R$ depends on a single external momentum $Q$, then the evolution of $R(Q)-$ or equivalently of $\alpha_{R}(Q)-$ on $Q$ can be studied self-consistently without ad-


#### Abstract

ditional inputs such as $\Lambda_{\mathrm{QCD}}$. This is usually stated as the renormalization-scheme-independent calculation. However, we should bear in mind that implicitly a particular scheme has been preferred: the $R$-scheme. The $R$-scheme is, in a sense, a natural scheme for the study of the evolution properties of a given field theory, because the coupling constant itself in this case is experimentally measured, hence there is no need for an outside coupling constant. But this method has its own limitations. For instance, the total hadron decay width of heavy quarkonia should be predictable from QCD, despite that it contains no lab controllable momentum and thus has no evolution to work with. Another problem with the RSI method is a proliferation of coupling constants: one coupling constant is introduced for each physical process. The problem of scale ambiguity comes back whenever we try to relate one effective coupling to another.


For a single-scale process

$$
\begin{equation*}
R_{N}(Q)=r_{0}(Q) \alpha^{p}(\mu)+r_{1}(Q, \mu) \alpha^{p+1}(\mu)+\ldots+r_{N}(Q, \mu) \alpha^{p+N}(\mu) \tag{2.16}
\end{equation*}
$$

the usual impression is that as long as the coupling scale $\mu$ is chosen near the typical scale $Q$ of a given process, its perturbation series will give a reasonable result. We should notice, however, that due to dimensional transmutation (i.e., the presence of $\Lambda_{Q C D}$ ) the correct scale might in some cases not be proportional to $Q$, but rather to some other power of $Q$, or an even more complicated form. So the assignment of coupling scale with typical physical scales runs the risk of being
too simplistic. For processes involving many scales, in general it is not clear how a "typical scale" can be defined.

For multi-scale processes, the assignment of a uniform coupling throughout all the vertices becomes questionable. Consider for instance the exclusive process $e^{+} e^{--} \rightarrow \mu^{+} \mu^{-} \gamma$ (Fig. 2.2). In QED the vertices $\mathbf{a}$ and $\mathbf{b}$ should have a coupling strength $\sim \alpha^{1 / 2}\left(Q^{2}\right)$, whereas the vertex involving the radiated photon should have a. strength $\sim \alpha^{1 / 2}(0) \sim 1 / \sqrt{137}$.


Fig. 2.2 A typical QED process, where the coupling strength at vertices $a$ and $b$ is expected to be stronger than the coupling strength at $c$.

Similarly, for the scattering of two electrons in QED (Fig. 2.3), the appropriate coupling constant for the first diagram should be $\alpha(t)$ and for the second diagram $\alpha(u)$, where $t$ and $u$ are the Mandelstam variables shown in the figure.

This observation and controversy on the various scale-setting procedures prompted the consideration of the Dressed Skeleton Expansion (DSE) as an alternative to the conventional power series expansion. The details of this calculation method will be the subject of the next chapter.


Fig. 2.3 Feynman diagrams for the scattering amplitude of $e^{-} e^{-} \rightarrow$ $e^{-} e^{-}$. The Mandelstam variables $t$ and $u$ are the squared momentum transfers carried by the exchanged photons.

## CHAPTER 3:

DRESSED SKELETON EXPANSION

The skeleton expansion is a method of organizing Feynman diagrams, where one groups together all diagrams differing only by self-energy and vertex insertions. The original idea can be traced back to Dyson [9] in the late 1940s. In the early development of quantum field theories, this expansion became a common technique in the proof of perturbative renormalizability $[16,17,18]$, although its role in this area has now been largely replaced by the BPHZ [19,20,21] formalism. Interest in the skeleton expansion has also come from the hope that some non-perturbative features can be revealed through the study of its coupled integral equations.

In this chapter we will explore the use the skeleton expansion as a calculation tool which hạs no scale ambiguity. We shall call this method the "Dressed Skeleton Expansion". In order to explain this method, we will review here first the concept of skeleton graphs.

### 3.1. Skeleton Expansion

Skeleton graphs represent Feynman diagrams stripped of vertex and selfenergy insertions. To fix the idea, let us consider the Feynman diagram in $\phi_{6}^{3}$ theory [22] depicted in Fig. 3.1. To obtain the skeleton graph of this particular Feynman diagram, we perform the following sequence of operations:


Fig. 3.1 An example of a Feynman diagram where the self-energy insertions have been boxed.

1) Draw a box around each self-energy correction. A self-energy correction is a subgraph with two legs coming out of it. See Fig. 3.1.
2) Replace these boxes by single lines (propagators). See Fig. 3.2.


Fig. 3.2 Result of the Feynman diagram in Fig. 3.1 after the removal of the self-energy insertions. The remaining vertex insertions have been boxed.
3) Draw a box around each vertex correction. A vertex correction is a subgraph with three legs coming out of it. See Fig. 3.2.
4) Replace these boxes by simple vertices. See Fig. 3.3.

We can therefore associate in a unique way with each graph $G$ another graph called the skeleton graph of G. For instance, Fig. 3.3 is the skeleton graph of Fig. 3.1.


Fig. 3.3 Skeleton graph of Fig. 3.1.

In Fig. 3.4 we give some more examples to clarify the concept of skeleton graphs. Notice that by definition, skeleton graphs are those that cannot be further reduced by removing vertex or self-energy insertions. Thus, the diagram (b) in Fig. 3.4 is a skeleton graph, while the diagrams $(a)$ and (c) are not skeleton graphs.


Fig. 3.4 (b) is a skeleton graph, where as (a) and (c) are not skeleton graphs.

We can generate the complete set of Feynman diagrams corresponding to a particular scattering amplitude (i.e., any connected and amputated $n$-point Green's function with $n>3$ ) by

1) first drawing all the possible skeleton graphs with $n$ external legs, then
2) replacing the propagators in the skeleton graphs by full propagators and the vertices by full vertex functions. (The full vertex function contains only one-particle-irreducible diagrams, since the one-particle-reducible parts are already included in the full propagator.)

As an example, the two-body scattering amplitude in $\phi_{6}^{3}$ theory will contain the skeleton graphs shown in Fig. 3.5. The full propagator and full vertex graphs are shown in Fig. 3.6. We can convince ourselves that all Feynman diagrams of this scattering amplitude are effectively contained once and only once in the skeleton expansion.


Fig. 3.5 The skeleton expansion for the two-particle scattering amplitude in $\phi_{6}^{3}$ theory.


Fig. 3.6 The full propagator and full vertex function in $\phi^{3}$ theory.

Skeleton expansion can be used to renormalize scattering amplitudes.

1) Let $Z_{O S}$ be the on-shell wavefunction renormalization constant, that is, the residue of the full propagator at the mass pole [23]. Define the renormalized propagator by dividing the full propagator by $Z_{\mathrm{OS}}$ (see Fig. 3.7).
2) Define the renormalized vertex function as the full vertex function multiplied by $Z_{\mathrm{OS}}^{3 / 2}$ (see Fig. 3.7).
3) More generally, according to LSZ [24] reduction formula, for a scattering amplitude with $n$ external legs ( $n \geq 4$ ), we multiply the overall amplitude by $Z_{\mathrm{OS}}^{\eta / 2}$ in order to renormalize it.

$$
\begin{aligned}
& \underset{R}{\longrightarrow}] \\
& Q_{R}^{\infty}=Z_{0 s}^{3 / 2}[\text { Q } \\
& \text { Qhe }=z_{0,}^{4 / 2}\left[\infty_{0}^{\infty}+\cdots+\infty\right. \\
& =
\end{aligned}
$$

Fig. 3.7 Renormalized skeleton expansion in $\phi^{3}$ theory for twoparticle scattering amplitude.
4) As a consequence, the renormalized scattering amplitudes can be written in terms of renormalized skeleton graphs consisting only of renormalized propagators and renormalized vertex functions. This is shown schematically in Fig. 3.7 for the two-particle scattering amplitude.

It is a rather involved task to rigorously prove that skeleton expansion indeed leads to a finite theory for all renormalized Green's functions. We shall assume in the following that the set of skeleton graphs to a given order in the number of vertices yields a finite result. The proof of this statement in QED is given in Bjorken and Drell [17], and in $\phi_{4}^{4}$ theory in Zinn-Justin [18].

### 3.2. Dressed Skeleton Expansion

The Dressed Skeleton Expansion (DSE) is an adaptation of the standard skeleton expansion. Two essential modifications are introduced:

1) Local Effective Wavefunction Renormalization: Instead of a unique, onshell wavefunction renormalization constant $Z_{\mathrm{OS}}$, there is a diagrammatically local effective wavefunction renormalization "constant" $Z\left(p^{2}\right)$ for each full propagator. The idea is to absorb all self-energy renormalization effects into effective wavefunction renormalization constants. More precisely, the full unrenormalized propagator is defined to be:

$$
\begin{equation*}
i \Delta\left(p^{2}\right)=\frac{i Z\left(p^{2}\right)}{p^{2}-m_{\stackrel{\mathrm{P}}{2}}^{2}} \tag{3.1}
\end{equation*}
$$

where $m_{\mathrm{P}}$ is the pole mass. The effective wavefunction renormalization constant coming from a particular full propagator is used to renormalize the vertices immediately adjacent to it. That is, at each vertex, the full, unrenormalized vertex function is to be multiplied by

$$
\begin{equation*}
Z^{1 / 2}\left(p^{2}\right) Z^{1 / 2}\left(q^{2}\right) Z^{1 / 2}\left(r^{2}\right) \tag{3.2}
\end{equation*}
$$

being $p^{2}, q^{2}$ and $r^{2}$ the squared momentum of the three legs attached to the particular vertex function (see Fig. 3.8).

$$
\longmapsto \equiv Z\left(p^{2}\right) \longmapsto
$$



Fig. 3.8 Propagator and vertex function renormalization in DSE.

Observe that this prescription is consistent with the LSZ prescription of multiplying $Z_{\mathrm{OS}}^{1 / 2}$ (the on-shell wavefunction renormalization constant) for each external leg. Effectively, for all external legs we have $Z^{1 / 2}\left(p^{2}=m_{\mathrm{P}}^{2}\right)=Z_{\mathrm{OS}}^{1 / 2}$; thus, all vertices attached to external legs are (and hence the overall amplitude, too, is) multiplied by the correct power of $Z_{\mathrm{OS}}^{1 / 2}$.


Fig. 3.9 DSE graphs for the two-particle scattering amplitude in $\phi^{3}$ theory.

The two-body scattering amplitude in DSE to one-loop skeleton order consists of the skeleton graphs given in Fig. 3.9. Notice now that in order to compute the skeleton graphs in Fig. 3.9, we have to know only one single function: the DS (Dressed Skeleton) vertex function. Obviously this holds true for any $n$-point $(n>3)$ connected and amputated Green's functions: once the DS vertex function is known, all higher order Green's functions can be expressed as functional expansions in terms of the DS vertex function, graphically represented by the DSE graphs. In DSE there is no coupling constant and thus no indeterminate coupling scale. Instead of a coupling constant we have a DS vertex function, and the (known) momenta flowing into it effectively fulfill the role of the (unknown) coupling scale of the conventional power series expansion.
2) Vertex function through renormalization group equation: The vertex function can be obtained by the multi-momentum renormalization group equation or other suitable techniques.

The multi-momentum renormalization group equation is an intuitive generalization of the usual $\beta$-function formalism. If the perturbative expansion of the DS
vertex function $\lambda_{\mathrm{DS}}\left(k_{1}, k_{2}, k_{3}\right)$ in terms of the bare coupling constant is

$$
\begin{equation*}
\lambda_{\mathrm{DS}}\left(k_{1}, k_{2}, k_{3}\right)=\lambda_{0}+\lambda_{0}^{3} f_{1}\left(k_{1}, k_{2}, k_{3}\right)+\lambda_{0}^{5} f_{2}\left(k_{1}, k_{2}, k_{3}\right)+\ldots \tag{3.3}
\end{equation*}
$$

where the coefficient functions $f_{i}$ contain both divergent and finite parts, then we can obtain the multi-momentum renormalization group equation by

1) taking the derivative of Eq. (3.3) with respect to the external momenta

$$
\begin{equation*}
\frac{\partial \lambda_{\mathrm{DS}}}{\partial k_{i}^{\mu}}=\lambda_{0}^{3} \frac{\partial f_{1}}{\partial k_{i}^{\mu}}+\lambda_{0}^{5} \frac{\partial f_{2}}{\partial k_{i}^{\mu}}+\ldots \tag{3.4}
\end{equation*}
$$

2) formally inverting Eq. (3.3) to expand $\lambda_{0}$ in power series of $\lambda_{\mathrm{DS}}$

$$
\begin{equation*}
\lambda_{0}=\lambda_{\mathrm{DS}}-\lambda_{\mathrm{DS}}^{3} f_{1}-\lambda_{\mathrm{DS}}^{5}\left(f_{2}-3 f_{1}^{2}\right)-\ldots \tag{3.5}
\end{equation*}
$$

3) substituting Eq. (3.5) into the right hand side of Eq. (3.4)

$$
\begin{align*}
\frac{\partial \lambda_{\mathrm{DS}}}{\partial k_{i}^{\mu}} & =\lambda_{\mathrm{DS}}^{3} \frac{\partial f_{1}}{\partial k_{i}^{\mu}}+\lambda_{\mathrm{DS}}^{5}\left(\frac{\partial f_{2}}{\partial k_{i}^{\mu}}-3 f_{1} \frac{\partial f_{1}}{\partial k_{i}^{\mu}}\right)+\ldots  \tag{3.6}\\
& \equiv \lambda_{\mathrm{DS}}^{3} \beta_{1}\left(\left\{k_{i}\right\}\right)+\lambda_{\mathrm{DS}}^{5} \beta_{2}\left(\left\{k_{i}\right\}\right)+\ldots
\end{align*}
$$

When the underlying theory is renormalizable, all the coefficient functions $\beta_{i}\left(\left\{k_{i}\right\}\right)$ of the multi-momentum $\beta$-function will be finite. The DS vertex function $\lambda_{\text {DS }}$ can be obtained by solving this equation with the specification of an integration constant (usually in the guise of a quantity analogous to $\Lambda_{Q C D}$ ).

To one-loop order, however, the renormalization group equation is equivalent to the well-known trick of eliminating the bare coupling constant through the
introduction of an integration constant. That is, if

$$
\begin{equation*}
\lambda_{\mathrm{DS}}\left(\left\{k_{i}\right\}\right)=\lambda_{0}+\lambda_{0}^{3}\left[\frac{c}{\epsilon}+f\left(\left\{k_{i}\right\}\right)\right]+O\left(\lambda_{0}^{5}\right) \tag{3.7}
\end{equation*}
$$

where the divergent part in the order $\lambda_{0}^{3}$ term is contained in the $1 / \epsilon$ pole term, then formally

$$
\begin{equation*}
\frac{1}{\lambda_{\mathrm{DS}}^{2}\left(\left\{k_{i}\right\}\right)}=\frac{1}{\lambda_{0}^{2}}-\frac{2 c}{\epsilon}-2 f\left(\left\{k_{i}\right\}\right)+O\left(\lambda_{0}^{2}\right) . \tag{3.8}
\end{equation*}
$$

This last equation is valid for any set of values of $\left\{k_{i}\right\}$; therefore, we also have

$$
\begin{equation*}
\frac{1}{\lambda_{\mathrm{DS}}^{2}\left(\left\{k_{i}^{\prime}\right\}\right)}=\frac{1}{\lambda_{0}^{2}}-\frac{2 c}{\epsilon}-2 f\left(\left\{k_{i}^{\prime}\right\}\right)+O\left(\lambda_{0}^{2}\right) \tag{3.9}
\end{equation*}
$$

Taking the difference between these last two equations and neglecting higher order terms, we obtain:

$$
\begin{equation*}
\lambda_{\mathrm{DS}}^{2}\left(\left\{k_{i}\right\}\right)=\frac{1}{C-2 f\left(\left\{k_{i}\right\}\right)}, \tag{3.10}
\end{equation*}
$$

where $C=2 f\left(\left\{k_{i}^{\prime}\right\}\right)+1 / \lambda_{\mathrm{DS}}^{2}\left(\left\{k_{i}^{\prime}\right\}\right)$ is effectively an integration constant.
The exact formula for the vertex function to 1 -loop order in $\phi_{6}^{3}$ theory is somewhat complicated, but in the small mass limit when some of the legs are on-shell, a simplified expression can be obtained (see next Chapter).

We notice that the expansion order in DSE is two-fold. We have to specify: 1) the number of vertices used in expanding a general scattering amplitude in terms of the DS vertex function, and 2) the number of terms used to compute the renormalization group equation for the DS vertex function. This actually also happens in the conventional method of perturbative calculations: the results in perturbative QCD also contain two expansion orders, one corresponding to the
order of the result itself in terms of the running coupling constant, and the other one corresponding to the order that is used in obtaining the running coupling constant through the $\beta$-function formalism. In principle there is no requirement that we match one order with the other, but in practice these two orders are often kept the same.

### 3.3. BLM as the Dressed Photon Expansion



Fig. 3.10 The photon charged propagator in QED.

In QED, due to the Ward Identity $Z_{1}=Z_{2}$, it is not necessary to perform a full skeleton expansion in order to renormalize the charge. In fact, as a consequence of this identity, the photon "charged propagator" (photon full propagator multiplied by the squared bare charge) (see Fig. 3.10)

$$
\begin{align*}
i e_{0}^{2} \Delta_{\mu \nu}(p) & =\frac{i e_{0}^{2}}{p^{2}}\left[g_{\mu \nu}-\frac{p_{\mu} p_{\nu}}{p^{2}}\right]+\ldots  \tag{3.11}\\
& \equiv \frac{i e^{2}\left(p^{2}\right)}{p^{2}}\left[g_{\mu \nu}-\frac{p_{\mu} p_{\nu}}{p^{2}}\right]
\end{align*}
$$

is a finite function by itself [25]. Therefore, in QED, we can use the dressed
photon expansion instead of the dressed vertex expansion. That is, the renormalization group equation is performed on the photon two-point function (effectively the "running coupling constant" $e\left(p^{2}\right)$ ) rather than on the fermion-photon-fermion vertex, and all other Green's functions (including the fermion full propagator and the fermion-photon-fermion vertex) are to be expressed in terms of dressed photon diagrams.

This is essentially the basis behind the BLM [2] "automatic scale setting" procedure in the case of QED. For instance, the lowest dressed-photon graph for the computation of the muon anomalous magnetic moment is given in Fig. 3.11. The result from this diagram can be expressed as

$$
\begin{equation*}
a_{\mu}=\frac{\alpha\left(Q^{* 2}\right)}{2 \pi}=\frac{e^{2}\left(Q^{* 2}\right)}{\delta \pi^{2}}, \tag{3.12}
\end{equation*}
$$

where $Q^{*}$ is the effective scale. Notice that the loop integral of this diagram is performed with the running coupling constant (i.e., the photon charged propagator) inside the integrand; therefore, $e^{2}\left(Q^{* 2}\right)$ effectively is the value obtained though the mean-value theorem by pulling the running coupling constant out of the integral. By using the running coupling constant $e\left(p^{2}\right)$ to one-loop order, this effective scale can be shown to be [26]

$$
\begin{equation*}
Q^{*}=m_{\mu} \exp (-5 / 4) \tag{3.13}
\end{equation*}
$$



Fig. 3.11 The lowest-order dressed-photon diagram for the muon anomalous magnetic moment calculation.

The fermion full propagator (mass and wavefunction renormalization effects) is also to be computed with dressed photon diagrams. The diagrammatic expansion is depicted in Fig. 3.12, but we shall not undertake detailed discussion of this subject here.


Fig. 3.12 The full fermion propagator in the dressed photon expansion.

The association of charge renormalization to a two-point function rather than a vertex function is not an exclusive property of QED. As we shall see shortly, the leading $1 / N$ Gross-Neveu model in the auxiliary field form also exhibits this feature.

