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THE NATURAL COORDINATE SYSTEM FOR STUDYING VECTOR PROBLEMS

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by

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TABLE OF CONTENTS

I. INTRODUCTION

In studying the dynamical properties of electron and ion beams moving through various kinds of electric and magnetic fields, it is usually convenient to choose one known trajectory as the reference axis and calculate the other neighbouring orbits surrounding it by the method of perturbation. Many authors in the instrumentation field of nuclear physics, notably, Herzog,[1] Klemm,[2] Streib and Brown,[3] have calculated in this manner charged-particle trajectories through bending magnets, quadrupole magnets, and magnetic spectrometers. These devices have a symmetry plane so that the chosen reference trajectory lies in this plane. A curved plane trajectory has no torsion, and the resulting coordinate system is orthogonal.

A more general coordinate system refers to some trajectory or vector line which is a space curve. A non-planar curve has both curvature and torsion. It seems clear that such a curvilinear coordinate system may be set up, but it is not at all obvious whether this system may be orthogonal or not.

The coordinates measured with reference to vector lines are called by Bjorgum[4] the "natural coordinates" for the vector lines. According to Bjorgum, these coordinates were used previously by Bjerknes[5] *et al.*, later by Ramsay,[6] and more recently by Milne-Thomson,[7] all in connection with hydrodynamical studies. Bjorgum generalized their method for his investigation of Beltrami vector fields and flows. (If $\vec{V} \times \vec{\nabla} \times \vec{V} = 0$, $\vec{V}$ is a Beltrami field.) This method of treating vector problems newly generalized by Bjorgum appears to be simple and effective, and generally useful for many vector problems in several branches of mathematical physics.

There are, however, some undesirable features associated with Bjorgum's method, which were clearly noted by him. The most restrictive one is that the coordinate variables used by him do not in general constitute "an ordinary system of coordinates." There exist no orthogonal surfaces to the coordinate lines except in very special cases, and the directional derivatives with respect to coordinate variables are usually not commutable with each other. Much skill and intuition are needed in order to obtain correct results. It seems, therefore, advisable to lay more groundwork for "natural coordinates" to facilitate the use of Bjorgum's analyses for varied purposes.

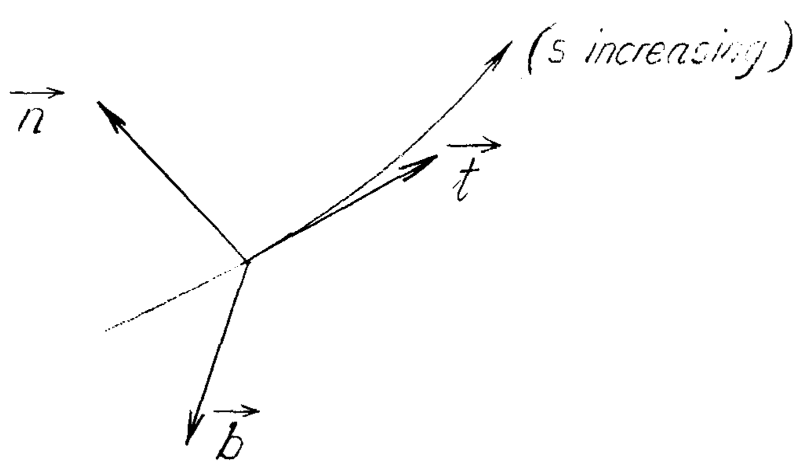
The objective of this paper is to set up the curvilinear coordinate system which has a curved non-planar vector line for its coordinate axis. This enables us to discuss "natural coordinates" in an elementary manner. We shall start from basic principles, collect and develop many relevant formulas including the metric and the Riemann-Christoffel symbols of this coordinate system, and elucidate how the afore-mentioned difficulties may be avoided. When the fundamental metric of a coordinate system is determined, all the relevant geometrical relations will be known in their standard forms and the carrying-out of analyses will become straightforward. By simply putting the torsion of the reference axis equal to zero, the coordinate system becomes orthogonal, the same as used by the authors mentioned in the first paragraph of this section.

II. NATURAL COORDINATES FOR VECTOR LINES

Consider a certain vector line (a space curve)

$$\vec{r} = \vec{r}\,(s) \tag{2.1}$$

shown in the following figure:



Here, s denotes the arc length along the vector line; $\vec{t}$, $\vec{n}$, $\vec{b}$ are unit vectors, $\vec{t}$ being the unit tangent vector ($\vec{t} = d\vec{r}/ds$), $\vec{n}$ the unit principal normal pointing in the direction of $d\vec{t}/ds$, and $\vec{b}$ the unit binormal. These three unit vectors, in the above-enumerated order, form a right-handed orthogonal set of coordinate axes at any given point on the vector line; they vary in directions from point to point along the vector line, all being functions of s.

Since $\vec{t}$, $\vec{n}$, $\vec{b}$ are orthogonal unit vectors, the dyadic

$$\vec{t}\vec{t} + \vec{n}\vec{n} + \vec{b}\vec{b} = \overleftrightarrow{I} \tag{2.2}$$

is an idemfactor,[a] which satisfies, for any arbitrary vector $\vec{A}$, the identity

$$\vec{A} \cdot \overleftrightarrow{I} = \overleftrightarrow{I} \cdot \vec{A} = \vec{A} \, . \tag{2.3}$$

Thus, the gradient operator $\vec{\nabla}$ becomes, on decomposition into its components,

$$\begin{aligned} \vec{\nabla} &= \vec{t}\vec{t} \cdot \vec{\nabla} + \vec{n}\vec{n} \cdot \vec{\nabla} + \vec{b}\vec{b} \cdot \vec{\nabla} \\ &\equiv \vec{t}\,\frac{\delta}{\delta s} + \vec{n}\,\frac{\delta}{\delta n} + \vec{b}\,\frac{\delta}{\delta b} \quad . \end{aligned} \tag{2.4}$$

The coordinates (s, n, b) measured in the directions ($\vec{t}$, $\vec{n}$, $\vec{b}$) are called by Bjorgum the natural coordinates. Here, we may note that the directional derivatives, $\delta/\delta s$, $\delta/\delta n$, and $\delta/\delta b$, are neither the ordinary partial derivatives, nor the ordinary covariant derivatives. These directional derivatives are usually not commutable with each other. Despite this inconvenience, Bjorgum made fruitful use of the natural coordinates in his investigation of Beltrami vector fields.

III. FRENET'S FORMULAS

The variation of the three orthonormal vectors $(\vec{t}, \vec{n}, \vec{b})$ along the tangential direction of the vector line are governed by Frenet's formulas.[9] These are as follows:

$$\delta\vec{t}/\delta s = \kappa\vec{n} \; , \tag{3.1a}$$

$$\delta\vec{n}/\delta s = -\kappa\vec{t} + \tau\vec{b} \tag{3.1b}$$

$$\delta\vec{b}/\delta s = -\tau\vec{n} \; . \tag{3.1c}$$

Here, κ and τ are scalar functions of s; κ denotes the curvature of the curve, always positive; τ denotes the torsion of the curve, either positive or negative. The sign of τ is a property of the space curve, independent of the choice of the direction of increasing s. For a plane curve, $\tau = 0$.

Equation (3.1a) is, in fact, a definition of the curvature κ and of the normal direction $\vec{n}$, because $\delta\vec{t}/\delta s$ must lie in the plane perpendicular to $\vec{t}$ at the point (s) under consideration. Similarly, Eq. (3.1c) is a definition of the torsion τ. This equation is proved by showing that $\delta\vec{b}/\delta s$ is parallel to $\vec{n}$, because $\delta\vec{b}/\delta s$ is perpendicular to $\vec{b}$ and $\vec{b} = \vec{t} \times \vec{n}$. The remaining formula, Eq. (3.1b), follows readily from these two definitions and the relation $\vec{n} = \vec{b} \times \vec{t}$.

The three unit vectors $(\vec{t}, \vec{n}, \vec{b})$ are defined in the whole domain of a vector field $\vec{V} = v\vec{t}$, except where $v = 0$ and $\kappa = 0$. When $v = 0$, $\vec{t}$ is indeterminate; when $\kappa = 0$, $\vec{n}$ is indeterminate. It may be assumed that the difficulty at such singular points can be removed by the conditions of continuity. These unit vectors vary not only along the s-direction of one vector line, as stated by Frenet's formulas, but also along transverse directions from one vector line to another. The functional dependence of the <u>reference</u> vectors $(\vec{t}, \vec{n}, \vec{b})$ on coordinate variables are determined by the characteristics of the vector field. This is, apparently, the reason that Bjorgum called the coordinates (s, n, b) natural coordinates.