Finally, here is a personal opinion. In QED to one-loop level, the BLM method
is equivalent to the absorption of the light-fermion family number $n_{f}$ (from vacuumpolarization effects) into the running coupling constant. This useful coincidence is peculiar to QED, and the $n_{f}$ criterion probably should not be regarded as the strict definition of the BLM scale setting method in extending to other field theories.

Recently S.J. Brodsky has studied a correspondence principle between QED and QCD [27]. Basically, QED can be interpreted as the limit of QCD when the number of colors tends to zero. This ensures the same scale in QCD as QED for corresponding processes.

APPLICATIONS OF DRESSED SKELETON EXPANSION

## 4.1. $\phi_{6}^{3}$ Theory

As a concrete application of the dressed skeleton expansion, let us first analyze the two-particle elastic scattering amplitude in $\phi_{6}^{3}$ theory. We will consider the limit when all the relevant momentum scales are much larger than the mass of the particles.

We use dimensional regularization [28] in $d=6+2 \epsilon$ to regulate ultraviolet divergences. The bare coupling constant of the theory can be expanded in terms of the $\overline{\mathrm{MS}}$ scheme [11] dimensionless coupling constant $\lambda_{\mu}$

$$
\begin{equation*}
\lambda_{0}=\mu^{-\epsilon}\left[\lambda_{\mu}+\frac{3}{8 \hat{\epsilon}} \frac{\lambda_{\mu}^{2}}{(4 \pi)^{3}}+\ldots\right] \tag{4.1}
\end{equation*}
$$

with

$$
\begin{equation*}
\frac{1}{\hat{\epsilon}}=\frac{1}{\epsilon}-\log (4 \pi)+\gamma_{E} . \tag{4.2}
\end{equation*}
$$

As pointed out in the previous section, the basic building block of the dressed skeleton expansion is the renormalized vertex function. In order to obtain this function, we first need to obtain the cffective wavefunction renormalization constant from the renormalized propagator.

$$
\stackrel{p^{2}}{=}=z\left(p^{2}\right) \stackrel{p^{2}}{ }
$$

Fig. 4.1. Propagator renormalization in $\phi_{6}^{3}$ theory.

The propagator to 1-loop order (Fig. 4.1) in the massless limit is given by

$$
\begin{equation*}
\Delta\left(p^{2}\right)=Z\left(p^{2}\right) \frac{i}{p^{2}-m_{\mathrm{ph}}^{2}} \tag{4.3}
\end{equation*}
$$

where $Z\left(p^{2}\right)$ is the effective wavefunction renormalization constant and $m_{\mathrm{ph}}$ the physical mass. We will need the expression of $Z\left(p^{2}\right)$ in the on-shell $\left(p^{2}=m_{\mathrm{ph}}^{2}\right)$ and large-momentum $\left(p^{2} \gg m^{2}\right)$ limits. In the on-shell limit, we have

$$
\begin{equation*}
Z_{\mathrm{OS}}=Z\left(p^{2}=m_{\mathrm{ph}}^{2}\right)=1+\frac{\lambda_{\mu}^{2}}{12(4 \pi)^{3}}\left[\frac{1}{\hat{\epsilon}}+\log \left(\frac{m_{p h}^{2}}{\mu^{2}}\right)-\frac{17}{3}+\pi \sqrt{3}\right] \tag{4.4}
\end{equation*}
$$

and in the large- $p^{2}$ limit

$$
\begin{equation*}
Z\left(p^{2}\right)=1+\frac{\lambda_{\mu}^{2}}{12(4 \pi)^{3}}\left[\frac{1}{\hat{\epsilon}}+\log \left(-\frac{p^{2}}{\mu^{2}}-i \epsilon\right)-\frac{8}{3}\right] \tag{4.5}
\end{equation*}
$$

Notice that we have absorbed all renormalization effects into $Z\left(p^{2}\right)$ and $Z_{\text {OS }}$. That is, the propagator retains its bare form, with only the bare mass replaced by the physical mass.


Fig. 4.2. Vertex renormalization in $\phi_{6}^{\mathbf{3}}$ theory.

The unrenormalized three-point function with one off-shell leg (Fig. 4.2) in the massless limit to 1-loop order is given by

$$
\begin{equation*}
\Gamma=-i \lambda_{0}\left\{1-\frac{\lambda_{\mu}^{2}}{2(4 \pi)^{3}}\left[\frac{1}{\hat{\epsilon}}+\log \left(-\frac{p^{2}}{\mu^{2}}-i \epsilon\right)-3\right]\right\} \tag{4.6}
\end{equation*}
$$

Its renormalized version is

$$
\begin{align*}
& \left.\Gamma_{R}=Z_{\mathrm{OS}} Z^{1 / 2}\left(p^{2}\right) \Gamma \equiv-i \lambda_{( } p^{2}\right) \\
& =-i \lambda_{0}\left\{1+\frac{\lambda_{\mu}^{2}}{(4 \pi)^{3}}\left[-\frac{3}{8 \hat{\epsilon}}+\frac{11}{12}+\frac{\pi \sqrt{3}}{12}+\frac{1}{12} \log \left(\frac{m_{p h}^{2}}{\mu^{2}}\right)-\frac{11}{24} \log \left(-\frac{p^{2}}{\mu^{2}}-i \epsilon\right)\right]\right\} \tag{4.7}
\end{align*}
$$

The renormalization group equation to this order is an algebraic equation that simply states that $\lambda_{0}$ is unique:

$$
\begin{equation*}
\frac{1}{\lambda_{0}^{2}}=\frac{1}{\lambda^{2}\left(p^{2}\right)}-\frac{1}{(4 \pi)^{3}}\left[\frac{3}{4 \hat{\epsilon}}-\frac{11}{6}-\frac{\pi \sqrt{3}}{6}-\frac{1}{6} \log \left(\frac{m_{p h}^{2}}{\mu^{2}}\right)+\frac{11}{12} \log \left(-\frac{p^{2}}{\mu^{2}}-i \epsilon\right)\right] \tag{4.8}
\end{equation*}
$$

The solution to this equation is

$$
\begin{equation*}
\lambda^{2}\left(p^{2}\right)=\frac{12(4 \pi)^{3}}{11 \log \left(-p^{2} / \Lambda_{\mathrm{DS}}^{2}-i \epsilon\right)} \tag{4.9}
\end{equation*}
$$

where $\Lambda_{\mathrm{DS}}\left(\mathrm{DS}=\mathrm{Dressed}\right.$ Skeleton) is a parameter that mimics the role of $\Lambda_{\mathrm{QCD}}$.

Similarly, when there are two off-shell legs, the 3-point vertex function can be shown to be

$$
\begin{align*}
\lambda^{2}\left(p^{2}, q^{2}\right)=12(4 \pi)^{3}\{ & 12 \frac{p^{2} \log \left(-p^{2} / \Lambda_{\mathrm{DS}}^{2}-i \epsilon\right)-q^{2} \log \left(-q^{2} / \Lambda_{\mathrm{DS}}^{2}-i \epsilon\right)}{p^{2}-q^{2}} \\
& \left.-\log \left(-\frac{p^{2}}{\Lambda_{\mathrm{DS}}^{2}}-i \epsilon\right)-\log \left(-\frac{q^{2}}{\Lambda_{\mathrm{DS}}^{2}}-i \epsilon\right)\right\}^{-1} \tag{4.10}
\end{align*}
$$

Notice that when $\left|p^{2}\right| \gg\left|q^{2}\right|$, we recover the one off-shell leg vertex; that is,

$$
\begin{equation*}
\lim _{\left|p^{2}\right| \gg\left|q^{2}\right|} \lambda\left(p^{2}, q^{2}\right)=\lambda\left(p^{2}\right) . \tag{4.11}
\end{equation*}
$$



Fig. 4.3. Tree-skeletons for two-particle scattering.

The two-particle scattcring amplitudc to tree skeleton level (Fig. 4.3) and to 1-loop renormalization in the fundamental vertices is given by

$$
\begin{align*}
i M_{\text {tree }} & =[-i \lambda(s)]^{2} \frac{i}{s}+[-i \lambda(t)]^{2} \frac{i}{t}+[-i \lambda(u)]^{2} \frac{i}{u} \\
& =-i \frac{12}{11}(4 \pi)^{3}\left[\frac{1}{s\left(\log \left|s / \Lambda_{\mathrm{DS}}^{2}\right|-i \pi\right)}+\frac{1}{t \log \left|t / \Lambda_{\mathrm{DS}}^{2}\right|}+\frac{1}{u \log \left|u / \Lambda_{\mathrm{DS}}^{2}\right|}\right] \tag{4.12}
\end{align*}
$$



Fig. 4.4. One-loop skeleton for two-particle scattering.

The box diagrams are calculated by inserting the renormalized vertex functions in the momentum integrals. For the box diagram shown in Fig. 4.4, we have the expression

$$
\begin{equation*}
i M_{b o x}=\int \frac{d^{6} k}{(2 \pi)^{6}} \frac{\lambda\left(k_{1}^{2}, k_{2}^{2}\right) \lambda\left(k_{2}^{2}, k_{3}^{2}\right) \lambda\left(k_{3}^{2}, k_{4}^{2}\right) \lambda\left(k_{4}^{2}, k_{1}^{2}\right)}{\left(k_{1}^{2}+i \epsilon\right)\left(k_{2}^{2}+i \epsilon\right)\left(k_{3}^{2}+i \epsilon\right)\left(k_{4}^{2}+i \epsilon\right)}, \tag{4.13}
\end{equation*}
$$

Observe that this expression contains no undetermined momentum scales. That is, higher order skeleton diagrams in general are also scale ambiguity free.

The numerical evaluation of the box diagram integral is complicated by the unusual presence of the vertex function $\lambda\left(p^{2}, q^{2}\right)$. However, we can formally expand the integral in power series of $1 / \log \left(s / \Lambda_{\mathrm{DS}}^{2}\right)$ (by expanding first $\lambda\left(p^{2}, q^{2}\right)$ in power series of $\left.1 / \log \left(s / \Lambda_{\mathrm{DS}}^{2}\right)\right)$. We shall content ourselves with the first two terms in this series expansion. In asymptotic free region ( $s \gg \Lambda_{\mathrm{DS}}^{2}$ ) this series would give a good approximation to the exact integral $[29,30]$.

To order $\log ^{-3}\left(s / \Lambda_{\mathrm{DS}}^{2}\right)$ the box diagram of Fig. 4.4 gives

$$
\begin{align*}
i M_{b o x}(s ; \hat{s}, \hat{t}) & =\frac{i(4 \pi)^{3}}{s \log ^{2}\left(s / \Lambda_{\mathrm{DS}}^{2}\right)}\left(\frac{6}{5}\right)^{2}\left\{I_{1}(\hat{s}, \hat{t})\right. \\
& \left.+\frac{1}{5 \log \left(s / \Lambda_{\mathrm{DS}}^{2}\right)}\left[-2 I_{1}(\hat{s}, \hat{t})+11 I_{2}(\hat{s}, \hat{t})+2 I_{3}(\hat{s}, \hat{t})\right]\right\} \tag{4.14}
\end{align*}
$$

where

$$
\begin{align*}
& \begin{aligned}
& \hat{s}=1 ; \hat{t}= t / s ; \hat{u}=u / s \\
& I_{1}(\hat{s}, \hat{t})= \frac{-1}{2(\hat{s}+\hat{t})}\left\{\log ^{2}\left(\frac{\hat{s}+i \epsilon}{\hat{t}+i \epsilon}\right)+\pi^{2}\right\} \\
& I_{2}(\hat{s}, \hat{t})= \frac{1}{\hat{s}+\hat{t}}\left\{\log \left(\frac{\hat{s}+i \epsilon}{\hat{t}+i \epsilon}\right)\left[\underline{L i}_{2}\left(\frac{\hat{s}+\hat{t}}{\hat{s}}\right)-\underline{L i}_{2}\left(\frac{\hat{s}+\hat{t}}{\hat{t}}\right)\right]\right. \\
&\left.+2\left[\underline{L i}_{3}\left(\frac{\hat{s}+\hat{t}}{\hat{s}}\right)+\underline{L i}_{3}\left(\frac{\hat{s}+\hat{t}}{\hat{t}}\right)\right]\right\} \\
& I_{3}(\hat{s}, \hat{t})=\frac{1}{\hat{s}+\hat{t}}\left\{-\frac{1}{2} \log \left|\frac{\hat{s}}{\hat{t}}\right|\left[\log ^{2}(-\hat{s}-i \epsilon)-\log ^{2}(-\hat{t}-i \epsilon)\right]\right. \\
&-\log (-\hat{s}-i \epsilon) \underline{L i}_{2}\left(\frac{\hat{s}+\hat{t}}{\hat{t}}\right)-\log (-\hat{t}-i \epsilon) \underline{L i}_{2}\left(\frac{\hat{s}+\hat{t}}{\hat{s}}\right) \\
&\left.+\underline{L i}_{3}\left(\frac{\hat{s}+\hat{t}}{\hat{s}}\right)+\underline{L i}_{3}\left(\frac{\hat{s}+\hat{t}}{\hat{t}}\right)\right\}
\end{aligned}
\end{align*}
$$

and $\underline{L i}_{2}(x)$ and $\underline{L i}_{3}(x)$ are respectively the real part of the dilogarithm and the real part of the trilogarithm functions [29].

Adding up contributions from tree and box skeletons, the total scattering amplitude is given by

$$
\begin{align*}
i M_{\mathrm{DSE}}= & i M_{t r e e}(s, t, u)+i M_{b o x}(s ; \hat{t}, \hat{u}) \\
& +i M_{b o x}(s ; \dot{\hat{u}}, \hat{s})+i M_{b o x}(s ; \hat{s}, \hat{t}) . \tag{4.16}
\end{align*}
$$

We will compare this result with the two-particle scattering amplitude obtained by standard methods of scale fixing, which is derived next.


Fig. 4.5 Feynman diagrams for two-particle scattering.

The $\overline{\mathrm{MS}}$-scheme running coupling constant to second order in $\beta$-function [31] is given by the solution of the following equation:

$$
\begin{equation*}
\frac{1}{\alpha_{\overline{\mathrm{MS}}}(\mu)}+\frac{125}{108} \log \left(\frac{125 \alpha_{\overline{\mathrm{MS}}}(\mu)}{125 \alpha_{\overline{\mathrm{MS}}}(\mu)+108}\right)=\frac{3}{4} \log \left|\mu^{2} / \Lambda_{\overline{\mathrm{MS}}}^{2}\right| \tag{4.17}
\end{equation*}
$$

and the squared renormalized scattering amplitude to 1-loop order (Fig. 4.5) is [29]

$$
\begin{equation*}
|M|^{2}=(4 \pi)^{6} \alpha_{\overline{\mathrm{MS}}}^{2}(\mu)\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)^{2}\left\{1+\alpha_{\overline{\mathrm{MS}}}(\mu)\left[\frac{3}{2} \log \left|\frac{\mu^{2}}{\Lambda_{\overline{\mathrm{MS}}}^{2}}\right|-\frac{11}{6} \mathrm{H}\left(s, t, u, \tilde{\Lambda}^{2}\right)\right]\right\} \tag{4.18}
\end{equation*}
$$

where

$$
\begin{align*}
\tilde{\Lambda} & =\Lambda_{\overline{M S}} \exp \left\{1+\frac{\pi \sqrt{3}}{11}+\frac{1}{11} \log \left|\frac{m_{\mathrm{ph}}^{2}}{\Lambda \frac{2}{\mathrm{MS}}}\right|\right\} \\
\mathrm{H}\left(s, t, u, \tilde{\Lambda}^{2}\right) & =\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)^{-1}\left\{\frac{1}{s}\left(\log \left|\frac{s}{\tilde{\Lambda}^{2}}\right|+\frac{6}{11} \log ^{2}\left|\frac{t}{u}\right|+\frac{6 \pi^{2}}{11}\right)\right.  \tag{4.19}\\
+ & \left.\frac{1}{t}\left(\log \left|\frac{t}{\tilde{\Lambda}^{2}}\right|+\frac{6}{11} \log ^{2}\left|\frac{u}{s}\right|\right)+\frac{1}{u}\left(\log \left|\frac{u}{\tilde{\Lambda}^{2}}\right|+\frac{6}{11} \log ^{2}\left|\frac{s}{t}\right|\right)\right\}
\end{align*}
$$

The $\alpha \frac{3}{\text { MS }}$ term in Eq. (4.18) comes from the interference of the 1 -loop diagrams with the tree diagrams.

We now apply the usual scale setting prescriptions. The result in PMS can be expressed as

$$
\begin{equation*}
|M|_{P M S}^{2}=(4 \pi)^{6}\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)^{2} \alpha^{2} \frac{324+125 \alpha}{324+375 \alpha} \tag{4.20}
\end{equation*}
$$

with $\alpha$ given self-consistently by

$$
\begin{equation*}
\alpha=\frac{12}{11 \mathrm{H}\left(s, t, u, \tilde{\Lambda}^{2}\right)}\left[\frac{324+500 \alpha}{324+375 \alpha}+\frac{125}{108} \alpha \log \left(\frac{125 \alpha}{125 \alpha+108}\right)\right] \tag{4.21}
\end{equation*}
$$

The result in FAC can be expressed as

$$
\begin{equation*}
|M|_{F A C}^{2}=(4 \pi)^{6}\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)^{2} \alpha^{2} \tag{4.22}
\end{equation*}
$$

with $\alpha$ given self-consistently by

$$
\begin{equation*}
\alpha=\frac{12}{11 \mathrm{H}\left(s, t, u, \tilde{\Lambda}^{2}\right)-\frac{125}{9} \log \left(\frac{125 \alpha}{125 \alpha+108}\right)} . \tag{4.23}
\end{equation*}
$$

(The BLM scale setting method does not apply here, since the charge renormalization in $\phi_{6}^{3}$ theory involves no fermion loops. The dressed skeleton calculation is effectively the extension of BLM in this case, in the sense that the coupling constant renormalization effects are absorbed into a vertex function and the effective coupling scale is set automatically.)

To compare the results of PMS and FAC with DSE, we need to know the relationship between $\tilde{\Lambda}$ and $\Lambda_{D S}$. Let us use $\Lambda_{\text {DS }}$ as our unit of momentum $\Lambda_{D S}=1$, and express all other momentums in unit of $\Lambda_{D S}$. We take the physical point
$s=2|t|=2|u|=10^{6}$ as the matching point. This leads to

$$
\begin{align*}
& \tilde{\Lambda}_{P M S}=3.5805,  \tag{4.24}\\
& \tilde{\Lambda}_{F A C}=3.6149 .
\end{align*}
$$

However, in practice the numerical results of PMS and FAC are indistinguishable. In Fig. 4.6 we show the $s$-dependence of $|M|_{D S E}^{2}$ and $|M|_{P M S-F A C}^{2}$ for the "symmetrical point" $s=2|t|=2|u|$, assuming that $\Lambda_{\mathrm{DS}}=1$. In Fig. 4.7 we show the $t$ dependence for fixed values of $s$. For this multi-scale process, Fig. 4.6 depicts the dependence of the scattering amplitude on the overall scale of the system, while Fig. 4.7 illustrates the dependence on the relative scale $(t / s)$.


Fig. 4.6 The $s$-dependence of the probability amplitude along the "symmetric" line $s=-2 t=-2 u$.


Fig. 4.7 The $t$-dependence of the probability amplitude for fixed values of $s$.

From these figures we can observe that, in this example

1) There is no qualitative difference between the results calculated by DSE or conventional methods (PMS-FAC) in the relative-scale dependence (Fig. 4.7 ), as long as we are in the deep asymptotic free region (say, $s>10^{4}$ ).
2) The main difference between the DSE and the PMS-FAC predictions come from the overall-scale dependence. If $\phi_{6}^{3}$ theory were a realistic model, this difference potentially is large enough to allow an "experiment" to check the performance of the two methods at somewhat lower scale.( We must stress here that the agreement of disagreement of either method does not imply that a certain method is correct or incorrect, but simply means that the particular
method is more or less efficient in organizing the perturbative series so that higher order terms yield negligible contributions.)