To circumvent the attendant difficulty mentioned earlier, we shall, from here on, refer to one and the same vector line, for which the orthonormal set of vectors will be denoted by $(\vec{t}_o, \vec{n}_o, \vec{b}_o)$. The coordinates measured with reference to this vector line along these three directions will be denoted by (ζ, μ, ν). On the reference line $(\mu = \nu = 0)$, the ζ- , μ- , and ν-directions coincide, respectively, with the s- , n- , and b-directions; off this line, the corresponding two directions are different from each other. As will be shown later, the coordinates (ζ, μ, ν) do constitute a proper system of curvilinear coordinates, though, generally, non-orthogonal. These coordinates will also be called, conveniently, natural coordinates.

IV. FUNDAMENTAL GEOMETRIC RELATIONS

Before proceeding to discuss the natural coordinate system specifically, we consider a general system of coordinates (x^1, x^2, x^3), curvilinear and non-orthogonal. In this general system, the infinitesimal displacement vector $d\vec{r}$ of components (dx^1, dx^2, dx^3) is given by

$$d\vec{r} = \vec{e}_1\, dx^1 + \vec{e}_2\, dx^2 + \vec{e}_3\, dx^3 \ ,$$

i.e.,

$$d\vec{r} = \vec{e}_i\, dx^i \qquad (i = 1,2,3) \tag{4.1}$$

Here, the $\vec{e}_i$'s are the three non-coplanar (covariant) base vectors. Obviously,

$$\vec{e}_i = \partial\vec{r}/\partial x^i \tag{4.2}$$

and

$$\partial\vec{e}_i/\partial x^k = \partial\vec{e}_k/\partial x^i \ . \tag{4.3}$$

From Eq. (4.1) we obtain

$$dr^2 = d\vec{r} \cdot d\vec{r} = g_{ik}\, dx^i dx^k \ , \tag{4.4}$$

where

$$g_{ik} = \vec{e}_i \cdot \vec{e}_k = g_{ki} \tag{4.5}$$

are called the covariant components of the metric tensor.

Reciprocal to this covariant set of base vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ there exists the contravariant set of base vectors $(\vec{e}^1, \vec{e}^2, \vec{e}^3)$. These are called reciprocal sets, because

$$\vec{e}_i \cdot \vec{e}^k = \delta_i^k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \tag{4.6}$$

Thus,

$$\vec{e}_i = \vec{e}_i \cdot \vec{e}^k \, \vec{e}_k \quad \text{and} \quad \vec{e}^i = \vec{e}^i \cdot \vec{e}_k \, \vec{e}^k \ .$$

In other words,

$$\vec{e}_i \, \vec{e}^i = \vec{e}^i \, \vec{e}_i = \overleftrightarrow{I} \ . \tag{4.7}$$

The contravariant components g^{ik} of the metric tensor are defined in a similar manner, namely,

$$g^{ik} = \vec{e}^i \cdot \vec{e}^k = g^{ki} \ . \tag{4.8}$$

If we denote by g the determinant of the matrix (g_{ik}) and by G_{ik} the minor of g_{ik} in the determinant g, so that

$$\sum_{\ell} g_{i\ell} \, G_{k\ell} = \begin{cases} g \ , & \text{if } i = k \\ 0 \ , & \text{if } i \neq k \end{cases} \ , \tag{4.9}$$

then,

$$g^{ik} = G_{ik}/g \quad , \tag{4.10}$$

$$g_{i\ell}\, g^{k\ell} = \delta_i^k \quad , \tag{4.11}$$

and

$$\det\,(g^{ik}) = 1/g = 1/\det\,(g_{ik}) \quad . \tag{4.12}$$

Here, we may note that the metric tensor (or dyadic) $\overleftrightarrow{g}$ is just the idemfactor,

$$\overleftrightarrow{g} = \vec{e}^{\,i}\, \vec{e}^{\,k}\, g_{ik} = \vec{e}_i\, \vec{e}_k\, g^{ik} = \vec{e}^{\,i}\, \vec{e}_k\, (g_{i\ell}\, g^{k\ell})$$

$$= \vec{e}_i\, \vec{e}^{\,k}\, (g^{i\ell}\, g_{k\ell}) = \overleftrightarrow{I} \quad .$$

Apparently, it is the current usage to write $\overleftrightarrow{I}$ for $\overleftrightarrow{g}$ and δ_i^k for g_i^k .

From the properties of reciprocal base vectors as given by Eq. (4.6), it also follows that

$$\vec{e}^{\,i} = (1/\sqrt{g})\, \vec{e}_k \times \vec{e}_\ell \tag{4.13a}$$

and

$$\vec{e}_i = \sqrt{g}\, \vec{e}^{\,k} \times \vec{e}^{\,\ell} \quad . \tag{4.13b}$$

Here, (i, k, ℓ) are even permutations of (1, 2, 3).

In terms of these base vectors we may decompose any vector $\vec{A}$ into its components, either covariant or contravariant. Thus,

$$\vec{A} = \vec{e}_i\, A^i = \vec{e}^{\,i}\, A_i \quad , \tag{4.14}$$

$A^i = \vec{A} \cdot \vec{e}^{\,i}$ being the contravariant components and $A_i = \vec{A} \cdot \vec{e}_i$ the covariant components.

In particular, the gradient vector $\vec{\nabla}$ may be decomposed as

$$\vec{\nabla} = \vec{e}_i \, (\vec{e}^{\,i} \cdot \vec{\nabla}) = \vec{e}^{\,i} \, (\vec{e}_i \cdot \vec{\nabla}) \quad . \tag{4.15}$$

Since

$$\vec{\nabla}\vec{r} = (\vec{\nabla} x^i) \, \partial\vec{r}/\partial x^i = \overleftrightarrow{I} \tag{4.16a}$$

and

$$\vec{r}\vec{\nabla} = (\partial\vec{r}/\partial x^i) \vec{\nabla} x^i = \overleftrightarrow{I} \tag{4.16b}$$

are identities independent of the coordinate system, we obtain by using Eq. (4.2)

$$\vec{e}^{\,i} = \vec{\nabla} x^i \quad . \tag{4.17}$$

Hence,

$$\vec{\nabla} = \vec{e}^{\,i} \frac{\partial}{\partial x^i} = \vec{e}_i \, g^{ik} \frac{\partial}{\partial x^k} \quad . \tag{4.18}$$

Similarly, we may decompose the reference vectors $(\vec{t}, \vec{n}, \vec{b})$ into their covariant (t_i, n_i, b_i) and contravariant (t^i, n^i, b^i) components. Since $(\vec{t}, \vec{n}, \vec{b})$ are orthogonal unit vectors, we must have

$$t_i t^i = n_i n^i = b_i b^i = 1 \quad . \tag{4.19a}$$

and

$$t_i n^i = t_i b^i = n_i b^i = 0 \quad . \tag{4.19b}$$

Furthermore, we obtain from $\vec{b} = \vec{t} \times \vec{n}$

$$b^i = (1/\sqrt{g})(t_k n_\ell - t_\ell n_k) \tag{4.20a}$$

and

$$b_i = \sqrt{g} \quad (t^k n^\ell - t^\ell n^k) \quad , \tag{4.20b}$$

(i, k, ℓ being even permutations of 1, 2, 3). Two similar pairs of equations may be obtained by cyclic permutations of (t, n, b).

To use Frenet's formulas in curvilinear coordinates we need perform the indicated differentiations. Thus,

$$\begin{aligned}
\vec{t} \cdot \vec{\nabla}\vec{t} &= \vec{t} \cdot \vec{e}^{k} \frac{\partial \vec{t}}{\partial x^{k}} = t^{k} \frac{\partial}{\partial x^{k}} (\vec{e}_{i} t^{i}) \\
&= t^{k} \left(\vec{e}_{i} \frac{\partial t^{i}}{\partial x^{k}} + t^{i} \frac{\partial \vec{e}_{i}}{\partial x^{k}} \right) \\
&= \vec{e}_{i} \; t^{k} \left\{ \frac{\partial t^{i}}{\partial x^{k}} + \left(\vec{e}^{i} \cdot \frac{\partial \vec{e}_{m}}{\partial x^{k}} \right) t^{m} \right\} \\
&\equiv \vec{e}_{i} \; t^{k} t^{i}_{\;;k} \quad ,
\end{aligned}$$

where $t^{i}_{\;;k}$ is the covariant derivative of t^{i} with respect to x^{k}. The scalar product of $\vec{e}^{i}$ and $\partial\vec{e}_{m}/\partial x^{k}$ is usually denoted by the Riemann-Christoffel symbol,[10] namely,

$$\Gamma^{i}_{km} = \vec{e}^{i} \cdot \frac{\partial \vec{e}_{m}}{\partial x^{k}} = - \frac{\partial \vec{e}^{i}}{\partial x^{k}} \cdot \vec{e}_{m} \quad . \tag{4.21}$$