### 4.2. Gross-Neveu Model in Leading $1 / N$ Expansion

This section is inspired by P. M. Stevenson's analysis of the PMS method in the Gross-Neveu model [32]. We will consider this model with the auxiliary scalar field $\sigma[33,34]$. The Lagrangian density of this model is given by

$$
\begin{equation*}
\mathcal{L}=\bar{\Psi}^{a}(i \not \partial) \Psi_{a}-\frac{1}{2} \sigma^{2}-g_{o} \bar{\Psi}^{a} \Psi_{a} \sigma, \quad a=1,2, \ldots, N \tag{4.25}
\end{equation*}
$$

The bare propagators and vertex functions of this theory are depicted in Fig. 4.8.


Fig. 4.8 Bare propagators and coupling vertex in the massless GrossNeveu model.

Let us analyze the off-shell fermion four-point function. Although this is not a "physical" quantity in the usual sense (because it is off-shellness), it nevertheless provides a simple Green's function where various ideas about scale fixing methods can be tested. For our purpose, we will only deal with perturbative quantities and bypass all non-perturbative effects arising from dynamical symmetry breaking [35]. The fermion four-point function to leading order in $1 / N$ has the structure (Fig.

$$
G\left(p_{1}, p_{2}, p_{3}, p_{4}\right)_{c d}^{a b}=-i g_{o}^{2}\left[\Delta(s) \delta^{a}{ }_{c} \delta^{b}{ }_{d}-\Delta(u) \delta^{a}{ }_{d} \delta^{b}{ }_{c}\right],
$$

where $\Delta(s)$ is the full propagator of the scalar particle to leading order in $1 / N$, $s=\left(p_{1}+p_{3}\right)^{2}$ and $u=\left(p_{1}+p_{2}\right)^{2}$.


Fig. 4.9 The fermion four-point function to leading order in $1 / N$. The double dashed line represents the full scalar propagator to leading order in $1 / N$.

Notice that for the Gross-Neveu model in the auxiliary-scalar-field context, every vertex in a given Feynman diagram counts as a negative unit power in $N$, while every scalar propagator counts as a positive unit power in $N$. Vertex and fermion self-energy corrections are thus absent in the leading $1 / N$ expansion, since
these effects are higher order in $1 / N$ [34].

Therefore, only the full scalar propagator multiplied by the squared bare charge needs renormalization. In the following we shall refer to this function as the "charged scalar propagator". That is, we can choose to "dress up" the charged scalar propagator instead of the three-point vertex function. This resembles the case of QED, where due to the fact that $Z_{1}=Z_{2}$ [2] the charge renormalization can be performed on the charged photon propagator.

As we will see later, in the leading $1 / N$ limit the dressed three-point function coincides with the charged propagator function. We will first consider the charged propagator here. Let us obtain its renormalization group equation, for the moment up to the sixth-order in the bare coupling constant.

$$
-i g_{D S}\left(p^{2}\right)=g_{0}^{2} \quad \cdot======
$$




Fig. 4.10 The charged scalar propagator and the vacuum polarization diagrams in Gross-Neveu model to leading order in $1 / N$.

The charged scalar propagator to this order is (see Fig. 4.10)

$$
\begin{align*}
-i g_{o}^{2} \Delta\left(p^{2}\right) & \equiv-i g_{\mathrm{DS}}^{2}\left(p^{2}\right) \\
& =g_{o}^{2}\left\{-i+(-i)\left[i g_{o}^{2} \Pi\left(p^{2}\right)\right](-i)+(-i)\left(\left[i g_{o}^{2} \Pi\left(p^{2}\right)\right](-i)\right)^{2}\right\}  \tag{4.27}\\
& =-i g_{o}^{2}\left\{1+g_{o}^{2} \Pi\left(p^{2}\right)+g_{o}^{4} \Pi^{2}\left(p^{2}\right)\right\}
\end{align*}
$$

where the subscript DS stands for Dressed Skeleton. The vacuum polarization correction (see Fig. 4.10) is given by

$$
\begin{equation*}
i g_{o}^{2} \Pi\left(p^{2}\right)=-g_{o}^{2} N \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{\operatorname{Tr}\{k(k+p)\}}{k^{2}(k+p)^{2}} . \tag{4.28}
\end{equation*}
$$

A straightforward calculation leads to

$$
\begin{equation*}
\Pi\left(p^{2}\right)=-\frac{N}{2 \pi}\left(\frac{1}{\hat{\epsilon}}+\log \left(-p^{2}-i \varepsilon\right)\right), \quad \frac{1}{\hat{\epsilon}}=\frac{1}{c}-\log 4 \pi+\gamma_{E}, \tag{4.29}
\end{equation*}
$$

where we have used dimensional regularization in $d=2+2 \epsilon$. Eq. (4.27) can be rewritten as

$$
\begin{equation*}
g_{\mathrm{DS}}^{2}\left(p^{2}\right)=g_{o}^{2}+g_{o}^{4} \Pi\left(p^{2}\right)+g_{o}^{6} \Pi^{2}\left(p^{2}\right) \tag{4.30}
\end{equation*}
$$

and by formally inverting this power series we can expand $g_{o}^{2}$ in power series of $g_{\mathrm{DS}}^{2}\left(p^{2}\right)$

$$
\begin{equation*}
g_{o}^{2}=g_{\mathrm{DS}}^{2}\left(p^{2}\right)-g_{\mathrm{DS}}^{4} \Pi\left(p^{2}\right)+g_{\mathrm{DS}}^{6}\left(p^{2}\right) \Pi^{2}\left(p^{2}\right)+O\left(g_{\mathrm{DS}}^{8}\right) \tag{4.31}
\end{equation*}
$$

Now let us obtain the renormalization group equation for $g_{\mathrm{DS}}\left(p^{2}\right)$. We first differentiate Eq. (4.30) with respect to the scale variable $x=\log \left(-p^{2}-i \varepsilon\right)$. Noting
that from Eq. (4.29)

$$
\begin{equation*}
\frac{d \Pi}{d x}=-\frac{N}{2 \pi} \tag{4.32}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
\frac{d g_{\mathrm{DS}}^{2}}{d x}=g_{o}^{4}\left(-\frac{N}{2 \pi}\right)+2 g_{o}^{6} \Pi\left(p^{2}\right)\left(-\frac{N}{2 \pi}\right) \tag{4.33}
\end{equation*}
$$

The next step is to use Eq. (4.31) here to replace $g_{o}$ by $g_{\mathrm{DS}}$. After this substitution we obtain the renormalization group equation for $g_{\mathrm{DS}}^{2}\left(p^{2}\right)$

$$
\begin{equation*}
\frac{d g_{\mathrm{DS}}^{2}}{d x}=-\frac{N}{2 \pi}\left\{\left(g_{\mathrm{DS}}^{2}\left(p^{2}\right)-g_{\mathrm{DS}}^{4}\left(p^{2}\right) \Pi\left(p^{2}\right)+\ldots\right)^{2}+2 g_{\mathrm{DS}}^{6}\left(p^{2}\right) \Pi\left(p^{2}\right)\right\} \tag{4.34}
\end{equation*}
$$

To order $g_{\mathrm{DS}}^{6}$, this equation reduces to

$$
\begin{equation*}
\frac{d g_{\mathrm{DS}}^{2}}{d x}=-\frac{N}{2 \pi} g_{\mathrm{DS}}^{4}+O\left(g_{\mathrm{DS}}^{8}\right) \tag{4.35}
\end{equation*}
$$

Notice that the order $g_{\mathrm{DS}}^{6}$ coefficient has vanished completely. This is a general result for this model: independent of the initial number of terms, all higher-order coefficients in Eq. (4.35) will vanish. (This result would have been obvious if we had applied the renormalization group equation to $g_{\mathrm{DS}}^{-2}$ instead of $g_{\mathrm{DS}}^{2}$, but we have chosen to present the equation for $g_{\mathrm{DS}}^{2}$ here to indicate the procedure for a general field theory.) In other words, we always obtain the exact infinite order solution

$$
\begin{equation*}
g_{\mathrm{DS}}^{2}\left(p^{2}\right)=\frac{2 \pi}{N \log \left(-p^{2} / \Lambda_{\mathrm{GN}}^{2}-i \varepsilon\right)} \tag{4.36}
\end{equation*}
$$

independent of the number of terms we have included in the original equation for the charged scalar propagator (Eq. (4.27)). This is true even if we have only included the lowest loop correction.

Naturally, we could choose to dress up the three-point vertex function instead of the two-point scalar function. However, in this particular model these two approaches are completely equivalent. Let us show next this equivalence.

In order to dress up the vertex function, we first need to obtain the effective wavefunction renormalization constant of the scalar propagator

$$
\begin{align*}
-i \Delta\left(p^{2}\right) & =-i+(-i)\left[i g_{o}^{2} \Pi\left(p^{2}\right)\right](-i)+\ldots  \tag{4.37}\\
& \equiv(-i) Z\left(p^{2}\right)
\end{align*}
$$

Since there are no fermion self-energy nor vertex corrections, we simply have to multiply the bare vertex function by the square root of the effective scalar wavefunction renormalization constant to renormalize the three-point function (see Fig. 4.11). If we designate the renormalized three-point function by $\tilde{g}_{\mathrm{DS}}\left(p^{2}\right)$, then

$$
\begin{equation*}
-i \tilde{g}_{\mathrm{DS}}\left(p^{2}\right) \equiv-i g_{o} Z^{1 / 2}\left(p^{2}\right) \tag{4.38}
\end{equation*}
$$



Fig. 4.11 Dressed three-point function in Gross-Neveu model to leading order in $1 / N$.

However, this implies

$$
\begin{equation*}
\tilde{g}_{\mathrm{DS}}^{2}\left(p^{2}\right)=g_{o}^{2} Z\left(p^{2}\right)=g_{\mathrm{DS}}^{2}\left(p^{2}\right) \tag{4.39}
\end{equation*}
$$

Thus dressing up the three-point vertex amounts exactly to dressing up the charged scalar two-point function.

The result for the fermion four-point function is obtained by replacing the $\Delta$ function in Eq. (4.26), using

$$
\begin{equation*}
g_{o}^{2} \Delta(s)=g_{\mathrm{DS}}^{2}(s)=\frac{2 \pi}{N \log \left(-s / \Lambda_{\mathrm{GN}}^{2}-i \varepsilon\right)} \tag{4.40}
\end{equation*}
$$

As shown in Ref. [32] and [33], this is also the exact answer. That is, for the leading $1 / N$ massless Gross-Neveu model, the exact answer is equivalent to the DSE expansion. This should be contrasted with conventional scale setting methods, where the results are not exact. In Fig. 4.12 we plot the symmetrized and the antisymmetrized four-point function [36] for spacelike $s$ and $u(s<0, u<0)$, where the scale has been fixed by applying second (without scheme variation) and third (with scheme variation) order PMS scale-scheme setting method. According to the convention of Ref. [32], these functions are defined by

$$
\begin{align*}
& R_{+}(s, u)=\frac{g_{o}^{2} N}{2 \pi}[\Delta(s)+\Delta(u)] \\
& R_{-}(s, u)=\frac{2 g_{o}^{2} N}{\pi \log (u / s)}[\Delta(s)-\Delta(u)] \tag{4.41}
\end{align*}
$$

and they are calculated in power series of a running coupling constant $g\left(\mu^{2}\right)$.


Fig. 4.12 Symmetrized $\left(R_{+}\right)$and antisymmetrized ( $R_{-}$) fermion four-point functions to leading order in $1 / N$ in GrossNeveu model. The dashed lines represent the exact results. The solid lines are the results obtained by applying the PMS optimization method. Fig. (a) and Fig. (b) correspond respectively to the second and the third order approximant.

We notice from the figure that the third-order approximant does improve over the second order approximant. This is especially true for the $R_{-}$component, which
is almost indistinguishable from the exact result in the range plotted. However, these approximants differ from the exact result at higher value of $u / s$. The conventional scale setting methods do not give the exact result in this simple model because they assign a single coupling scale to both skeleton graphs. In fact, if the conventional scale setting procedures (FAC, PMS) are applied to the two skeleton graphs individually, they will also give the exact result.

The lesson of this exercise is that different skeleton diagrams possess individual renormalization properties, and that by separating different skeleton graphs, at least in this case, one obtains a more exact answer.

## 4.3. $\mathrm{N}=2$ Gross-Neveu Model

In the following we will consider the $\mathrm{N}=2$ Gross-Neveu model without the $1 / N$ expansion (this is effectively a two-flavor Thirring model $[37,38]$ ). The main purpose of considering this model here is to illustrate the DSE calculation beyond the tree skeleton level. As before, we will only be interested in performing perturbative calculations, and all non-perturbative effects (dynamical mass generation, spontaneous symmetry breaking $[34,37,38]$, etc.) shall be bypassed. Since the vertex correction is no longer trivial, we cannot choose to dress up the charged two-point function. Instead, we have to perform the renormalization group equation on the three-point vertex function. We shall carry out our calculation within the context of dimensional regularization, with $d=2+2 \epsilon$. To one-loop order, the fermion self-energy correction remains zero (see Fig. 4.13),

$$
\begin{equation*}
i g_{o}^{2} \Sigma(p)=(-i)\left(-i g_{o}\right)^{2} \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{i}{k}=0 \tag{4.42}
\end{equation*}
$$

Hence there is no fermion wavefunction renormalization to this order:

$$
\begin{equation*}
Z_{f}(p)=1+O\left(g_{o}^{4}\right) . \tag{4.43}
\end{equation*}
$$





Fig. 4.13 One-loop self-energy, vacuum polarization and vertex correction diagrams in the $\mathrm{N}=2$ Gross-Neveu model.

From the previous section, the scalar propagator for $N=2$ is (see also Fig. 4.13)

$$
\begin{align*}
-i \Delta\left(k^{2}\right) & =-i+(-i)\left(i g_{0}^{2} \Pi\left(k^{2}\right)\right)(-i) \\
& =-i\left[1-\frac{g_{o}^{2}}{\pi}\left(\frac{1}{\hat{\epsilon}}+\log \left(-k^{2}-i \varepsilon\right)\right)\right] \tag{4.44}
\end{align*}
$$

From here, we obtain the effective scalar wavefunction renormalization constant

$$
\begin{equation*}
Z_{b}\left(k^{2}\right)=1-\frac{g_{o}^{2}}{\pi}\left(\frac{1}{\hat{\epsilon}}+\log \left(-k^{2}-i \varepsilon\right)\right) \quad, \quad \frac{1}{\hat{\epsilon}}=\frac{1}{\epsilon}-\log 4 \pi+\gamma_{E} \tag{4.45}
\end{equation*}
$$

The vertex correction (Fig. 4.12) is given by

$$
\begin{align*}
-i g_{o}^{3} \Gamma_{1}(p, q) & =(-i)\left(-i g_{o}\right)^{3} \int \frac{d^{d} r}{(2 \pi)^{d}} \frac{i}{k+t} \frac{i}{p+t}  \tag{4.46}\\
\Gamma_{1}(p, q) & =\Gamma_{1}\left(k^{2}\right)=\frac{1}{4 \pi}\left(\frac{1}{\hat{\epsilon}}+\log \left(-k^{2}-i \varepsilon\right)\right)
\end{align*}
$$

Combining the self-energy, vacuum polarization and vertex corrections, we obtain the renormalized vertex function

$$
\begin{equation*}
-i g_{\mathrm{DS}}\left(k^{2}\right) \equiv-i g_{o} Z_{f}^{1 / 2}(q)\left(1+g_{o}^{2} \Gamma_{1}\left(k^{2}\right)\right) Z_{f}^{1 / 2}(p) Z_{b}^{1 / 2}\left(k^{2}\right) \tag{4.47}
\end{equation*}
$$

This equation can be put into the following form:

$$
\begin{equation*}
\frac{1}{g_{\mathrm{DS}}^{2}\left(k^{2}\right)}=\frac{1}{g_{o}^{2}}+\frac{1}{2 \pi}\left(\frac{1}{\hat{\epsilon}}+\log \left(-k^{2}-i \varepsilon\right)\right) \tag{4.48}
\end{equation*}
$$

and its solution is given by

$$
\begin{equation*}
g_{\mathrm{DS}}^{2}\left(k^{2}\right)=\frac{2 \pi}{\log \left(-k^{2} / \Lambda_{\mathrm{DS}}^{2}-i \varepsilon\right)} \tag{4.49}
\end{equation*}
$$

Notice that if we had used the $1 / N$ expansion (compare with Eq. (4.36)), we would have erred by an overall factor of 2 . Also notice that the vertex function to this order depends exclusively on the squared momentum of the scalar particle. Now, let us use this vertex function to study the elastic scattering amplitude of two particles of the same flavor. Consider the process indicated in Fig. 4.14, where we have chosen the center-of-mass frame to express our kinematics. The corresponding tree skeleton diagrams are indicated in Fig. 4.15.

## before collision


$p_{1}=(p, p) \quad p_{2}=(p,-p)$
after collision


Fig. 4.14 Kinematics of the elastic collision between two same-flavor particles in the $\mathrm{N}=2$ Gross-Neveu model in the center-ofmass frame.


Fig. 4.15 Tree-skeleton diagrams for two-particle elastic scattering amplitude in the $\mathrm{N}=2$ Gross-Neveu model.

We shall use here $\gamma_{0}=\sigma_{x}$ and $\gamma_{1}=i \sigma_{y}$. The external fermion wavefunctions are given by

$$
\begin{gather*}
u_{1}=\sqrt{2 p}\binom{1}{0}, \quad u_{2}=\sqrt{2 p}\binom{0}{1}, \\
\bar{u}_{3}=\sqrt{2 p}\left(\begin{array}{ll}
0 & 1
\end{array}\right), \quad \bar{u}_{4}=\sqrt{2 p}\left(\begin{array}{ll}
1 & 0
\end{array}\right), \tag{4.50}
\end{gather*}
$$

and the tree level amplitude is simply

$$
\begin{align*}
i M_{t r e e} & =i g_{\mathrm{DS}}^{2}(t)\left(\bar{u}_{3} u_{1}\right)\left(\bar{u}_{4} u_{2}\right)+i g_{\mathrm{DS}}^{2}(u)\left(\bar{u}_{4} u_{1}\right)\left(\bar{u}_{3} u_{2}\right) \\
& =i 4 p^{2} g_{\mathrm{DS}}^{2}(u)  \tag{4.51}\\
& =\frac{i 8 \pi p^{2}}{\log \left(4 p^{2} / \Lambda_{\mathrm{DS}}^{2}\right)}
\end{align*}
$$

The Mandelstam variables have the following values

$$
\begin{equation*}
s=4 p^{2}, \quad t=0, \quad u=-4 p^{2} \tag{4.52}
\end{equation*}
$$

The one-loop order skeleton diagram is given in Fig. 4.16. Let us spend some time to discuss these diagrams. First of all, let us compute the box diagrams in the usual perturbation theory, i.e., using the bare coupling constant at the vertices instead of the DS vertex function. By simple power counting argument, one can see that the two diagrams are individually ultraviolet divergent. However, it turns out that the divergences coming from the two diagrams cancel each other, as one would expected from the renormalizability of the theory. The Feynman integral of these box diagrams is given by

$$
\begin{align*}
i M_{b o x} & =g_{o}^{4} \int \frac{d^{2} k}{(2 \pi)^{2}}\left(\bar{u}_{3} \frac{1}{p_{1}-k} u_{1}\right)\left(\bar{u}_{4} \frac{1}{p_{2}+k} u_{2}\right)+\left(\bar{u}_{3} \frac{1}{p_{1}-\not k} u_{1}\right)\left(\bar{u}_{4} \frac{1}{p_{2}-k} u_{2}\right) \\
& =-4 p^{2} g_{o}^{4} \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{k^{2}}{\left(k-p_{1}\right)^{2}}\left\{\frac{1}{\left(k+p_{2}\right)^{2}}-\frac{1}{\left(k-p_{2}\right)^{2}}\right\} . \tag{4.53}
\end{align*}
$$



Fig. 4.16 One-loop skeleton diagram for two-particle elastic scattering amplitude in the $\mathrm{N}=2$ Gross-Neveu model.