Thus,

$$t^{i}_{\;;k} = \frac{\partial t^{i}}{\partial x^{k}} + \Gamma^{i}_{km} \, t^{m} \quad . \tag{4.22}$$

In Cartesian coordinates every such symbol Γ^{i}_{km} vanishes.

The s-derivatives ($\vec{t} \cdot \vec{\nabla}$) of $\vec{n}$ and $\vec{b}$ are similarly calculated. Thus, Frenet's formulas become, in curvilinear coordinate systems, as follows:

$$t^{k}\, t^{i}_{\;;k} = \kappa n^{i} \quad , \tag{4.23a}$$

$$t^{k}\, n^{i}_{\;;k} = - \kappa t^{i} + \tau b^{i} \quad , \tag{4.23b}$$

$$t^{k}\, b^{i}_{\;;k} = - \tau n^{i} \quad . \tag{4.23c}$$

V. TRANSFORMATION OF COORDINATES

Consider the transformation from one general coordinate system (x^1, x^2, x^3) into another (x'^1, x'^2, x'^3):

$$dx'^i = \frac{\partial x'^i}{\partial x^k} dx^k \; ; \qquad dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k \quad . \tag{5.1}$$

Since

$$d\vec{r} = \vec{e}_i \, dx^i = \vec{e}'_k \, dx'^k \quad , \tag{5.2}$$

$d\vec{r}$ being a vector, an invariant quantity under any coordinate transformation,

$$dx'^i = (\vec{e}'^i \cdot \vec{e}_k) \, dx^k \quad , \quad \text{i.e.,} \quad (\vec{e}'^i \cdot \vec{e}_k) = \frac{\partial x'^i}{\partial x^k} \tag{5.3a}$$

and

$$dx^i = (\vec{e}^i \cdot \vec{e}'_k) \, dx'^k \quad , \quad \text{i.e.,} \quad (\vec{e}^i \cdot \vec{e}'_k) = \frac{\partial x^i}{\partial x'^k} \quad . \tag{5.3b}$$

Noting that the idemfactor $\overleftrightarrow{I}$ is an invariant, namely,

$$\overleftrightarrow{I} = \vec{e}_i \, \vec{e}^i = \vec{e}^i \, \vec{e}_i = \vec{e}'_k \, \vec{e}'^k = \vec{e}'^k \, \vec{e}'_k \quad ,$$

we obtain

$$\vec{e}^i = \vec{e}^i \cdot \vec{e}'_k \, \vec{e}'^k = \frac{\partial x^i}{\partial x'^k} \vec{e}'^k \quad , \tag{5.4a}$$

$$\vec{e}_i = \vec{e}_i \cdot \vec{e}'^k \, \vec{e}'_k = \frac{\partial x'^k}{\partial x^i} \vec{e}'_k \quad , \tag{5.4b}$$

$$\vec{e}'^i = \vec{e}'^i \cdot \vec{e}_k \, \vec{e}^k = \frac{\partial x'^i}{\partial x^k} \vec{e}^k \quad , \tag{5.4c}$$

and

$$\vec{e}'_i = \vec{e}'_i \cdot \vec{e}^k \, \vec{e}_k = \frac{\partial x^k}{\partial x'^i} \vec{e}_k \quad . \tag{5.4d}$$

In other words, the contravariant base vectors transform like contravariant vector components and the covariant base vectors transform like covariant vector components. Using the foregoing transformation laws we may easily transform any kind of vector and tensor components.

In this connection, it is very helpful to note that not only scalars but also vectors (e.g. $d\vec{r}$, $\vec{\nabla}\varphi$, etc.), dyadics (e.g. $\overleftrightarrow{g}$, $\overrightarrow{\overrightarrow{\nabla A}}$, etc.), triadics (e.g. $\vec{B}(\overrightarrow{\overrightarrow{\nabla A}})$, $\overrightarrow{\overrightarrow{\overrightarrow{\nabla\nabla A}}}$, etc.), etc. are invariant quantities under coordinate transformations. Vectors, dyadics, triadics, etc. are, respectively, tensors of rank one, two, three, etc. For example,

$$\vec{A} = \vec{e}_i\, A^i = \vec{e}^{\,i} A_i = \vec{e}\,'_k\, A'^k = \vec{e}\,'^k A'_k \quad ,$$

$$\overleftrightarrow{\Phi} = \vec{e}_i\, \vec{e}_k\, \Phi^{ik} = \vec{e}^{\,i}\, \vec{e}^{\,k}\, \Phi_{ik} = \vec{e}_i\, \vec{e}^{\,k}\, \Phi^i_k = \vec{e}^{\,i}\, \vec{e}_k\, \Phi^k_i$$

$$= \vec{e}\,'_m\, \vec{e}\,'_n\, \Phi'^{mn} = \vec{e}\,'^m\, \vec{e}\,'^n\, \Phi'_{mn} = \vec{e}\,'_m\, \vec{e}\,'^n \Phi'^m_n = \vec{e}\,'^m\, \vec{e}\,'_n\, \Phi'^n_m \quad .$$

These identities lead to the following equations of transformations:

$$A'^i = \frac{\partial x'^i}{\partial x^k} A^k \quad ; \qquad A'_i = \frac{\partial x^k}{\partial x'^i} A_k \quad . \tag{5.5}$$

$$\left.\begin{aligned} \Phi'^{mn} &= \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^n}{\partial x^k} \Phi^{ik} \quad , \\ \Phi'_{mn} &= \frac{\partial x^i}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \Phi_{ik} \quad , \\ &\text{etc.} \end{aligned}\right\} \tag{5.6}$$

VI. THE NATURAL COORDINATE SYSTEM

Now we come to consider (ζ, μ, ν) to be the curvilinear coordinates (x^1, x^2, x^3). The two reciprocal sets of base vectors will be denoted by $(\vec{e}_\zeta, \vec{e}_\mu, \vec{e}_\nu)$ and $(\vec{e}^\zeta, \vec{e}^\mu, \vec{e}^\nu)$. According to Eq. (4.2), we have

$$\vec{e}_\zeta = \vec{a}_x \frac{\partial x}{\partial \zeta} + \vec{a}_y \frac{\partial y}{\partial \zeta} + \vec{a}_z \frac{\partial z}{\partial \zeta} \tag{6.1a}$$

$$\vec{e}_\mu = \vec{a}_x \frac{\partial x}{\partial \mu} + \vec{a}_y \frac{\partial y}{\partial \mu} + \vec{a}_z \frac{\partial z}{\partial \mu} \tag{6.1b}$$

$$\vec{e}_\nu = \vec{a}_x \frac{\partial x}{\partial \nu} + \vec{a}_y \frac{\partial y}{\partial \nu} + \vec{a}_z \frac{\partial z}{\partial \nu} \quad . \tag{6.1c}$$

Here, (x, y, z) are Cartesian coordinates and $(\vec{a}_x, \vec{a}_y, \vec{a}_z)$ are constant unit vectors. The contravariant set of base vectors may be obtained from the covariant set by using Eq. (4.13a).

Let the equations of coordinate transformation be as follows:

$$x = \Phi(\zeta, \mu, \nu), \quad y = \Psi(\zeta, \mu, \nu), \quad z = \Omega(\zeta, \mu, \nu) \; . \tag{6.2}$$

When $\mu = 0$ and $\nu = 0$,

$$\left.\begin{aligned} x &= \Phi\ (\zeta, 0, 0) = \varphi\ (\zeta) \quad , \\ y &= \Psi\ (\zeta, 0, 0) = \psi\ (\zeta) \quad , \\ z &= \Omega\ (\zeta, 0, 0) = \omega\ (\zeta) \quad . \end{aligned}\right\} \tag{6.3}$$

These equations represent the reference trajectory or vector line in terms of its arc length ζ .