The propagators in these expressions come with the $+i \varepsilon$ prescription, and in the language of distribution theory they should be interpreted as the sum of a principal-value part and a delta function

$$
\begin{equation*}
\frac{1}{k^{2}+i \varepsilon}=\text { P.V. } \frac{1}{k^{2}}-i \pi \delta\left(k^{2}\right) \tag{4.54}
\end{equation*}
$$

The terms in the integrand in Eq. (4.53) can thus be classified into the following three types

1) product of two principal-value parts,
2) product of a principal value part with a delta function, and
3) product of two delta functions.


Fig. 4.17 Locations of the double-delta function singularities of the box diagrams (a) and (b) of Fig. 4.16 in the $k_{0}-k_{1}$ plane. The hyperbola indicates the location of the Landau singularity at $k^{2}=-\Lambda_{\mathrm{DS}}^{2}$.

By direct calculation, it can be shown that the contribution from the first two types of terms vanish; thus, the net contribution of the box diagrams comes entirely from the double delta function terms. In Fig. 4.17 we plot the locations of the singularities of the double delta functions. The result after integration has a simple expression:

$$
\begin{equation*}
i \mathcal{M}_{b o x}=-p^{2} g_{o}^{4} \tag{4.55}
\end{equation*}
$$

Now let us return to the dressed skeleton case. We have to replace the bare coupling vertex $-i g_{o}$ by the dressed vertex function $-i g_{\mathrm{DS}}\left(k^{2}\right)$. At high energies ( $p>\Lambda_{\mathrm{DS}}$ ) the dominant contribution is from the two double-delta points, because these two points are located in deep-spacelike and deep-timelike regions, i.e., far
away from the light-cone, and because $g_{\mathrm{DS}}\left(k^{2}\right)$ is a slow varying function at large $\left|k^{2}\right|$. Therefore, the corrections coming from the infrared behavior of the vertex function $g_{\mathrm{DS}}\left(k^{2}\right)$ are expected to be higher-twist in nature [39]. In a sense, we can interpret the two points shown in Fig. 4.17 as the "scalc-setting centers" of the skeleton box diagrams. The Landau singularity at $k^{2}=\Lambda_{\mathrm{DS}}^{2}$ might cause concern about the box integral, but one should bear in mind that this pole actually is located off the real axis due to the presence of the $+i \varepsilon$ term, and as long as we respect this prescription, this pole poses no threat to the finiteness of the box integral. It turns out that the box skeleton diagram can be calculated exactly (see Appendix A for the calculation and discussion about the box integral and its renormalons)

$$
\begin{align*}
i \mathcal{M}_{b o x} & =-4 p^{2} \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{k^{2}}{\left(k-p_{1}\right)^{2}}\left\{\frac{1}{\left(k+p_{2}\right)^{2}}-\frac{1}{\left(k-p_{2}\right)^{2}}\right\} g_{\mathrm{DS}}^{4}\left(k^{2}\right)  \tag{4.56}\\
& =-S p^{2} \beta^{\prime}\left(-\frac{2 i}{\pi} \log \left(2 p / \Lambda_{\mathrm{DS}}\right)\right),
\end{align*}
$$

where $\beta^{\prime}(z)$ is the derivative of the $\beta(z)$ function [40]

$$
\begin{equation*}
\beta(z)=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{z+n}, \quad \beta^{\prime}(z)=\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(z+n)^{2}} \tag{4.57}
\end{equation*}
$$

Needless to say, the box amplitude is totally free of scale ambiguity: the result of the skeleton box diagrams is directly expressed in terms of $p$ and $\Lambda_{\mathrm{DS}}$, and no extraneous coupling has been invoked in the calculation.

One can associate an "effective coupling" and an "effective scale" to the box diagram. These functions are defined by (see Eq. (4.49) and (4.55)):

$$
\begin{align*}
i \mathcal{M}_{l o x}(p) & \equiv-p^{2} g_{e f f}^{4}(p) \\
& \equiv p^{2} g_{\mathrm{DS}}^{4}\left(-p_{e f f}^{2}\right) \equiv p^{2}\left(\frac{2 \pi}{\log \left(p_{e f f}^{2} / \Lambda_{\mathrm{DS}}^{2}\right)}\right)^{2} \tag{4.58}
\end{align*}
$$

In Fig. 4.18 we plot the real and imaginary part of the effective coupling constant $g_{\text {eff }}(p)$, and in Fig. 4.19 we plot the Bode diagrams of amplitude and phase for the effective scale $p_{e f f}(p)$. We observe that at high energies the effective scale has, in the language of phasors, a reactive (negative) angle of $45^{\circ}$. This is expected since one box diagram probes into the deep timelike region while the other box diagram probes into the deep spacelike region (see Fig. 4.17); thus the effective scale is expected to be half reactive and half resistive. In contrast with conventional scale setting methods, the effective scales and the effective coupling constants in DSE are in general complex numbers.


Fig. 4.18 Real and imaginary parts of the effective coupling constant for the box amplitude of the $\mathrm{N}=2$ Gross-Neveu model. Notice that at high energy $g_{e f j}(p) \rightarrow g_{\mathrm{DS}}(p)$.


Fig. 4.19 (a) Bode diagram of amplitude for the effective scale of the box amplitude in the $N=2$ Gross-Neveu model. The dashed line represents $p_{\text {eff }}=p$. (b) Bode diagram of phase (measured in degrees) for the effective scale of the same amplitude. The dashed line indicates $-45^{\circ}$.

The total amplitude to one-loop skeleton level is given by the simple addition of the tree-level amplitude (Eq. (4.51)) and the box amplitude (Eq. (4.56)),

$$
\begin{equation*}
i \mathcal{M}_{\text {tot }}=i \mathcal{M}_{\text {tree }}+i \mathcal{M}_{\text {box }} . \tag{4.59}
\end{equation*}
$$

Notice that different order skeletons in general have different effective coupling scales, a feature that has been pointed out in the BLM paper [2].

### 4.4. Yukawa Interaction in $1+1$ Dimension

The main purpose in using the Yukawa model here is to present the subtleties related to the mass renormalization of propagators and to the matrix structure of vertex functions. While the usage of the skeleton technique for massless scalar bosons is straightforward, the presence of mass terms and the existence of matrix structure in the various basic vertex functions make the extension of the DSE not immediately trivial. The Yukawa model is chosen because it presents these two features at one-loop level. Although the Yukawa model in $1+1$ dimension is a superrenormalizable theory, this does not affect our discussion of the Dirac structure. The Yukawa theory describes the interaction between a fermion field and a scalar boson field according to the following Lagrangian density:

$$
\begin{equation*}
\mathcal{L}=\bar{\Psi}\left(i \not \partial-m_{f}\right) \Psi+\frac{1}{2}\left(\partial^{2}-m_{b}^{2}\right) \phi^{2}+\lambda_{o} m \bar{\Psi} \Psi \phi, \tag{4.60}
\end{equation*}
$$

where a mass unit $m$ has been inserted in the interaction term to make the bare coupling $\lambda_{0}$ dimensionless. To simplify our discussion, we shall assume that both the fermion physical mass and the boson physical mass are equal to $m$. The bare interaction vertex is scalar, in the sense that it is given by $-\lambda_{0} m$ and is thus proportional to the identity matrix (this will be the meaning of the word "scalar"
throughout this section). However, this feature is spoiled by the presence of higher order corrections. The full vertex function will in general contain a non-trivial Dirac structure (Fig. 4.20)

$$
\begin{equation*}
-i m \mathbf{\Lambda}(p, q)=-i m\left\{\tilde{\Lambda}_{o} 1+\tilde{\Lambda}_{1}(p, q) p+\tilde{\Lambda}_{2}(p, q) \notin+\tilde{\Lambda}_{3}(p, q) p \phi\right\} . \tag{4.61}
\end{equation*}
$$

In general the vertex function $\boldsymbol{\Lambda}(p, q)$ will be an $N \times N$ matrix, where $N$ is the dimension of the representation of the Dirac algebra, and an immediate question is how to apply the DSE method to obtain all the $N^{2}$ components of this vertex function. A first approach would be to write down the renormalization group equations for all the components and solve them separately. But this would introduce $N^{2}$ integration constants, that is, $N^{2}$ quantities analogous to $\Lambda_{\mathrm{QCD}}$. This is hardly necessary, for we know that, aside from the masses of the particles, we only need one more parameter to fix the entire theory. Therefore we can solve the equation for only one component, and then expand the other components in terms of the one we have solved for.

The next question is how to choose the component for the renormalization group equation. One obvious selection is $\tilde{\Lambda}_{o}$, for we know that in weak coupling regime the vertex function should somehow resemble the bare coupling, which is scalar (i.e., proportional to the identity matrix). More precisely, we

1) solve the renormalization group equation for $\tilde{\Lambda}_{o}(p, q)$

$$
\begin{equation*}
\tilde{\Lambda}_{o}(p, q)=\lambda_{o}\left(1+f_{1}(p, q) \lambda_{o}^{2}+f_{2}(p, q) \lambda_{o}^{4}+\ldots\right) \tag{4.62}
\end{equation*}
$$

2) expand the other components in power series of $\tilde{\Lambda}_{o}(p, q)$ by inverting Eq. (4.62). For example, $\tilde{\Lambda}_{1}$ will have the expression

$$
\begin{align*}
\tilde{\Lambda}_{1}(p, q) & =h_{1}(p, q) \lambda_{o}^{3}+h_{2}(p, q) \lambda_{o}^{5}+\ldots \\
& =h_{1}(p, q) \Lambda_{o}^{3}(p, q)+\left(h_{2}(p, q)-3 f_{1}(p, q)\right) \Lambda_{o}^{5}(p, q)+\ldots \tag{4.63}
\end{align*}
$$

While this procedure is formally valid, we shall argue that the four matrices $\{1, p, \not, p \not, p\}$ are not the most desirable basis for decomposing $\boldsymbol{\Lambda}$. The problem is that when $p$ and $q$ are on-shell and the vertex function is multiplied by the external fermion wavefunctions, the matrices $p$ and $\phi$ can be formally replaced by the scalar matrix $m \cdot 1$ because the wavefunctions satisfy the Dirac equation: $(p-m) u(p)=\bar{u}(q)(\phi-m)=0$. This means that, on-shell, the matrices $p$ and $\phi$ are indistinguishable from a scalar matrix. Thus it is highly unnatural to perform the RGE on $\tilde{\Lambda}_{a}$, for it means that its on-shell value will not be representative of the entire vertex function. Therefore, we are led to the more natural choice of basis matrices given by: $\{1, p-m, \notin-m,(h-m)(p-m)\}[41]$. Notice that now the non-scalar components $\{p-m, \phi-m,(k-m)(p-m)\}$ vanish on-shell upon contraction with the external fermion wavefunctions because of the Dirac equation; thus, the on-shell value of the vertex function is completely contained in the scalar component.

Let us explicitly compute of these components of the vertex function in DSE to 1-loop order. The scalar boson propagator offers no major difficulty: we simply absorb all the renormalization effect into the effective wavefunction renormalization constant $Z_{b}$ (Fig. 4.20):


Fig. 4.20 One-loop scalar and fermion propagator and vertex correction diagrams for the Yukawa model.

$$
\begin{gather*}
i \Delta_{b}\left(p^{2}\right)=\frac{i}{p^{2}-m_{b}^{2}}+\frac{i}{p^{2}-m_{b}^{2}} i \lambda_{o}^{2} m^{2} \Pi\left(p^{2}\right) \frac{i}{p^{2}-m_{b}^{2}}+\ldots  \tag{4.64}\\
i \lambda_{o}^{2} m^{2} \Pi\left(r^{2}\right)=(-1)\left(-i \lambda_{o} m\right)^{2}(i)^{2} \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{\operatorname{Tr}\left[\left(k+\nmid+m_{f}\right)\left(k+m_{f}\right)\right]}{\left[(k+r)^{2}-m_{f}^{2}\right]\left[k^{2}-m_{f}^{2}\right]} . \tag{4.65}
\end{gather*}
$$

To lowest order, we can replace the bare fermion mass $m_{f}$ by $m$ in the previous expression, and obtain

$$
\begin{equation*}
\Pi\left(r^{2}\right)=-\frac{1}{2 \pi}\left(\frac{1}{\hat{\epsilon}}+2+\int_{0}^{1} d x \log \left(-x(1-x) r^{2}+m^{2}-i \varepsilon\right)\right) \tag{4.66}
\end{equation*}
$$

The bare boson mass to order $\lambda_{o}^{2}$ is given by

$$
\begin{equation*}
m_{b}^{2}=m^{2}\left(1+\lambda_{o}^{2} c_{b}\right) \tag{4.67}
\end{equation*}
$$

where $c_{b}$ is the lowest order counterterm [42]. Substituting (4.66) and (4.67) into (4.64), and retaining only terms to order $\lambda_{o}^{2}$, we obtain the expression

$$
\begin{equation*}
i \Delta_{b}\left(r^{2}\right)=\frac{i}{r^{2}-m^{2}}\left(1+\lambda_{o}^{2} m^{2} \frac{c_{b}-\Pi\left(r^{2}\right)}{r^{2}-m^{2}}\right) \tag{4.68}
\end{equation*}
$$

On mass shell $\left(r^{2}=m^{2}\right), \Delta_{b}$ has a simple pole, therefore $c_{b}=\Pi\left(m^{2}\right)$, and

$$
\begin{equation*}
i \Delta_{b}\left(r^{2}\right)=\frac{i}{r^{2}-m^{2}}\left(1-\lambda_{o}^{2} m^{2} \frac{\Pi\left(r^{2}\right)-\Pi\left(m^{2}\right)}{r^{2}-m^{2}}\right) \equiv \frac{i}{r^{2}-m^{2}} Z_{b}\left(r^{2}\right) \tag{4.69}
\end{equation*}
$$

The effective wavefunction renormalization constant is given by

$$
\begin{equation*}
Z_{b}\left(r^{2}\right)=1-\lambda_{o}^{2} m^{2} \frac{\Pi\left(r^{2}\right)-\Pi\left(m^{2}\right)}{r^{2}-m^{2}} \tag{4.70}
\end{equation*}
$$

In particular, the on-shell renormalization constant is

$$
\begin{equation*}
Z_{b-O S}=Z_{b}\left(m^{2}\right)=1-\left.\lambda_{o}^{2} m^{2} \frac{d \Pi}{d r^{2}}\right|_{r^{2}=m^{2}}=1-\frac{\lambda_{o}^{2}}{2 \pi}\left(\frac{2 \pi}{3 \sqrt{3}}-1\right) \tag{4.71}
\end{equation*}
$$

For the fermion propagator we apply a similar procedure (Fig. 4.20). To one-loop order

$$
\begin{equation*}
i \mathbf{D}_{f}(p)=\frac{i}{\not p-m_{f}}+\frac{i}{p-m_{f}}\left(i \lambda_{o}^{2} m^{2} \Sigma(p)\right) \frac{i}{p-m_{f}} \tag{4.72}
\end{equation*}
$$

where the self-energy is given by

$$
\begin{equation*}
i \lambda_{o}^{2} m^{2} \Sigma(p)=\left(-i \lambda_{o} m\right)^{2}(i)^{2} \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{\not k+m}{\left[(k-p)^{2}-m_{b}^{2}\right]\left[k^{2}-m_{f}^{2}\right]} \tag{4.73}
\end{equation*}
$$

For $\mathbf{D}_{f}(p)$ to order $\lambda_{o}^{2}$, we can replace the boson mass $m_{b}$ by $m$. After removing
the mass counterterm for the fermion mass

$$
\begin{equation*}
m_{f}^{2}=m^{2}\left(1+\lambda_{o}^{2} c_{f}\right) \tag{4.74}
\end{equation*}
$$

by requiring $\mathbf{D}_{f}(p)$ to have a simple pole at $p^{2}=m^{2}$, we obtain the following expression for the full fermion propagator

$$
\begin{align*}
& i \mathbf{D}_{f}(p)= \mathrm{Z}_{f}(p) \frac{i}{p-m} \\
& \begin{aligned}
\mathrm{Z}_{f}^{1 / 2}(p)= & 1-\frac{\lambda_{o}^{2}}{8 \pi}\left(f\left(p^{2}\right)+6 \frac{m^{2} f\left(p^{2}\right)-m^{2} f\left(m^{2}\right)}{p^{2}-m^{2}}\right) \\
& -\frac{3 \lambda_{o}^{2}}{8 \pi} \frac{m^{2} f\left(p^{2}\right)-m^{2} f\left(m^{2}\right)}{p^{2}-m^{2}}\left(\frac{p-m}{m}\right), \\
f\left(p^{2}\right)= & \frac{m^{2}}{\sqrt{p^{2}\left(p^{2}-4 m^{2}+i \varepsilon\right)}}\left[\log \left(1-\sqrt{\frac{p^{2}}{p^{2}-4 m^{2}+i \varepsilon}}\right)\right. \\
& \left.-\log \left(1+\sqrt{\frac{p^{2}}{p^{2}-4 m^{2}+i \varepsilon}}\right)\right]
\end{aligned}
\end{align*}
$$

Notice that instead of a scalar wavefunction renormalization constant, we have introduced an effective wavefunction renormalization matrix. The on-shell expression of this matrix is

$$
\begin{align*}
\mathrm{Z}_{f-O S}^{1 / 2}(p) & =\left.\mathrm{Z}_{f}^{1 / 2}(p)\right|_{p^{2}=m^{2}} \\
& =1+\frac{\lambda_{o}^{2}}{8 \pi}\left(\frac{\pi}{3 \sqrt{3}}-4\right)+\frac{\lambda_{o}^{2}}{8 \pi}\left(\frac{\pi}{3 \sqrt{3}}-2\right) \frac{p-m}{m} \tag{4.76}
\end{align*}
$$

where the scalar part ( the first two terms) is readily identified as the conventional on-shell wavefunction renormalization constant. The last term vanishes on-shell upon contraction with the associated external fermion wavefunction.

Let us study the full vertex function at on-shell boson, one spacelike fermion and one on-shell fermion configuration. That is, $p^{2}=r^{2}=m^{2}$ and $q^{2}=-Q^{2}<0$.

The vertex function at a completely general momentum configuration could be studied in the same manner, but the expressions involved would be much more complicated.