The unit tangent, the unit principal normal, and the unit binormal vectors of the reference trajectory are, respectively, $\vec{t}_o$, $\vec{n}_o$, and $\vec{b}_o$. As discussed in differential geometry,[11]

$$\vec{t}_o = \vec{a}_x \varphi' + \vec{a}_y \psi' + \vec{a}_z \omega' \quad , \tag{6.4a}$$

$$\kappa\vec{n}_o = \vec{a}_x \varphi'' + \vec{a}_y \psi'' + \vec{a}_z \omega'' \quad , \tag{6.4b}$$

$$\kappa\vec{b}_o = \vec{a}_x (\psi'\omega'' - \omega'\psi'') + \vec{a}_y (\omega'\varphi'' - \varphi'\omega'') + \vec{a}_z (\varphi'\psi'' - \psi'\varphi'') \; . \tag{6.4c}$$

In these equations, $\varphi' = d\varphi/d\zeta$, $\varphi'' = d^2\varphi/d\zeta^2$, etc. Evidently,

$$(\varphi')^2 + (\psi')^2 + (\omega')^2 = 1 \quad ; \tag{6.5}$$

$$\kappa^2 = (\varphi'')^2 + (\psi'')^2 + (\omega'')^2 \tag{6.6a}$$

$$= (\psi'\omega''-\omega'\psi'')^2+(\omega'\varphi''-\varphi'\omega'')^2+(\varphi'\psi''-\psi'\varphi'')^2 \quad . \tag{6.6b}$$

Furthermore, by using the third Frenet's formula, namely

$$d\,\vec{b}_o/d\zeta = -\,\tau\,\vec{n}_o \quad ,$$

it can easily be shown that the torsion of the reference curve is

$$\tau = \frac{1}{\kappa^2}\begin{vmatrix} \varphi' & \psi' & \omega' \\ \varphi'' & \psi'' & \omega'' \\ \varphi''' & \psi''' & \omega''' \end{vmatrix} \quad . \tag{6.7}$$

In order to set up the simplest coordinate system (ζ, μ, ν) we impose the condition that the functions Φ, Ψ, and Ω are linear functions of μ and ν. Under this condition,

$$\left.\begin{aligned} x &= \varphi(\zeta) + \mu\,\frac{\partial\Phi}{\partial\mu} + \nu\,\frac{\partial\Phi}{\partial\nu} \quad , \\ y &= \psi(\zeta) + \mu\,\frac{\partial\Psi}{\partial\mu} + \nu\,\frac{\partial\Psi}{\partial\nu} \quad , \\ z &= \omega(\zeta) + \mu\,\frac{\partial\Omega}{\partial\mu} + \nu\,\frac{\partial\Omega}{\partial\nu} \quad . \end{aligned}\right\} \tag{6.8}$$

All the μ- and ν-derivatives of Φ, Ψ, and Ω are independent of both μ and ν. Hence,

$$\vec{e}_\mu = \vec{a}_x\,\frac{\partial\Phi}{\partial\mu} + \vec{a}_y\,\frac{\partial\Psi}{\partial\mu} + \vec{a}_z\,\frac{\partial\Omega}{\partial\mu} \tag{6.9a}$$

and

$$\vec{e}_\nu = \vec{a}_x \frac{\partial \Phi}{\partial \nu} + \vec{a}_y \frac{\partial \Psi}{\partial \nu} + \vec{a}_z \frac{\partial \Omega}{\partial \nu} \tag{6.9b}$$

are functions of ζ only. Now we note that we are at liberty to choose the scale factors for μ and ν so that both $\vec{e}_\mu$ and $\vec{e}_\nu$ are unit vectors. Since $\vec{e}_\mu$ is along the principal normal and $\vec{e}_\nu$ along the binormal, we may simply require:

$$\vec{e}_\mu = \vec{n}_o \quad ; \tag{6.10a}$$

$$\vec{e}_\nu = \vec{b}_o \quad . \tag{6.10b}$$

We thus obtain

$$\frac{\partial \Phi}{\partial \mu} = \frac{1}{\kappa} \varphi'' \ , \quad \frac{\partial \Psi}{\partial \mu} = \frac{1}{\kappa} \psi'' \ , \quad \frac{\partial \Omega}{\partial \mu} = \frac{1}{\kappa} \omega'' \ ; \tag{6.11}$$

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial \nu} &= \frac{1}{\kappa} (\psi'\omega'' - \omega'\psi'') \ , \\ \frac{\partial \Psi}{\partial \nu} &= \frac{1}{\kappa} (\omega'\varphi'' - \varphi'\omega'') \ , \\ \frac{\partial \Omega}{\partial \nu} &= \frac{1}{\kappa} (\varphi'\psi'' - \psi'\varphi'') \ . \end{aligned}\right\} \tag{6.12}$$