The vertex correction (Fig. 4.20) is given by

$$
\begin{equation*}
-i \lambda_{o}^{3} m \Gamma_{1}=\left(-i \lambda_{o} m\right)^{3} \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{(i)^{3}(k+\not k+m)(k+p+m)}{\left[(k+q)^{2}-m^{2}\right]\left[(k+p)^{2}-m^{2}\right]\left[k^{2}-m^{2}\right]}, \tag{4.77}
\end{equation*}
$$

where we have set $m_{f}=m_{b}=m$. The decomposition of $\Gamma_{1}$ into the various components is given by

$$
\begin{equation*}
\Gamma_{1}\left(q^{2}\right)=\frac{1}{4 \pi}\left\{h_{o}\left(q^{2}\right) \mathbf{1}+h_{1}\left(q^{2}\right) \frac{p-m}{m}+h_{2}\left(q^{2}\right) \frac{h-m}{Q}+h_{3}\left(q^{2}\right) \frac{(q-m)(p-m)}{Q \cdot m}\right\} \tag{4.78}
\end{equation*}
$$

with

$$
\begin{align*}
h_{o}\left(q^{2}\right) & =3 \int_{0}^{1} d x \int_{0}^{x} d y \frac{x-y}{D^{2}} \\
h_{1}\left(q^{2}\right) & =\int_{0}^{1} d x \int_{0}^{x} d y \frac{1+x-3 y}{D^{2}} \\
h_{2}\left(q^{2}\right) & =\frac{Q}{m} \int_{0}^{1} d x \int_{0}^{x} d y \frac{-1+3 x-y}{D^{2}}  \tag{4.79}\\
h_{3}\left(q^{2}\right) & =\frac{Q}{m} \int_{0}^{1} d x \int_{0}^{x} d y \frac{x-y}{D^{2}} \\
D & =1-y+y^{2}-(1-x)(x-y) \frac{q^{2}}{m^{2}}-i \varepsilon
\end{align*}
$$

The renormalized vertex function is given by

$$
\begin{equation*}
-i m \boldsymbol{\Lambda}(p, q)=-i \lambda_{o} m \mathbf{Z}_{f}^{1 / 2}(q)\left(1+\lambda_{o}^{2} \Gamma_{1}\right) Z_{f-O S}^{1 / 2}(p) Z_{b-O S}^{1 / 2} \tag{4.80}
\end{equation*}
$$

Upon decomposition we have

$$
\begin{align*}
\Lambda(p, q)=\lambda_{\mathrm{DS}}\left(q^{2}\right) 1 & +\lambda_{1}\left(q^{2}\right) \frac{p-m}{m}+\lambda_{2}\left(q^{2}\right) \frac{\not q-m}{Q} \\
& +\lambda_{3}\left(q^{2}\right) \frac{(\not q-m)(p-m)}{Q m} \tag{4.81}
\end{align*}
$$

where we have named the scalar component the dressed skeleton effective coupling constant $\lambda_{\mathrm{DS}}\left(q^{2}\right)$. It satisfies the RGE

$$
\begin{equation*}
\frac{1}{\lambda_{\mathrm{DS}}^{2}\left(q^{2}\right)}=\frac{1}{\lambda_{o}^{2}}+\frac{1}{4 \pi}\left[f\left(q^{2}\right)-2 h_{o}\left(q^{2}\right)+6 \frac{m^{2} f\left(q^{2}\right)-m^{2} f\left(m^{2}\right)}{q^{2}-m^{2}}+\frac{\pi}{\sqrt{3}}+2\right] \tag{4.82}
\end{equation*}
$$

with the solution

$$
\begin{align*}
\lambda_{\mathrm{DS}}^{2}\left(q^{2}\right) & =\frac{\lambda^{2}}{1+\left(\lambda^{2} / 4 \pi\right) L\left(q^{2}\right)}, \\
L\left(q^{2}\right) & =\frac{q^{2}+5 m^{2}}{q^{2}-m^{2}} f\left(q^{2}\right)-2 h_{o}\left(q^{2}\right)-\frac{2 \pi}{\sqrt{3}} \frac{m^{2}}{q^{2}-m^{2}}-\frac{2 \pi}{3 \sqrt{3}}-\frac{7}{2} \tag{4.83}
\end{align*}
$$

where we have chosen the integration constant $\lambda$ such that $L\left(q^{2}=0^{-}\right)=0$. That is, $\lambda$ is the effective coupling at zero spacelike momentum

$$
\begin{equation*}
\lambda_{\mathrm{DS}}\left(q^{2}=0^{-}\right)=\lambda \tag{4.84}
\end{equation*}
$$

The general procedure to obtain the other three components involves an expansion of $\lambda_{o}$ in terms of $\lambda_{\operatorname{DS}}\left(q^{2}\right)$ by inverting Eq. (4.82), and then using this substitution in the various $\lambda_{i}\left(q^{2}\right)$ of Eq. (4.81). But to this order we simply need to replace $\lambda_{o}$ in Eq. (4.81) by $\lambda_{\text {DS }}$. The resulting expressions are

$$
\begin{align*}
& \lambda_{1}\left(q^{2}\right)=\frac{\lambda_{\mathrm{DS}}^{3}\left(q^{2}\right)}{4 \pi}\left(h_{1}\left(q^{2}\right)+\frac{\pi}{6 \sqrt{3}}-1\right) \\
& \lambda_{2}\left(q^{2}\right)=\frac{\lambda_{\mathrm{DS}}^{3}\left(q^{2}\right)}{4 \pi}\left(h_{2}\left(q^{2}\right)-\frac{3}{2} m Q \frac{f\left(q^{2}\right)-f\left(m^{2}\right)}{q^{2}-m^{2}}\right)  \tag{4.85}\\
& \lambda_{3}\left(q^{2}\right)=\frac{\lambda_{\mathrm{DS}}^{3}\left(q^{2}\right)}{4 \pi} h_{3}\left(q^{2}\right)
\end{align*}
$$



Fig. 4.21 Different components of the full vertex function of the Yukawa model in $1+1$ dimension as obtained by DSE. The external legs of the scalar boson and one of the fermions are on-shell; the second fermion has a spacelike momentum $q^{2}=-Q^{2}<0$. In Fig. (a) $\lambda^{2} / 4 \pi=0.15$. In Fig. (b) $\lambda^{2} / 4 \pi=0.1$.

In Fig. 4.21 we plot the different components of the full vertex function for two different values of $\lambda$. Notice that in the weak coupling regime (say, $\lambda^{2} / 4 \pi<0.1$ ),
the renormalization effects become small; namely, the scalar component at high energy only gets slightly renormalized, and the non-scalar ones become comparatively negligible.

## CHAPTER 5:

In multiple-scale processes, it is desirable to have a prescription for coupling scales from simple considerations of Feynman diagrams. For instance, in Fig. 5.1 (a) we have the elastic scattering of two quarks. We clearly have to assign $\mu^{2} \sim q^{2}$ for the coupling scales at the quark-gluon vertices $a$ and $b$. Similarly, in the case of the elastic scattering of three quarks via a three-gluon vertex as indicated in Fig. 5.1 (b), we would intuitively assign $\mu^{2} \sim p^{2}, q^{2}, r^{2}$ for the vertices $a, b$ and $c$. However, there is a priori no clear prescription for the coupling scale for the three-gluon vertex $d$.

(a)

(b)

Fig. 5.1 (a) Two-quark scattering process via one-gluon exchange.
(b) The non-Abelian part of three-gluon scattering process. The coupling scale at the vertex $d$ lacks a prescription.

The assignment of different coupling scales to different vertices cannot be done in an arbitrary fashion, though. Gauge invariance has to be observed; otherwise, the final result would be physically meaningless. The tree-level Feynman diagrams in Fig. 5.1 are gauge-invariant; hence, the assignment of different coupling constants for the various vertices is allowed to this order.

In the previous section we have applied the dressed skeleton expansion to a variety of field theoretical models. The extension of dressed-skeleton method to gauge theories is not straightforward, since the skeleton graphs in these theories are in general not gauge invariant. Unlike QED, where the dressed-photon expansion provides a gauge-invariant way of clustering Feynman diagrams, in QCD we lack of a systematic method of obtaining gauge-invariant skeletons.

Some time ago Cornwall and Papavassiliou obtained a gauge-invariant gluon propagator and three-gluon vertex function [43] to one-loop order through the application of the "pinch" technique. Essentially, these functions correspond to the gauge-invariant skeletons of QCD to one-loop level. In this Chapter, we apply the multi-momentum renormalization group equation of the dressed skeleton method to the gauge-invariant gluon two- and three-point functions and obtain their effective coupling scales.

In Section 5.1 we study the case of the quark-gluon vertex and recover the well known result of one-loop QCD running coupling constant.

In Section 5.2 we analyze the case of the three-gluon vertex. We obtain a somewhat more involved expression. However, the effective coupling scale is roughly given by

$$
\begin{equation*}
\mu^{2} \sim \frac{Q_{\min }^{2} Q_{\mathrm{med}}^{2}}{Q_{\max }^{2}} \tag{5.1}
\end{equation*}
$$

being $Q_{\text {min }}^{2}, Q_{\text {med }}^{2}$ and $Q_{\text {max }}^{2}$ respectively the smallest, the next-to-smallest and the largest gluon virtuality of the three-gluon vertex. We show that the functional form for the effective coupling supports the BLM ansatz [2] of using fermion loops as probes of coupling scales.

### 5.1. Quark-Gluon Coupling


. Fig. 5.2 Diagrams involved in the gauge-invariant gluon propagator calculation to one-loop order. The definition of the pinched diagrams are given in Ref. [43].

The gauge-invariant gluon propagator is calculated by using the pinch technique in Ref. [43]. The one-loop Feynman diagrams are indicated in Fig. 5.2. The interpretation of the pinched diagrams is explained in Ref. [43].

To illustrate the principle of pinch technique, let us analyze the pinched diagram in the upper right corner of Fig. 5.2. In Fig. 5.3 we show the same diagram before and after the pinching process.


Fig. 5.3 (a) A diagram involving three-gluon vertex in two-quark scattering amplitude. (b) The pinched part of the diagram in (a).

Although the pinch technique can be carried out in an arbitrary covariant gauge, the use of Feynman gauge offers enormous simplification. The bare propagator in a general covariant gauge is given by:

$$
\begin{equation*}
-i \Delta_{\mu \nu}^{o}\left(q^{2}\right)=-\frac{i}{q^{2}}\left(g_{\mu \nu}-\eta \frac{q_{\mu} q_{\nu}}{q^{2}}\right) \tag{5.2}
\end{equation*}
$$

and Feynman gauge corresponds to the choice $\eta=0$.
The three-gluon vertex in Fig. 5.3 can be decomposed into two parts (the group theoretical factor will be omitted for clarity)

$$
\begin{align*}
\Gamma_{\alpha \beta \gamma}(p, q, r) & =g_{\alpha \beta}(p-q)_{\gamma}+g_{\beta \gamma}(q-r)_{\alpha}+g_{\gamma \alpha}(r-p)_{\beta}  \tag{5.3}\\
& =\mathrm{I}_{\alpha \beta \gamma}^{\mathrm{F}}(p, q, r)+\Gamma_{\alpha \beta \gamma}^{\mathrm{P}}(p, q, r),
\end{align*}
$$

with

$$
\begin{align*}
& \Gamma_{\alpha \beta \gamma}^{\mathrm{F}}(p, q, r)=-2 g_{\alpha \beta} q_{\gamma}+2 g_{\beta \gamma} q_{\alpha}+g_{\gamma \alpha}(r-p)_{\beta}  \tag{5.4}\\
& \Gamma_{\alpha \beta \gamma}^{\mathrm{P}}(p, q, r)=-g_{\alpha \beta} r_{\gamma}+g_{\beta \gamma} p_{\alpha}
\end{align*}
$$

The Feynman part $\Gamma^{\mathrm{F}}$ is obtained by requiring it to satisfy a Feynman gauge Ward
identity:

$$
\begin{equation*}
q^{\beta} \Gamma_{\alpha \beta \gamma}^{\mathrm{F}}(p, q, r)=\left(p^{2}-r^{2}\right) g_{\alpha \gamma} \tag{5.5}
\end{equation*}
$$

(For $\eta \neq 0$ gauge, this must be replaced by the corresponding Ward identity in that gauge. See Ref. [44].)

The pinched part $\Gamma^{P}$ contains only components that carry longitudinal momenta $p_{\alpha}$ and $r_{\gamma}$. Notice that these momenta effectively pinch out the internal quark propagator in Fig. 5.3(a). For instance,

$$
\begin{equation*}
p_{\alpha} \gamma^{\alpha}=p=(k-m)-\left(k_{1}-m\right) \rightarrow \not k-m . \tag{5.6}
\end{equation*}
$$

The term $k_{1}-m$ vanishes upon multiplying the external quark wavefunction. The remaining term exactly cancels the internal quark propagator. Similarly,

$$
\begin{equation*}
r_{\gamma} \gamma^{\gamma}=t=\left(k_{2}-m\right)-(k-m) \rightarrow-(k-m) . \tag{5.7}
\end{equation*}
$$

Thus, when the pinched part $\Gamma^{p}$ is contracted with the vertices on the upper quark line, all the dependence on the quark mass $m$ is effectively removed. The pinch technique therefore extracts from the total amplitude the part that only depends on the momentum transfer $q^{2}$.

For more details on the use of pinch technique, the reader is referred to the original paper in Ref. [43].

The expression for the full gauge-invariant propagator can be parametrized as

$$
\begin{equation*}
-i \Delta_{\mu \nu}\left(q^{2}\right)=-\frac{i}{q^{2}}\left(g_{\mu \nu}-\frac{q_{\mu} q_{\nu}}{q^{2}}\right) Z\left(q^{2}\right)+i(1-\eta) \frac{q_{\mu} q_{\nu}}{q^{2}}, \tag{5.8}
\end{equation*}
$$

where $Z\left(q^{2}\right)$ is the gauge-invariant gluon wavefunction renormalization constant and $\eta$ the gauge parameter.

To one-loop order we have

$$
\begin{equation*}
Z\left(q^{2}\right)=1-\frac{g_{o}^{2}}{(4 \pi)^{2}}\left[\left(11-\frac{2}{3} N_{f}\right)\left(\frac{1}{\hat{\epsilon}}+\ln \left(-q^{2}\right)\right)-22+\frac{10}{9} N_{f}\right] \tag{5.9}
\end{equation*}
$$

where we have employed dimensional regularization with $D=4+2 \epsilon$ and $1 / \hat{\epsilon}=$ $1 / \epsilon+\gamma_{\mathrm{E}}-4 \pi . N_{f}$ is the number of light quark flavors. In analogy with QED, we define the effective quark-gluon running coupling constant to be

$$
\begin{equation*}
g_{2}^{2}\left(q^{2}\right) \equiv g_{o}^{2} Z\left(q^{2}\right) \tag{5.10}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\frac{1}{g_{2}^{2}\left(q^{2}\right)}=\frac{1}{g_{o}^{2}}+\frac{1}{(4 \pi)^{2}}\left[\left(11-\frac{2}{3} N_{f}\right)\left(\frac{1}{\hat{\epsilon}}+\ln \left(-q^{2}\right)\right)-22+\frac{10}{9} N_{f}\right] \tag{5.11}
\end{equation*}
$$

Upon solving this renormalization group equation we obtain the familiar expression

$$
\begin{equation*}
\alpha_{2}\left(q^{2}\right) \equiv \frac{g_{2}^{2}\left(q^{2}\right)}{4 \pi}=\frac{4 \pi}{\left(11-\frac{2}{3} N_{f}\right) \ln \left(-q^{2} / \Lambda_{2}^{2}\right)} \tag{5.12}
\end{equation*}
$$

The scale $\Lambda_{2}$ is formally an integration constant to be fixed by experimental measurement. We observe that the gauge-invariant gluon propagator effectively introduces a renormalization scheme with itself. To this order, the relationship between $\Lambda_{2}$ and the more conventional $\Lambda_{\overline{\mathrm{MS}}}$ can be obtained by noting that in the $\overline{\mathrm{MS}}$ scheme

$$
\begin{equation*}
\frac{1}{g_{\overline{\mathrm{MS}}}^{2}\left(\mu^{2}\right)}=\frac{1}{g_{o}^{2}}+\frac{1}{(4 \pi)^{2}}\left[\left(11-\frac{2}{3} N_{f}\right)\left(\frac{1}{\hat{\epsilon}}+\ln \left(\mu^{2}\right)\right)\right] . \tag{5.13}
\end{equation*}
$$

By comparing equations (5.11) and (5.13) at $-q^{2}=\Lambda_{2}^{2}$ and $\mu^{2}=\Lambda \frac{2}{\overline{M S}}$, and noting that the left-hand sides of both equations vanish, we obtain the relationship

$$
\begin{equation*}
\Lambda_{2}=\exp \left(\frac{99-5 N_{f}}{99-6 N_{f}}\right) \Lambda_{\overline{\mathrm{MS}}} \tag{5.14}
\end{equation*}
$$

For $N_{f}=4$ and $N_{f}=5$ we have respectively $\Lambda_{2}=2.867 \Lambda_{\overline{\mathrm{MS}}}$ and $\Lambda_{2}=2.923 \Lambda_{\overline{\mathrm{MS}}}$.

### 5.2. Three-Gluon Coupling

The effective coupling of the three-gluon vertex has been studied previously by a number of authors [45]. However, previous studies have been focused on the gauge-dependent three-gluon vertex. The presence of the gauge parameter impeded a reliable physical interpretation of the effective charge.

The gauge-invariant three-gluon vertex to one-loop order was first obtained by Cornwall and Papavassiliou [43]. The renormalized version of this vertex function is given below, where we have added the quark-loop contribution absent in Ref. [43].

$$
\begin{align*}
-g_{o} f^{a b c} \Gamma_{\lambda \mu \nu}(p, q, r)= & Z^{1 / 2}\left(p^{2}\right) Z^{1 / 2}\left(q^{2}\right) Z^{1 / 2}\left(r^{2}\right) f^{a b c}\{ \\
& -g_{o}\left[(p-q)_{\nu} g_{\lambda \mu}+(q-r)_{\lambda} g_{\mu \nu}+(r-p)_{\mu} g_{\nu \lambda}\right] \\
& -\frac{3}{2} i g_{o}^{3} \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k_{1}^{2} k_{2}^{2} k_{3}^{2}} \times \\
& {\left[\Gamma_{1 \lambda}^{\mathrm{F}} \Gamma_{2}^{\mathrm{F}} \Gamma_{3 \nu}^{\mathrm{F}}+2\left(k_{2}+k_{3}\right)_{\lambda}\left(k_{3}+k_{1}\right)_{\mu}\left(k_{1}+k_{2}\right)_{\nu}\right] } \\
& -12 i g_{o}^{3}\left(p_{\nu} g_{\lambda \mu}-p_{\mu} g_{\nu \lambda}\right) \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k^{2}(k+p)^{2}} \\
& -12 i g_{o}^{3}\left(q_{\lambda} g_{\mu \nu}-q_{\nu} g_{\lambda \mu}\right) \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k^{2}(k+q)^{2}} \\
& -12 i g_{o}^{3}\left(r_{\mu} g_{\nu \lambda}-r_{\lambda} g_{\mu \nu}\right) \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k^{2}(k+r)^{2}} \\
& \left.-\frac{N_{f}}{2} i g_{o}^{3} \int \frac{d^{4} k}{(2 \pi)^{4}} \frac{\operatorname{Tr}\left[k_{1} \gamma_{\nu} k_{2} \gamma_{\lambda} k_{3} \gamma_{\mu}\right]}{k_{1}^{2} k_{2}^{2} k_{3}^{2}}\right\}, \tag{5.15}
\end{align*}
$$

where $Z\left(p^{2}\right)$ is the gauge-invariant gluon wavefunction renormalization as given in

Eq. (5.9), and the Feynman parts [43] of the three-gluon vertex are given by

$$
\begin{align*}
& \Gamma_{1 \beta \lambda \gamma}^{\mathrm{F}}=2 p_{\gamma} g_{\lambda \beta}-2 p_{\beta} g_{\gamma \lambda}-\left(k_{2}+k_{3}\right)_{\lambda} g_{\beta \gamma} \\
& \Gamma_{2 \gamma \mu \alpha}^{\mathrm{F}}=2 q_{\alpha} g_{\mu \gamma}-2 q_{\gamma} g_{\alpha \mu}-\left(k_{3}+k_{1}\right)_{\mu} g_{\gamma \alpha}  \tag{5.16}\\
& \Gamma_{3 \alpha \nu \beta}^{\mathrm{F}}=2 r_{\beta} g_{\nu \alpha}-2 r_{\alpha} g_{\beta \nu}-\left(k_{1}+k_{2}\right)_{\nu} g_{\alpha \beta}
\end{align*}
$$

The definition of the various momenta and indices is given in Fig. 5.4.


Fig. 5.4 The definition of the various momenta, Lorentz indices and color indices involved in the one-loop three-gluon vertex calculation.

The gluon vertex has a complicated tensor structure. We can classify the various tensor components of this vertex into

$$
\begin{align*}
\Gamma_{\lambda \mu \nu}= & \Gamma^{1} g_{\lambda \mu}(p-q)_{\nu}+\Gamma^{2} g_{\mu \nu}(q-r)_{\lambda}+\Gamma^{3} g_{\nu \lambda}(r-p)_{\mu} \\
& +\Gamma^{4}(q-r)_{\lambda}(r-p)_{\mu}(p-q)_{\nu}+\Gamma_{\lambda \mu \nu}^{\mathrm{long}}(p, q, r) \tag{5.17}
\end{align*}
$$

where the longitudinal part $\Gamma_{\lambda \mu \nu}^{\text {long }}$ contains all the terms that vanish upon contracting with the projector operator

$$
\begin{equation*}
\Pi_{\lambda \mu \nu}^{\lambda^{\prime} \mu^{\prime} \nu^{\prime}}(p, q, r)=\left(g_{\lambda \lambda^{\prime}}-\frac{p_{\lambda} p_{\lambda^{\prime}}}{p^{2}}\right)\left(g_{\mu \mu^{\prime}}-\frac{q_{\mu} q_{\mu^{\prime}}}{q^{2}}\right)\left(g_{\nu \nu^{\prime}}-\frac{r_{\nu} r_{\nu^{\prime}}}{r^{2}}\right) \tag{5.18}
\end{equation*}
$$

That is

$$
\begin{equation*}
\Pi \cdot \Gamma^{\text {long }}=\Pi_{\lambda \mu \nu}^{\lambda^{\prime} \mu^{\prime} \nu^{\prime}}(p, q, r) \Gamma_{\lambda^{\prime} \mu^{\prime} \nu^{\prime}}^{\mathrm{long}}(p, q, r)=0 \tag{5.19}
\end{equation*}
$$

The Born component, i.e., the component proportional to the tree-level tensor, is given by

$$
\begin{equation*}
\Gamma^{0}=\frac{1}{3}\left(\Gamma^{1}+\Gamma^{2}+\Gamma^{3}\right) \tag{5.20}
\end{equation*}
$$

We can calculate this component by using the tensor method. Namely, we first obtain a set of linearly-independent equations by contracting the three-gluon vertex in Eq. (5.17) with a complete set of basis tensors, and then we solve for $\Gamma^{0}$ from this set of equations. Fortunately, the outcome of this lengthy analysis can be expressed in a rather compact form,

$$
\begin{align*}
\Gamma^{0} & =\frac{1}{48 \mathcal{R}} S \cdot \Pi \cdot \Gamma \\
& =\frac{1}{4 \delta \mathcal{R}} S_{\lambda^{\prime} \mu^{\prime} \nu^{\prime}} \Pi_{\lambda \mu \nu}^{\lambda^{\prime} \mu^{\prime} \nu^{\prime}} \Gamma^{\lambda \mu \nu} \tag{5.21}
\end{align*}
$$

with $S$ the projection tensor given by

$$
\begin{align*}
S_{\lambda \mu \nu}= & 2 p^{2}(q-r)_{\lambda} g_{\mu \nu}+2 q^{2}(r-p)_{\mu} g_{\nu \lambda}+2 r^{2}(p-q)_{\nu} g_{\lambda \mu}  \tag{5.22}\\
& +(q-r)_{\lambda}(r-p)_{\mu}(p-q)_{\nu}
\end{align*}
$$

and

$$
\begin{equation*}
\mathcal{R}=\frac{1}{4}\left(2 p^{2} q^{2}+2 q^{2} r^{2}+2 r^{2} p^{2}-p^{4}-q^{4}-r^{4}\right) \tag{5.23}
\end{equation*}
$$

The effective three-gluon coupling is defined in terms of the Born component by

$$
\begin{equation*}
g_{3}\left(p^{2}, q^{2}, r^{2}\right) \equiv g_{0} \Gamma^{0} \tag{5.24}
\end{equation*}
$$

This is a natural choice since in the perturbative regimen the Born structure dominates. All the non-Born components are formally higher-order in $g_{o}$ and hence
are subleading. Also notice that, to one-loop order, all the ultraviolet divergences are contained within the Born component; therefore, it is the only component responsible for the coupling constant renormalization.

Upon inverting and squaring the previous equation,

$$
\begin{align*}
& \frac{1}{g_{3}^{2}\left(p^{2}, q^{2}, r^{2}\right)}=\frac{1}{g_{o}^{2}}+\frac{1}{(4 \pi)^{2}}\left[\left(11-\frac{2}{3} N_{f}\right) \times\right. \\
&\left.\left(\frac{1}{\hat{\epsilon}}+\mathrm{L}\left(-p^{2},-q^{2},-r^{2}\right)-\frac{16}{3 \sqrt{3}} \mathrm{~L} \sin 2\left(\frac{\pi}{3}\right)\right)-22+\frac{2}{3} N_{f}\right] \tag{5.25}
\end{align*}
$$

where

$$
\begin{align*}
\mathrm{L}\left(-p^{2},-\right. & \left.q^{2},-r^{2}\right)=\frac{r \cdot p p \cdot q}{\mathcal{R}} \ln \left(-p^{2}\right)+\frac{p \cdot q q \cdot r}{\mathcal{R}} \ln \left(-q^{2}\right) \\
& +\frac{q \cdot r r \cdot p}{\mathcal{R}} \ln \left(-r^{2}\right)+\frac{p^{2} q^{2} r^{2}}{\mathcal{R}} F\left(p^{2}, q^{2}, r^{2}\right)+\frac{16}{3 \sqrt{3}} \operatorname{Lsin} 2\left(\frac{\pi}{3}\right) . \tag{5.26}
\end{align*}
$$

The various dot products are expressible in terms of the gluon virtualities, e.g., $p \cdot q=\left(r^{2}-p^{2}-q^{2}\right) / 2$. The functions $\mathrm{F}\left(p^{2}, q^{2}, r^{2}\right)$ and $\operatorname{Lsin}_{2}(x)$ are fully described in Appendix B; for completeness, we reproduce a summary here.

$$
\begin{align*}
\mathrm{F}\left(p^{2}, q^{2}, r^{2}\right) & =\frac{i}{\pi^{2}} \int d^{4} k \frac{1}{k_{1}^{2} k_{2}^{2} k_{3}^{2}} \\
& =\frac{1}{\rho}\left[\operatorname{Lsin}_{2}\left(2 \phi_{1}\right)+\operatorname{Lsin}_{2}\left(2 \phi_{2}\right)+\operatorname{Lsin}_{2}\left(2 \phi_{3}\right)\right] \\
\rho & =\sqrt{\mathcal{R}}, \\
\phi_{1} & =\arctan \left(\frac{\rho}{q \cdot r-i \epsilon}\right)=\frac{i}{2} \ln \left(\frac{q \cdot r-i \rho-i \epsilon}{q \cdot r+i \rho-i \epsilon}\right) \\
\phi_{2} & =\arctan \left(\frac{\rho}{r \cdot p-i \epsilon}\right)=\frac{i}{2} \ln \left(\frac{r \cdot p-i \rho-i \epsilon}{r \cdot p+i \rho-i \epsilon}\right)  \tag{5.27}\\
\phi_{3} & =\arctan \left(\frac{\rho}{p \cdot q-i \epsilon}\right)=\frac{i}{2} \ln \left(\frac{p \cdot q-i \rho-i \epsilon}{p \cdot q+i \rho-i \epsilon}\right) \\
\operatorname{Lsin}_{2}(z) & =\frac{1}{2 i}\left[\operatorname{Li}_{2}\left(e^{i z}\right)-\operatorname{Li}_{2}\left(e^{-i z}\right)\right]=\sum_{1} \frac{\sin n z}{n^{2}} \\
\operatorname{Lsin} 2\left(\frac{\pi}{3}\right) & =1.01494160 \ldots .
\end{align*}
$$

The function $\mathrm{L}(x, y, z)$ can be considered as a three-variable extension of the logarithmic function. In fact, on the symmetric axis $x=y=z$, the function $\mathrm{L}(x, y, z)$ reduces to

$$
\begin{equation*}
\mathrm{L}(x, x, x)=\ln (x) \tag{5.28}
\end{equation*}
$$

The function $\mathrm{L}(x, y, z)$ also satisfies the simple scaling property

$$
\begin{equation*}
\mathrm{L}(\lambda x, \lambda y, \lambda z)=\ln \lambda+\mathrm{L}(x, y, z), \text { for } \lambda>0 \tag{5.29}
\end{equation*}
$$

We can interpret Eq. (5.25) as a multi-momentum renormalization group equation (see Chapter 3). It solution is given by

$$
\begin{align*}
\alpha_{3}\left(p^{2}, q^{2}, r^{2}\right) & =\frac{g_{3}^{2}\left(p^{2}, q^{2}, r^{2}\right)}{4 \pi} \\
& =\frac{4 \pi}{\left(11-\frac{2}{3} N_{f}\right) \mathrm{L}\left(-p^{2} / \Lambda_{3}^{2},-q^{2} / \Lambda_{3}^{2},-r^{2} / \Lambda_{3}^{2}\right)} \tag{5.30}
\end{align*}
$$

where the function $\Lambda_{3}$ is a quantity to be fixed by experimental measurement. Notice the similarity between this formula and the familiar form of the strong coupling constant as given in (5.12). In both cases, the factor $11-\frac{2}{3} N_{f}$ multiplies a single function. The functional form of the fermion contribution thus is identical to the pure-gluon contribution. In the three-gluon vertex, this feature is a surprise given the complicated form of the integrals in Eq. (5.15). This strongly supports BLM's proposal [2] of using fermion loops as probes of QCD coupling scales, since the scale obtained via fermion-loop analysis is identical to the one obtained by a more complete analysis.

The scale $\Lambda_{3}$ can be expressed in terms of $\Lambda_{2}$ or $\Lambda_{\overline{\mathrm{MS}}}$ since the bare couplingconstant of QCD is unique. By comparing Eqs. (5.11), (5.13) and (5.25), we
obtain

$$
\begin{align*}
\Lambda_{3} & =\exp \left(\frac{33-N_{f}}{33-2 N_{f}}+\frac{8}{3 \sqrt{3}} \operatorname{Lsin}_{2}\left(\frac{\pi}{3}\right)\right) \Lambda_{\overline{\mathrm{MS}}}  \tag{5.31}\\
& =\exp \left(\frac{2 N_{f}}{99-6 N_{f}}+\frac{8}{3 \sqrt{3}} \operatorname{Lsin} 2\left(\frac{\pi}{3}\right)\right) \Lambda_{2}
\end{align*}
$$

For $N_{f}=4$ and $N_{f}=5$ we have respectively $\Lambda_{3}=15.22 \Lambda_{\overline{\mathrm{MS}}}=5.308 \Lambda_{2}$ and $\Lambda_{3}=16.12 \Lambda_{\overline{\mathrm{MS}}}=5.515 \Lambda_{2}$.

In what follows we will consider only the case where $p^{2}, q^{2}$ and $r^{2}$ are all spacelike [46]. In Fig. 5.5 we plot the equal-coupling surfaces of $\alpha_{3}\left(p^{2}, q^{2}, r^{2}\right)$ in this kinematic region.

In the limit when one of the momentum scales is much larger than the other two, we have

$$
\begin{align*}
\mathrm{L}\left(-p^{2} / \Lambda_{3}^{2},-q^{2} / \Lambda_{3}^{2},-r^{2} / \Lambda_{3}^{2}\right) & \rightarrow \ln \left(\frac{Q_{\min }^{2} Q_{\mathrm{med}}^{2}}{Q_{\max }^{2} \Lambda_{3}^{2}}\right)+\frac{16}{3 \sqrt{3}} \operatorname{Lsin} 2\left(\frac{\pi}{3}\right) \\
& =\ln \left(\frac{Q_{\min }^{2} Q_{\operatorname{med}}^{2}}{Q_{\max }^{2} \tilde{\Lambda}_{3}^{2}}\right) . \tag{5.32}
\end{align*}
$$

with $Q_{\min }^{2}, Q_{\text {med }}^{2}$ and $Q_{\max }^{2}$ respectively the smallest, the next-to-smallest and the largest scales among $-p^{2},-q^{2}$ and $-r^{2}$, and

$$
\begin{align*}
\tilde{\Lambda}_{3} & =\exp \left(-\frac{8}{3 \sqrt{3}} \operatorname{Lsin} 2\left(\frac{\pi}{3}\right)\right) \Lambda_{3} \\
& =\exp \left(\frac{33-N_{f}}{33-2 N_{f}}\right) \Lambda_{\overline{\mathrm{MS}}}  \tag{5.33}\\
& =\exp \left(\frac{2 N_{f}}{99-6 N_{f}}\right) \Lambda_{2}
\end{align*}
$$

For $N_{f}=4$ and $N_{f}=5$ we have respectively $\tilde{\Lambda}_{3}=3.190 \Lambda_{\overline{\mathrm{MS}}}=1.113 \Lambda_{2}$ and $\tilde{\Lambda}_{3}=3.378 \Lambda_{\overline{\mathrm{MS}}}=1.156 \Lambda_{2}$. From Eq. (5.32) we see that the effective coupling
scale of the three-gluon vertex is essentially given by

$$
\begin{equation*}
Q_{\mathrm{eff}}^{2} \sim \frac{Q_{\min }^{2} Q_{\operatorname{med}}^{2}}{Q_{\max }^{2}} \tag{5.34}
\end{equation*}
$$



Fig. 5.5 Equal-coupling surfaces for the effective three-gluon coupling constant in the completely spacelike region.

Next, we define the scale correction factor $K$ through the relation

$$
\begin{equation*}
\mathrm{L}\left(-p^{2} / \Lambda_{3}^{2},-q^{2} / \Lambda_{3}^{2},-r^{2} / \Lambda_{3}^{2}\right) \equiv \ln \left(K^{2} \frac{Q_{\min }^{2} Q_{\operatorname{med}}^{2}}{Q_{\max }^{2} \tilde{\Lambda}_{3}^{2}}\right) \tag{5.35}
\end{equation*}
$$

In Fig. 5.6 we plot $K(x, y)$ as function of the ratios $x=Q_{1}^{2} / Q_{\max }^{2}$ and $y=$ $Q_{2}^{2} / Q_{\max }^{2}$, where $Q_{\max }^{2}$ is the maximum scale among $-p^{2},-q^{2},-r^{2}$, and $Q_{1}^{2}$ and $Q_{2}^{2}$ are the two remaining scales. From the figure we see that the actual coupling scale is in general within a factor $0.2096 \sim 1 / 5$ of the simple expression given in Eq. (5.34).


Fig. 5.6 Scale-correction factor function, as defined in text. Note that this function takes values between $K_{\text {min }}=0.2096$ and $K_{\text {max }}=1$.

Note that formula (5.34) indicates that the coupling scale in general will be small when there is one scale disproportionately larger than the other two scales. Consider for instance the jet-production process indicated in Fig. 5.7. Formula (5.34) implies that, for fixed gluon-jet invariant masses $M_{1}^{2}$ and $M_{2}^{2}$, the three-gluon vertex becomes non-perturbative at high values of $Q^{2}$. That is, the three-gluon vertex is perturbative only if the invariant-masses all of the gluon jets are allowed to increase simultaneously with $Q^{2}$.

We know that in the conventional $\beta$-function analysis in QCD the coupling constant becomes large at low energies. This signals two things: 1) The renormalization group equation breaks down at low energies. That is, many higher order terms in the $\beta$-function need to be taken into account. 2) If the perturbative series
for a physical quantity contains a large coupling constant, higher order corrections become non-negligible and the tree-level result alone will not correctly reproduce the physics.


Fig. 5.7 A four-jet process involving a three-gluon vertex. The three-gluon coupling is expected to be large at large values of $Q^{2}$ and fixed values of invariant masses $M_{1}^{2}$ and $M_{2}^{2}$.

Exactly the same situation happens with the multi-momentum renormalization group analysis of the three-gluon coupling when the momentum configuration is asymmetric. 1) For highly asymmetric momentum configuration, the multimomentum renormalization group equation becomes inapplicable. This is reflected by the fact that the effective coupling becomes large. 2) The Born graphs containing highly asymmetric three-gluon vertices will not be enough to represent the physics. Higher-order diagrams should be considered. In particular, multi-particle emission effects become crucial (See Ref. [47]).

To conclude, we make the following observations.

1. The large values of $\Lambda_{3}$ and $\tilde{\Lambda}_{3}$ with respect to $\Lambda_{\overline{M S}}$ (see Eqs. (5.31) and (5.33)) indicate that in general one should choose a smaller-than-expected $\dot{\text { scale for }}$ the coupling constant $\alpha_{\overline{\mathrm{MS}}}\left(\mu^{2}\right)$ in four-jet physics, where the three-
gluon vertex plays an essential role. This, together with the fact that the effective scale for the three-gluon vertex as given by (5.34) is always smaller than the smallest scale, might help to explain the surprising smallness of the effective-coupling scale observed in four-jet cross-sections [48]

$$
\begin{equation*}
\mu_{\exp }^{2}=0.001 \sim 0.002 s, \tag{5.36}
\end{equation*}
$$

with $s$ the squared total center-of-mass energy.
2. In principle, the running of the three-gluon coupling could be studied by detailed measurements of four-jet events in $e^{+} e^{-}$annihilation [49]. In particular, these measurements would allow us to test the validity of the functional dependence of effective-coupling scale as given in Eqs. (5.34) and (5.35). Unfortunately, this is not possible in the presently available energy region, since the three-gluon vertex in this region is highly non-perturbative. Consider some typical invariant-mass values in $Z^{o}$ physics, say (see Fig. 5.7)

$$
\begin{equation*}
Q=20 \mathrm{GcV}, \quad M_{1}=M_{2}=7 \mathrm{GeV} . \tag{5.37}
\end{equation*}
$$

According to formula (5.35), this would give an effective coupling scale

$$
\begin{equation*}
\mu=\frac{K M_{1} M_{2} \Lambda_{\overline{\mathrm{MS}}}}{Q \tilde{\Lambda_{3}}} \sim 0.4 \mathrm{GeV} \tag{5.38}
\end{equation*}
$$

for the three-gluon vertex. Given a value of $\Lambda_{\overline{\mathrm{MS}}} \sim 175 \mathrm{MeV}$ for $N_{f}=5$ (See Review of Particle Properties, Ref. [12]), this would mean

$$
\begin{equation*}
\alpha_{3}=\frac{4 \pi}{\left(11-\frac{2}{3} N_{f}\right) \ln \left(\mu^{2} / \Lambda_{\overline{\mathrm{MS}}}^{2}\right)} \sim 1.0 . \tag{5.39}
\end{equation*}
$$

Therefore higher-order effects must be taken into account. In other words, the tree-level picture is not valid.
3. Observe that the Born-level description should remain valid for $q \bar{q} q \bar{q}$ events (see Fig. 5.8) since the scale of the strong coupling is given by the momentum squared of the exchanged gluon.


Fig. 5.8 A four-jet process with four final quarks. The effective coupling scale is expected to be given by the squared momentum transfer $Q^{2}$ of the exchanged gluon.
(Experimentally, heavy quarks can be tagged by their semileptonic decay products [50] or by using a vertex detector [51].)

If the skeleton picture is correct, the effective coupling $\mu^{2}$ for $q \bar{q} q \bar{q}$ processes should be larger than the corresponding scale observed for all four-jet events (q$q g g+$ $q \bar{q} q \bar{q})$. For a value of $y_{c u t}=0.05$, we would expect

$$
\begin{equation*}
\mu^{2}>\frac{y_{c u t} s \Lambda_{2}^{2}}{\Lambda_{\overline{M S}}^{2}}=0.006 s \tag{5.40}
\end{equation*}
$$

(The expected value of $\mu^{2}$ should be larger than the value shown here since the squared momentum transfer of the gluon usually is larger than $y_{c u t} s$.)

## CHAPTER 6:

## DISCUSSION AND CONCLUSION

This thesis represents an effort towards the clarification and generalization of automatic scale-setting methods in perturbative quantum field theory, first proposed by Brodsky, Lepage and Mackenzie [2].

We have pointed out that automatic scale-setting can be achieved in a number of field theoretical models by employing the dressed skeleton method discussed in the thesis. This method is based on the following two procedures:

1. Expand scattering amplitudes in skeleton graphs instead of conventional Feynman diagrams;
2. Obtain the coupling vertex function by the multi-momentum renormalization group equation.

The absence of the scale ambiguity is obtained naturally in this approach. In the dressed skeleton expansion, we do not rely on a coupling constant as the expansion parameter; hence, we do not encounter undetermined coupling scales. Instead, a functional expansion in the renormalized vertex function is employed. In contrast to the case of the conventional coupling constant where the coupling scale is an arbitrary parameter, in the case of the renormalized vertex function, the various scales are given by the physical momentum flows in the skeleton graphs. That is, the effective coupling scale in the dressed skeleton approach is automatically dictated by the kinematics of each problem.

The absence of scale ambiguity eliminates a typical nuisance of conventional methods: the fact that one can arbitrarily shift the weight of lower-order and higher-order contributions by simply manipulating the coupling scale of each problem. In dressed skeleton calculation, each term in the expansion has a well-defined magnitude. This allows us to judge objectively the validity of a particular perturbative expansion by analyzing its numerical convergence.

A fair question is whether the classification of Feynman diagrams into dressed skeleton graphs has some physical justification. As we showed in Chapter 4, among the various scale setting methods, the dressed skeleton expansion is the only method that provides the exact answer in the leading $1 / N$ Gross-Neveu model. This hints that separate skeletons renormalize independently; hence, they possess different effective coupling scales. Forcing different skeletons to share the same coupling scale in general will lead to inaccurate answers.

One drawback of the dressed skeleton method occurs in loop-skeleton calculations. Since the point-vertices of Feynman diagrams are now replaced by complicated vertex functions, the new loop integrals are much more difficult. However, we have seen in an explicit example that loop skeleton graphs in general absorb a large number of renormalon poles. This leads to an optimistic expectation that skeleton expansion might provide a better large-order convergence than conventional coupling constant expansion.

We have also discussed the generalizations of the dressed skeleton method to other field theories like the Yukawa model, QED and QCD. When a field theory contains more complicated vertices, generally we have to take additional care to make the method useful. This is especially true in the case of gauge field theories like QED or QCD, since their naïve skeleton graphs are not gauge-invariant.

In the case of QED this problem can be circumvented thanks to the Ward identity $Z_{1}=Z_{2}$. We only need the photon vacuum polarization to renormalize the bare coupling constant. The dressed-photon expansion thus provides a scale-ambiguity-free calculation method that respects gauge invariance. This is exactly the basis for the BLM automatic scale setting method in QED.

In QCD the situation is not as simple. Although the one-loop order gauge-
invariant gluon propagator and three-gluon vertex have been obtained [43], no general method of obtaining gauge-invariant QCD skeletons seems to be available. Despite this shortcoming, useful results have been obtained in the study of the lower-order vertices. In Chapter 5 we applied the multi-momentum renormalization group equation to the gauge-invariant three-gluon vertex and obtained an effective three-gluon coupling

$$
\begin{equation*}
\alpha_{s}\left(p^{2}, q^{2}, r^{2}\right)=\frac{4 \pi}{\left(11-\frac{2}{3} N_{f}\right) \mathrm{L}\left(-p^{2} / \Lambda_{3}^{2},-q^{2} / \Lambda_{3}^{2},-r^{2} / \Lambda_{3}^{2}\right)} . \tag{6.1}
\end{equation*}
$$

which bears a remarkable resemblance to the conventional single-scale coupling constant

$$
\begin{equation*}
\alpha_{s}\left(\mu^{2}\right)=\frac{4 \pi}{\left(11-\frac{2}{3} N_{f}\right) \ln \left(\mu^{2} / \Lambda_{\mathrm{QCD}}^{2}\right)} \tag{6.2}
\end{equation*}
$$

The function $\mathrm{L}(x, y, z)$ is given in Eq. (5.26) and it can be interpreted as the three-dimensional extension of the logarithmic function. As explained in Chapter 5 , the factorization of the coefficient $11-\frac{2}{3} N_{f}$ in the three-gluon coupling constant is non-trivial. This adds support to BLM's proposal of employing fermion loops to probe the coupling scale of physical processes.

For highly asymmetric momentum configurations, the effective coupling scale of the three-gluon vertex is shown to be given by

$$
\begin{equation*}
\mu^{2} \sim \frac{Q_{\min }^{2} Q_{\operatorname{med}}^{2}}{Q_{\max }^{2}} \tag{6.3}
\end{equation*}
$$

being $Q_{\text {min }}^{2}, Q_{\text {med }}^{2}$ and $Q_{\max }^{2}$ respectively the smallest, the next-to-smallest and the largest gluon virtuality of the three-gluon vertex. This functional form of the three-gluon coupling gives an effective coupling scale that is always smaller than
the smallest gluon virtuality. This feature might help explain the smallness of the effective coupling scale observed in four-jet physics in $e^{+} e^{-}$annihilation.

To conclude, we point out here two main limitations of the dressed skeleton method:

1. In QCD we lack a systematic method of constructing gauge-invariant skeleton graphs,
2. Higher-order skeleton calculations present great difficulties.

Despite these limitations, the study of lower-order skeletons allows us to gain insight to the scale setting mechanism of various physical processes, as exemplified in the case of the three-gluon coupling. Lower-order skeleton graphs also provide us an objective scale setting ansatz, in contrast to the many ad hoc scale setting solutions, which often involve guessing processes. However, the remaining unsettled issues make this field worth exploring.

## APPENDIX A:

BOX SKELETON CALCULATION

The skeleton box diagrams indicated in Fig. 4.16 gives the following Feynman integral:
$i \mathcal{M}_{b o x}=-4 p^{2} \int \frac{d^{2} k}{(2 \pi)^{2}} \frac{k^{2}}{\left(k-p_{1}\right)^{2}}\left\{\frac{1}{\left(k+p_{2}\right)^{2}}-\frac{1}{\left(k-p_{2}\right)^{2}}\right\} \frac{4 \pi^{2}}{\log ^{2}\left(-k^{2} / \Lambda_{\mathrm{DS}}^{2}-i \varepsilon\right)}$.

To perform this integral, let us first expand the inverse square of the logarithm into power series in $\log \left(p^{2} / \Lambda_{\mathrm{DS}}^{2}\right)$. Define

$$
\begin{equation*}
x=\log \left(p^{2} / \Lambda_{\mathrm{DS}}^{2}\right), \quad \hat{k}=k / p, \tag{A.2}
\end{equation*}
$$

we have

$$
\begin{align*}
\log ^{-2}\left(-\frac{k^{2}}{\Lambda_{\mathrm{DS}}^{2}}-i \varepsilon\right) & =\left[\log \left(\frac{p^{2}}{\Lambda_{\mathrm{DS}}^{2}}\right)+\log \left(-\frac{k^{2}}{p^{2}}-i \varepsilon\right)\right]^{-2} \\
& =\left[x+\log \left(-\hat{k}^{2}-i \varepsilon\right)\right]^{-2}  \tag{A.3}\\
& =\frac{1}{x^{2}} \sum_{n=0}^{\infty}\binom{-2}{n} \frac{\log ^{n}\left(-\hat{k}^{2}-i \varepsilon\right)}{x^{n}}
\end{align*}
$$

where

$$
\begin{equation*}
\binom{-2}{n}=\frac{(-2)(-3) \ldots(-1-n)}{1 \cdot 2 \cdot \ldots \cdot n}=(-1)^{n}(n+1) \tag{A.4}
\end{equation*}
$$

This expansion effectively corresponds to the expansion of the box skeleton diagrams into a power series in the coupling constant at scale $p$.

By applying the identity

$$
\begin{equation*}
\log ^{n}\left(-\hat{k}^{2}-i \varepsilon\right)=\left(\frac{\partial}{\partial \alpha}\right)_{\alpha=0}^{n}\left(-\hat{k}^{2}-i \varepsilon\right)^{\alpha} \tag{A.5}
\end{equation*}
$$

the Feynman integrals can be done exactly; the result is:

$$
\begin{align*}
i \mathcal{M}_{b o x} & =-\frac{4 \pi^{2} p^{2}}{x^{2}} \sum_{n=0}^{\infty}(-1)^{n} \frac{n+1}{x^{n}}\left(\frac{\partial f}{\partial \alpha}\right)_{\alpha=0}^{n}  \tag{A.6}\\
& =-\frac{4 \pi^{2} p^{2}}{x^{2}}\left\{1-2!f_{1} x^{-1}+\ldots+(-1)^{n}(n+1)!f_{n} x^{-n}+\ldots\right\}
\end{align*}
$$

where

$$
\begin{equation*}
f(\alpha)=(-4 i)^{\alpha} \sec \left(\frac{\pi \alpha}{2}\right)=f_{0}+f_{1} \alpha+f_{2} \alpha^{2}+\ldots \tag{A.7}
\end{equation*}
$$

We give here the numerical values of the first few coefficients:

$$
\begin{align*}
& f_{0}=1 \\
& f_{1}=1.38629-i 1.5708 \\
& f_{2}=0.960906-i 2.17759 \\
& f_{3}=0.444033-i 2.80132  \tag{A.8}\\
& f_{4}=0.15389-i 2.48848 \\
& f_{5}=0.0426674-i 2.75823 \\
& f_{6}=0.00985826-i 2.40832
\end{align*}
$$

The expansion (A.6) exhibits an $n$ ! divergence behavior (because $f_{i}$ is roughly constant for large value of $i$ ), typical of an asymptotic series that needs Borel resummation $[52,53]$ in order to yield a finite result [54]. Fortunately this series can be Borel resummed exactly, and the result obtained by a straightforward application of the Borel resummation formulas is given by:

$$
\begin{align*}
i \mathcal{M}_{b o x} & =-2 p^{2}\left[\Psi^{\prime}\left(\frac{1}{2}-\frac{i}{\pi} \log \left(2 p / \Lambda_{\mathrm{DS}}\right)\right)-\Psi^{\prime}\left(-\frac{i}{\pi} \log \left(2 p / \Lambda_{\mathrm{DS}}\right)\right)\right]  \tag{A.9}\\
& =-8 p^{2} \beta^{\prime}\left(-\frac{2 i}{\pi} \log \left(2 p / \Lambda_{\mathrm{DS}}\right)\right)
\end{align*}
$$

where $\Psi^{\prime}$ is the trigamma function [40] defined by

$$
\begin{align*}
\Psi^{\prime}(z) & =\frac{d \Psi}{d z}=\frac{d^{2}}{d z^{2}} \log \Gamma(z)  \tag{A.10}\\
& \sim \frac{1}{z}+\frac{1}{2 z^{2}}+\frac{1}{6 z^{3}}-\frac{1}{30 z^{5}}+\frac{1}{42 z^{7}}-\frac{1}{30 z^{9}}+\ldots
\end{align*}
$$

and $\beta^{\prime}(z)$ is the derivative of the $\beta(z)$ function [40]

$$
\begin{equation*}
\beta(z)=\frac{1}{2}\left[\Psi\left(\frac{z+1}{2}\right)-\Psi\left(\frac{z}{2}\right)\right]=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{z+n} \tag{A.11}
\end{equation*}
$$

It is interesting to observe that the Borel transform of the series (A.6):

$$
\begin{equation*}
G(y) \sim \delta(y)+\frac{d^{2}}{d y^{2}}(y f(-y)) \tag{A.12}
\end{equation*}
$$

possesses an infinite number of poles on the real axis (see Fig. A.1). These poles exhibit the typical features of renormalon singularities [52,55]. We notice that these poles lie exactly on the real axis, i.e., they do not have infinitesimal imaginary part. Thus, when performing the Borel integral, those poles on the positive real axis should be interpreted in the principal value sense. We note that the resulting integral under this prescription is finite, despite the presence of the infinite number of poles.

Notice that if the original integral in Eq. (A.1) were performed numerically, we would never have to worry about renormalons. In a sense, the renormalons of this example are effectively "eaten" by skeletons.


Fig. A. 1 Location of the singularities of the Borel transform of the box amplitude in the complex-y plane. There is a delta function at the origin and an infinite number of poles located at odd integer numbers, which correspond to renormalon singularities.

## APPENDIX B:

MASSLESS ONE-LOOP SCALAR THREE-POINT INTEGRAL AND ASSOCIATED CLAUSEN, GLAISHER AND L-FUNCTIONS

The one-loop three-point integral has been obtained by other authors previously [56]. The result is usually expressed in terms of dilogarithms, also known as Spence functions. However, the obtained formula lacks explicit symmetry under the permutation of the three external momenta, and conceals the structure of the real part of the integral.

Here, we obtain the massless one-loop three-point integral in terms of associated Clausen functions. Our expression manifests the symmetry under the permutation of the three external momenta and provides a transparent real part. (The real part of the integral is actually given by the imaginary part of the function $\mathrm{F}\left(p_{1}, p_{2}, p_{3}\right)$ defined below.) Since one-loop Feynman integrals are in increasing demand, and also since the various associated functions introduced here are not as well-documented as the polylogarithmic functions [57,58], we have decided to collect our results here to facilitate future reference.

We have employed only standard integration techniques in obtaining our formula; therefore, we shall present the result without derivation [59]. The massless one-loop three-point integral in question is (see also Fig. B.1):

$$
\begin{equation*}
\int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k_{1}^{2} k_{2}^{2} k_{3}^{2}} \equiv-\frac{i}{(4 \pi)^{2}} \mathrm{~F}\left(p_{1}, p_{2}, p_{3}\right) \tag{B.1}
\end{equation*}
$$

where $p_{1}, p_{2}, p_{3}$ are the external momenta of the three-point function. It is convenient to introduce the following variables:

$$
\begin{align*}
\delta_{i} & =p_{i-1} \cdot p_{i+1}=\left(p_{i}^{2}-p_{i-1}^{2}-p_{i+1}^{2}\right) / 2 \\
\mathcal{R} & =\delta_{1} \delta_{2}+\delta_{2} \delta_{3}+\delta_{3} \delta_{1}=\left(2 p_{1}^{2} p_{2}^{2}+2 p_{2}^{2} p_{3}^{2}+2 p_{3}^{2} p_{1}^{2}-p_{1}^{4}-p_{2}^{4}-p_{3}^{4}\right) / 4 \\
\rho & =\sqrt{|\mathcal{R}|} \tag{B.2}
\end{align*}
$$

The subindices are understood to be modulo-3. That is, $p_{4} \equiv p_{1}$ and $p_{0} \equiv p_{3}$.


Fig. B. 1 One-loop Feynman diagram associated to the massless threepoint function.
.- The exact form of the function $\mathrm{F}\left(p_{1}, p_{2}, p_{3}\right)$ depends on the kinematic region of the three external momenta. In general, we can classify a kinematic region as trigonometric or hyperbolic, according to the signature of the variable $\mathcal{R}$.


$$
\rho=2 \times \text { Area of Triangle }
$$

Fig. B. 2 Geometrical interpretation of the angles $\phi_{1}, \phi_{2}, \phi_{3}$ and the variable $\rho$ in the completely spacelike region.

1) Trigonometric case ( $\mathcal{R}>0)$

$$
\begin{align*}
\mathrm{F}\left(p_{1}, p_{2}, p_{3}\right) & =\frac{1}{\rho}\left[\mathrm{Cl}_{2}\left(2 \phi_{1}\right)+\mathrm{Cl}_{2}\left(2 \phi_{2}\right)+\mathrm{Cl}_{2}\left(2 \phi_{3}\right)\right] \\
\phi_{i} & =\arctan \left(\frac{\rho}{\delta_{i}}\right) \tag{B.3}
\end{align*}
$$

where $\mathrm{Cl}_{2}(x)$ is the Clausen function, which will be described later. The trigonometric case can happen only in the completely spacelike $\left(p_{1}^{2}, p_{2}^{2}, p_{3}^{2}<\right.$ 0 ) and the completely timelike ( $p_{1}^{2}, p_{2}^{2}, p_{3}^{2}>0$ ) regions. Geometrically, in the completely spacelike region the angles $\phi_{1}, \phi_{2}$ and $\phi_{3}$ correspond to the three internal angles of a triangle with sides $\sqrt{-p_{1}^{2}}, \sqrt{-p_{2}^{2}}$ and $\sqrt{-p_{3}^{2}}$ (see Fig. B.2), and $\rho$ is twice the area of the triangle. Thus, in the completely spacelike region we have

$$
\begin{equation*}
\phi_{1}+\phi_{2}+\phi_{3}=\pi . \tag{B.4}
\end{equation*}
$$

In the completely timelike region we have the same identity with the opposite sign:

$$
\begin{equation*}
\phi_{1}+\phi_{2}+\phi_{3}=-\pi . \tag{B.5}
\end{equation*}
$$

Note that $\mathrm{F}\left(p_{1}, p_{2}, p_{3}\right)$ contains no imaginary part in the trigonometric case, as one would expect in the completely spacelike and timelike regions.
2) Hyperbolic case ( $\mathcal{R}<0)$

$$
\begin{align*}
\mathrm{F}\left(p_{1}, p_{2}, p_{3}\right)=\frac{1}{\rho} & {\left[\widetilde{\mathrm{Clh}}_{2}\left(2 \phi_{1}\right)+\widetilde{\mathrm{Clh}}_{2}\left(2 \phi_{2}\right)+{\widetilde{\mathrm{Clh}_{2}}}_{2}\left(2 \phi_{3}\right)\right.}  \tag{B.6}\\
& \left.+i \pi \phi_{1} \theta\left(p_{1}^{2}\right)+i \pi \phi_{2} \theta\left(p_{2}^{2}\right)+i \pi \phi_{3} \theta\left(p_{3}^{2}\right)\right],
\end{align*}
$$

where $\theta(x)$ is the step function

$$
\begin{gather*}
\theta(x)=\left\{\begin{array}{ll}
1, & \text { if } x \geq 0 \\
0, & \text { if } x<0
\end{array},\right.  \tag{B.7}\\
\phi_{i}=\frac{1}{2} \ln \left|\frac{\delta_{i}+\rho}{\delta_{i}-\rho}\right|=\left\{\begin{array}{ll}
\operatorname{arctanh}\left(\rho / \delta_{i}\right), & \text { if } p_{i-1}^{2} p_{i+1}^{2}>0 \\
\operatorname{arctanh}\left(\delta_{i} / \rho\right), & \text { if } p_{i-1}^{2} p_{i+1}^{2}<0
\end{array},\right. \tag{B.8}
\end{gather*}
$$

and
where $\mathrm{Clh}_{2}(x)$ is the hyperbolic Clausen function and $\phi \mathrm{lh}_{2}(x)$ is the alternating hyperbolic Clausen function. The definitions and properties of these functions are discussed later. The hyperbolic case can happen in kinematic regions with any signature ( $p_{1}^{2}, p_{2}^{2}, p_{3}^{2} \gtrless 0$ ). For the hyperbolic case we have the following identity

$$
\begin{equation*}
\phi_{1}+\phi_{2}+\phi_{3}=0 . \tag{B.10}
\end{equation*}
$$

Thus, despite its appearance, Eq. (B.6) contains no imaginary part in the completely timelike region.

In summary, in the definite-signature regions (completely spacelike or timelike regions), we encounter both the trigonometric case and the hyperbolic case, whereas in the mixed-signature regions (some of the external momenta are spacelike and some are timelike), we can have only the hyperbolic case. The numerical evaluation of the various associated Clausen functions can be performed with the help of the series expansions given below. We have checked our result numerically against direct Feynman parameter integrals in all kinematic regions.

Next, we give the definition and the main properties of the associated Clausen functions [60].

1) (Trigonometric) Clausen function.

- definition

$$
\begin{equation*}
\mathrm{Cl}_{2}(x) \equiv-\int_{0}^{x} \ln |2 \sin (x / 2)| d x=\sum_{1} \frac{\sin n x}{n^{2}} \tag{B.11}
\end{equation*}
$$

- periodicity

$$
\begin{equation*}
\mathrm{Cl}_{2}(x+2 n \pi)=\mathrm{Cl}_{2}(x), \quad n=0, \pm 1, \pm 2, \ldots \tag{B.12}
\end{equation*}
$$

- parity

$$
\begin{equation*}
\mathrm{Cl}_{2}(-x)=-\mathrm{Cl}_{2}(x) \tag{B.13}
\end{equation*}
$$

zeros

$$
\begin{equation*}
x=n \pi, \quad n=0, \pm 1, \pm 2, \ldots \tag{B.14}
\end{equation*}
$$

- maxima

$$
\begin{align*}
x_{\max } & =\frac{\pi}{3}+2 n \pi, \quad n=0, \pm 1, \pm 2, \ldots  \tag{B.15}\\
\mathrm{Cl}_{2}\left(x_{\max }\right) & =1.01494160 \ldots
\end{align*}
$$

- minima

$$
\begin{align*}
x_{\min } & =-\frac{\pi}{3}+2 n \pi, \quad n=0, \pm 1, \pm 2, \ldots  \tag{B.16}\\
\mathrm{Cl}_{2}\left(x_{\min }\right) & =-1.01494160 \ldots
\end{align*}
$$

- duplication formula

$$
\begin{equation*}
\mathrm{Cl}_{2}(2 x)=2 \mathrm{Cl}_{2}(x)-2 \mathrm{Cl}_{2}(\pi-x) \tag{B.17}
\end{equation*}
$$

- special values

$$
\begin{align*}
& \mathrm{Cl}_{2}\left(\frac{\pi}{2}\right)=\frac{1}{1^{2}}-\frac{1}{3^{2}}+\frac{1}{5^{2}}-\ldots=G=0.91596559 \ldots \\
& \mathrm{Cl}_{2}\left(\frac{\pi}{3}\right)=\frac{3}{2} \mathrm{Cl}_{2}\left(\frac{2 \pi}{3}\right)=1.01494160 \ldots \tag{B.18}
\end{align*}
$$

where $G$ is Catalan's constant.

- expansion around $x=0$

$$
\begin{align*}
\mathrm{Cl}_{2}(x)= & -x \ln |x|+x+\sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_{2 k}}{2 k(2 k+1)!} x^{2 k+1} \\
= & -x \ln |x|+x+\frac{x^{3}}{72}+\frac{x^{5}}{14400}+\frac{x^{7}}{1270080}  \tag{B.19}\\
& +\frac{x^{9}}{87091200}+\frac{x^{11}}{5269017600}+\ldots
\end{align*}
$$

where $B_{n}$ are Bernoulli numbers [61],

- expansion around $x=\pi$; define $\bar{x}=x-\pi$

$$
\left.\begin{array}{rl}
\mathrm{Cl}_{2}(x)= & -(\ln 2) \bar{x}+\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\left(2^{2 k}-1\right) B_{2 k}}{2 k(2 k+1)!} x^{2 k+1} \\
= & -(\ln 2) \bar{x}
\end{array}\right)+\frac{\bar{x}^{3}}{24}+\frac{\bar{x}^{5}}{960}+\frac{\bar{x}^{7}}{20160}+\frac{\bar{x}^{9}}{5806080}+\frac{31 \bar{x}^{11}}{159667200}
$$

2) (Trigonometric) Alternating Clausen function. Although this function is not used in the scalar three-point integral, we have included it here for completeness.

- definition

$$
\begin{equation*}
\not \subset l_{2}(x) \equiv-\int_{0}^{x} \ln |2 \cos (x / 2)| d x=\sum_{1} \frac{(-1)^{n} \sin n x}{n^{2}} \tag{B.21}
\end{equation*}
$$

- relation to Clausen function.

$$
\begin{equation*}
\mathscr{C l} 1_{2}(x)=\mathrm{Cl}_{2}(x+\pi) \tag{B.22}
\end{equation*}
$$

Since $\mathscr{C l} l_{2}(x)$ is simply the half-period translation of $\mathrm{Cl}_{2}(x)$, all the properties of $\not \ell_{2}(x)$ can be easily obtained from those of $\mathrm{Cl}_{2}(x)$; therefore we will not give them separately here.
3) Hyperbolic Clausen function.

- definition

$$
\begin{equation*}
\mathrm{Clh}_{2}(x) \equiv-\int_{0}^{x} \ln |2 \sinh (x / 2)| d x=\sum_{1} \frac{\sinh n x}{n^{2}} \tag{B.23}
\end{equation*}
$$

The series should be considered formal, since it is not convergent for real values of $x$.

- parity

$$
\begin{equation*}
\mathrm{Clh}_{2}(-x)=-\mathrm{Clh}_{2}(x) \tag{B.24}
\end{equation*}
$$

- zeros

$$
\begin{equation*}
x=0, \pm 2.49879679 \ldots \tag{B.25}
\end{equation*}
$$

- maximum and minimum

$$
\begin{align*}
x_{\max } & =-x_{\min }=2 \ln (1 / 2+\sqrt{5} / 2)=0.96242365 \ldots  \tag{B.26}\\
\mathrm{Cl}_{2}\left(x_{\max }\right) & =-\mathrm{Cl}_{2}\left(x_{\min }\right)=0.98695978 \ldots
\end{align*}
$$

- expansion around $x=0$

$$
\begin{align*}
\mathrm{Clh}_{2}(x)= & -x \ln |x|+x-\sum_{k=1}^{\infty} \frac{B_{2 k}}{2 k(2 k+1)!} x^{2 k+1} \\
= & -x \ln |x|+x-\frac{x^{3}}{72}+\frac{x^{5}}{14400}-\frac{x^{7}}{1270080}  \tag{B.27}\\
& +\frac{x^{9}}{87091200}-\frac{x^{11}}{5269017600}+\ldots
\end{align*}
$$

- large- $x$ expansion. For $x>0$

$$
\begin{equation*}
\operatorname{Clh}_{2}(x)=-\frac{x^{2}}{4}+\pi^{2} / 6-\sum_{1} \frac{e^{-n x}}{n^{2}} \tag{B.28}
\end{equation*}
$$

4) Alternating Hyperbolic Clausen function.

- definition

$$
\begin{equation*}
\not \subset \operatorname{lh}_{2}(x) \equiv-\int_{0}^{x} \ln |2 \cosh (x / 2)| d x=\sum_{1} \frac{(-1)^{n} \sinh n x}{n^{2}} \tag{B.29}
\end{equation*}
$$

The series should be considered formal, since it is not convergent for real values of $x$.

- parity

$$
\begin{equation*}
\phi \operatorname{lh}_{2}(-x)=-\phi \operatorname{lh}_{2}(x) \tag{B.30}
\end{equation*}
$$

- zero

$$
\begin{equation*}
x=0 \tag{B.31}
\end{equation*}
$$

- expansion around $x=0$

$$
\begin{align*}
\mathscr{C l h _ { 2 }}(x)= & -(\ln 2) x-\sum_{k=1}^{\infty} \frac{\left(2^{2 k}-1\right) B_{2 k}}{2 k(2 k+1)!} x^{2 k+1} \\
= & -(\ln 2) x-\frac{x^{3}}{24}+\frac{x^{5}}{960}-\frac{x^{7}}{20160}+\frac{x^{9}}{5806080}-\frac{31 x^{11}}{159667200} \\
& +\frac{691 x^{13}}{49816166400}-\frac{5461 x^{15}}{5230697472000}+\ldots \tag{B.32}
\end{align*}
$$

- large- $x$ expansion. For $x>0$

$$
\begin{equation*}
\mathscr{C l h} 2(x)=-\frac{x^{2}}{4}-\pi^{2} / 12-\sum_{1} \frac{(-1)^{n} e^{-n x}}{n^{2}} \tag{B.33}
\end{equation*}
$$

In Fig. B. 3 we plot the functions $\mathrm{Cl}_{2}(x), \mathrm{Clh}_{2}(x)$ and $\mathscr{C l h} \mathrm{h}_{2}(x)$ in the interval $-6 \leq x \leq 6$. Notice the approximately sinusoidal nature of $\mathrm{Cl}_{2}(x)$. The derivative of $\mathrm{Cl}_{2}(x)$ at zero is infinite.


Fig. B. 3 Plot of the Clausen function, the hyperbolic Clausen function and the alternating hyperbolic Clausen function.

Another set of functions closely related to the associated Clausen functions are the associated Glaisher functions [57]. We include their basic features here for completeness. All these functions have even parity, and their defining series are given by

$$
\begin{align*}
\mathrm{Gl}_{2}(x) & =\sum_{1} \frac{\cos n x}{n^{2}}, \quad G \mathrm{l}_{2}(x)=\sum_{1} \frac{(-1)^{n} \cos n x}{n^{2}}, \\
\mathrm{GlH}_{2}(x) & =\sum_{1} \frac{\cosh n x}{n^{2}}, \quad G 1 \mathrm{~h}_{2}(x)=\sum_{1} \frac{(-1)^{n} \cosh n x}{n^{2}}, \tag{B.34}
\end{align*}
$$

where the two hyperbolic series are only formal. The trigonometric Glaisher functions are periodic with period $2 \pi$, and in the interval $[0, \pi]$ they are given by

$$
\begin{align*}
& \mathrm{Gl}_{2}(x)=\frac{1}{4}(\pi-x)^{2}-\frac{\pi^{2}}{12}  \tag{B.35}\\
& G \mathrm{Il}_{2}(x)=\frac{x^{2}}{4}-\frac{\pi^{2}}{12}
\end{align*}
$$

The hyperbolic Glaisher functions are explicitly given by

$$
\begin{align*}
& \mathrm{Glh}_{2}(x)=-\frac{x^{2}}{4}+\frac{\pi^{2}}{6} \\
& G \mathrm{Kh}_{2}(x)=-\frac{x^{2}}{4}-\frac{\pi^{2}}{12} \tag{B.36}
\end{align*}
$$

The massless three-point integral can also be expressed in terms of a complex analytic function, thus avoiding the division into subcases [62]. For all kinematic regions, the function $\mathrm{F}\left(p_{1}, p_{2}, p_{3}\right)$ has the following expression

$$
\begin{equation*}
\mathrm{F}\left(p_{1}, p_{2}, p_{3}\right)=\frac{1}{\rho}\left[\operatorname{Lsin}_{2}\left(2 \phi_{1}\right)+\operatorname{Lsin}_{2}\left(2 \phi_{2}\right)+\operatorname{Lsin}_{2}\left(2 \phi_{3}\right)\right] \tag{B.37}
\end{equation*}
$$

with

$$
\begin{equation*}
\phi_{i}=\arctan \left(\frac{\rho}{\delta_{i}}\right) \equiv \frac{i}{2} \ln \left(\frac{\delta_{i}-i \rho-i \epsilon}{\delta_{i}+i \rho-i \epsilon}\right) \tag{B.38}
\end{equation*}
$$

The variables $\delta_{i}$ are defined as before, $\rho=\sqrt{\mathcal{R}}$, and $\epsilon$ is infinitesimally small and
positive. The conventions for the imaginary part of logarithms and negative square roots can be taken to be $\operatorname{Im} \ln (-|x|)=i \pi$ and $\sqrt{-|x|}=i \sqrt{|x|}$.

The function $\operatorname{Lsin}_{2}(z)$ is the analytical extension of the function $\mathrm{Cl}_{2}(x)$ to the entire complex plane. For a number of reasons, we have introduced a new notation for this function and other analytically extended functions. First of all, the new notation emphasizes the form of the defining series of these functions. Secondly, Clausen and Glaisher functions are real functions whereas the L-functions are complex functions. This distinction is very clear in the case of the hyperbolic Glaisher function. For real $x$

$$
\begin{equation*}
\operatorname{Glh}_{2}(x)=-\frac{x^{2}}{4}+\frac{\pi^{2}}{6} \tag{B.39}
\end{equation*}
$$

whereas

$$
\begin{equation*}
\operatorname{Lcosh}_{2}(x)=-\frac{x^{2}}{4}+\frac{\pi^{2}}{6}-i \frac{\pi}{2}|x| \tag{B.40}
\end{equation*}
$$

Another argument in favor of a new notation is that, in the case of Clausen functions, their L-function partners are not the naive analytical continuation of their defining integrals as given in Eqs. (B.11), (B.21), (B.23) and (B.29). It seems best to keep Clausen-Glaisher functions real, and name their analytical partners differently.

Keeping the definition of L-functions separate from Clausen-Glaisher functions also avoids the staggered definition used in Ref. [57], for example

$$
\begin{equation*}
\mathrm{Cl}_{2 m}(x)=\sum_{1} \frac{\sin n x}{n^{2 m}} \tag{B.41}
\end{equation*}
$$

.but

$$
\begin{equation*}
\mathrm{Cl}_{2 m+1}(x)=\sum_{1} \frac{\cos n x}{n^{2 m+1}} \tag{B.42}
\end{equation*}
$$

Finally, as we will see shortly, all the L-functions are naturally defined in terms of the $\operatorname{Lexp}_{m}(x)$ function. It thus appears appropriate to use the new notation to reflect this relationship.

Next, we give the list of L-functions [63] and their basic properties.

$$
\begin{align*}
\operatorname{Lexp}_{m}(z) & \equiv \operatorname{Li}_{m}\left(e^{z}\right)=\sum_{1} \frac{e^{n z}}{n^{m}} \\
\operatorname{Lsin}_{m}(z) & \equiv \frac{1}{2 i}\left[\operatorname{Lexp}_{m}(i z)-\operatorname{Lexp}_{m}(-i z)\right]=\sum_{1} \frac{\sin n z}{n^{m}} \\
\operatorname{Lcos}_{m}(z) & \equiv \frac{1}{2}\left[\operatorname{Lexp}_{m}(i z)+\operatorname{Lexp}_{m}(-i z)\right]=\sum_{1} \frac{\cos n z}{n^{m}}  \tag{B.43}\\
\operatorname{Lsinh}_{m}(z) & \equiv \frac{1}{2}\left[\operatorname{Lexp}_{m}(z)-\operatorname{Lexp}_{m}(-z)\right]=\sum_{1} \frac{\sinh n z}{n^{m}} \\
\operatorname{Lcosh}_{m}(z) & \equiv \frac{1}{2}\left[\operatorname{Lexp}_{m}(z)+\operatorname{Lexp}_{m}(-z)\right]=\sum_{1} \frac{\cosh n z}{n^{m}}
\end{align*}
$$

The various series given above should be considered formal. All L-functions are periodic. The period of $\operatorname{Lexp}_{m}(z), \operatorname{Lsinh}_{m}(z)$ and $\operatorname{Lcosh}_{m}(z)$ is $2 \pi i$, whereas the period of $\operatorname{Lsin}_{m}(z)$ and $\operatorname{Lcos}_{m}(z)$ is $2 \pi$. The alternating L-functions ( $\nless$-functions) are defined as the half-period shifts of the L-functions,

$$
\begin{align*}
& \not \exp _{m}(z) \equiv \operatorname{Lexp}_{m}(z+i \pi)=\sum_{1} \frac{(-1)^{n} e^{n z}}{n^{m}}, \\
& \neq \sin _{m}(z) \equiv \operatorname{Lsin}_{m}(z+\pi)=\sum_{1} \frac{(-1)^{n} \sin n z}{n^{m}}, \\
& \notin \cos _{m}(z) \equiv \operatorname{Lcos} m(z+\pi)=\sum_{1} \frac{(-1)^{n} \cos n z}{n^{m}},  \tag{B.44}\\
& \nVdash \sinh _{m}(z) \equiv \operatorname{Lsinh}_{m}(z+i \pi)=\sum_{1} \frac{(-1)^{n} \sinh n z}{n^{m}}, \\
& \nVdash \cosh _{m}(z) \equiv \operatorname{Lcosh}_{m}(z+i \pi)=\sum_{1} \frac{(-1)^{n} \cosh n z}{n^{m}} .
\end{align*}
$$

The $\operatorname{Lexp}_{m}(z)$ function satisfies the following recursion relation

$$
\begin{equation*}
\operatorname{Lexp}_{m}(z)=\int_{-\infty}^{z} \operatorname{Lexp}_{m-1}(z) d z \tag{B.45}
\end{equation*}
$$

The first three $\operatorname{Lexp}_{2}(z)$ functions are given by

$$
\begin{align*}
& \operatorname{Lexp}_{0}(z)=\frac{e^{z}}{1-e^{z}} \\
& \operatorname{Lexp}_{1}(z)=-\ln \left(1-e^{z}\right)  \tag{B.46}\\
& \operatorname{Lexp}_{2}(z)=-\int_{-\infty}^{z} \ln \left(1-e^{z}\right) d z
\end{align*}
$$

Similarly, we have

$$
\begin{equation*}
\not \exp _{m}(z)=\int_{-\infty}^{z} \not \operatorname{kexp}_{m-1}(z) d z \tag{B.47}
\end{equation*}
$$

and

$$
\begin{align*}
& \not \exp _{0}(z)=-\frac{e^{z}}{1+e^{z}} \\
& \not \exp _{1}(z)=-\ln \left(1+e^{z}\right)  \tag{B.48}\\
& \not \exp _{2}(z)=-\int_{-\infty}^{z} \ln \left(1+e^{z}\right) d z
\end{align*}
$$

The explicit form of other L-functions can be similarly obtained. We will not reproduce them here.

In the following, we will concentratc on the case $m=2$. The function $\operatorname{Lexp}_{2}(z)$ has branch cuts on the positive scmiaxes where $\operatorname{Im} z=2 n \pi i, n=0, \pm 1, \pm 2, \ldots$, and the function $\not \exp _{2}(z)$ has branch cuts on the positive semiaxes where $\operatorname{Im} z=$ $(2 n+1) \pi i, n=0, \pm 1, \pm 2, \ldots$ On the real axis, we choose the imaginary part of
$\operatorname{Lexp}_{2}(x)$ to be

$$
\begin{equation*}
\operatorname{Im} \operatorname{Lexp}_{2}(x)=-i \pi x \theta(x) \tag{B.49}
\end{equation*}
$$

Around the origin, the two functions have the following series expansion

$$
\begin{align*}
& \operatorname{Lexp}_{2}(z)=-z \ln (-z)+z-\frac{z^{2}}{4}-\sum_{k=1}^{\infty} \frac{B_{2 k}}{2 k(2 k+1)!} z^{2 k+1}  \tag{B.50}\\
& \not \operatorname{kexp}_{2}(z)=-\frac{\pi}{12}-(\ln 2) z-\frac{z^{2}}{4}-\sum_{k=1}^{\infty} \frac{\left(2^{2 k}-1\right) B_{2 k}}{2 k(2 k+1)!} z^{2 k+1}
\end{align*}
$$

where $B_{n}$ are Bernoulli numbers [61] defined through the generating function

$$
\begin{equation*}
\frac{t}{c^{t}-1}=\sum_{0} B_{n} \frac{t^{n}}{n!} \tag{B.51}
\end{equation*}
$$

We have $B_{0}=1, B_{1}=-1 / 2, B_{2}=1 / 6, B_{4}=-1 / 30$, etc. The series expansions for other L-functions follow easily from those in Eqs. (B.50).

The real and imaginary parts of $\operatorname{Lexp}_{2}(z)$ can be obtained by Kummer's formula (see Ref. [57])

$$
\begin{align*}
\operatorname{Lexp}_{2}(x+i y)= & -\frac{1}{2} \int_{0}^{x} \ln \left(1-2 e^{x} \cos y+e^{2 x}\right) d x  \tag{B.52}\\
& +i\left\{x y^{\prime}+\frac{1}{2} \mathrm{Cl}_{2}(2 y)+\frac{1}{2} \mathrm{Cl}_{2}\left(2 y^{\prime}\right)+\frac{1}{2} \mathrm{Cl}_{2}\left(2 y^{\prime \prime}\right)\right\}
\end{align*}
$$

where

$$
\begin{equation*}
y^{\prime}=\arctan \left(\frac{e^{x} \sin y}{1-e^{x} \cos y}\right) \quad, \quad y^{\prime \prime}=\pi-y-y^{\prime} \tag{B.53}
\end{equation*}
$$

The separation of other L-functions into real and imaginary parts can be obtained by using the previous formula.

We have given here only some basic features of the L-functions. However, since they are defined from the polylogarithms, many other properties of polylogarithms are translated directly to L -functions. We refer the reader to Refs. [57, 58] for other potential properties of L-functions.

In summary, we have provided an analytically and numerically desirable expression for the massless three-point scalar integral in terms of associated Clausen functions and discussed the main features of these functions and their analytically extended partners, the L-functions. The simplicity shown in Eq. (B.37) hints at the potential usefulness of these functions in other Feynman diagram calculations.

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$$
\begin{equation*}
B\left(s, M^{2}\right)=\left(\frac{s+4 M^{2}}{s}\right)^{1 / 2} \log \left(\frac{\left(s+4 M^{2}\right)^{1 / 2}+\sqrt{s}}{\left(s+4 M^{2}\right)^{1 / 2}-\sqrt{s}}\right) \tag{B.54}
\end{equation*}
$$

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[^0]:    *Ph. D. thesis