Having obtained the equations of transformation from the curvilinear coordinates (ζ, μ, ν) to Cartesian coordinates (x, y, z) we may now determine the metric properties of the curvilinear system. From Eqs. (6.8), (6.11), and (6.12) it is clear that the curvilinear coordinate system is determined entirely by the geometrical properties of one chosen curve of reference represented by Eq. (6.3). Since

$$\varphi' \frac{\partial \Phi}{\partial \mu} + \psi' \frac{\partial \Psi}{\partial \mu} + \omega' \frac{\partial \Omega}{\partial \mu} = 0 \ , \tag{6.13a}$$

$$\varphi' \frac{\partial \Phi}{\partial \nu} + \psi' \frac{\partial \Psi}{\partial \nu} + \omega' \frac{\partial \Omega}{\partial \nu} = 0 \ , \tag{6.13b}$$

and

$$\frac{\partial\Phi}{\partial\mu}\frac{\partial\Phi}{\partial\nu} + \frac{\partial\Psi}{\partial\mu}\frac{\partial\Psi}{\partial\nu} + \frac{\partial\Omega}{\partial\mu}\frac{\partial\Omega}{\partial\nu} = 0 \quad , \tag{6.13c}$$

we easily obtain the following relations:

$$(x-\varphi)\,\varphi' + (y-\psi)\,\psi' + (z-\omega)\,\omega' = 0 \;, \tag{6.14a}$$

$$(x-\varphi)\,\frac{\partial\Phi}{\partial\mu} + (y-\psi)\,\frac{\partial\Psi}{\partial\mu} + (z-\omega)\,\frac{\partial\Omega}{\partial\mu} = \mu \;, \tag{6.14b}$$

and

$$(x-\varphi)\,\frac{\partial\Phi}{\partial\nu} + (y-\psi)\,\frac{\partial\Psi}{\partial\nu} + (z-\omega)\,\frac{\partial\Omega}{\partial\nu} = \nu \;. \tag{6.14c}$$

Squaring Eqs. (6.14b) and (6.14c) and adding them together, and simplifying the resulting equation by making use of Eqs. (6.5) and (6.6), we then obtain

$$\mu^2 + \nu^2 = (x-\varphi)^2 + (y-\psi)^2 + (z-\omega)^2 - \left\{(x-\varphi)\varphi' + (y-\psi)\psi' + (z-\omega)\omega'\right\}^2 \;.$$

Clearly, Eq. (6.14a) represents the plane, normal to the reference curve at the point $(\zeta, 0, 0)$. The square of the distance from the point $(\zeta, 0, 0)$ to the point (ζ, μ, ν) is

$$\mu^2 + \nu^2 = (x-\varphi)^2 + (y-\psi)^2 + (z-\omega)^2 \quad . \tag{6.15}$$

The remaining base vector in the covariant set is

$$\vec{e}_\zeta = \vec{a}_x \frac{\partial\Phi}{\partial\zeta} + \vec{a}_y \frac{\partial\Psi}{\partial\zeta} + \vec{a}_z \frac{\partial\Omega}{\partial\zeta} \quad , \tag{6.16}$$

having Cartesian components

$$\frac{\partial\Phi}{\partial\zeta} = \varphi' + \mu \frac{\partial}{\partial\zeta}\frac{\partial\Phi}{\partial\mu} + \nu \frac{\partial}{\partial\zeta}\frac{\partial\Phi}{\partial\nu} = \varphi' + \vec{a}_x\cdot\left(\mu \frac{d\vec{n}_o}{d\zeta} + \nu \frac{d\vec{b}_o}{d\zeta}\right) \quad ,$$

$$\frac{\partial\Psi}{\partial\zeta} = \psi' + \mu \frac{\partial}{\partial\zeta}\frac{\partial\Psi}{\partial\mu} + \nu \frac{\partial}{\partial\zeta}\frac{\partial\Psi}{\partial\nu} = \psi' + \vec{a}_y\cdot\left(\mu \frac{d\vec{n}_o}{d\zeta} + \nu \frac{d\vec{b}_o}{d\zeta}\right) \quad ,$$

and

$$\frac{\partial\Omega}{\partial\zeta} = \omega' + \mu \frac{\partial}{\partial\zeta}\frac{\partial\Omega}{\partial\mu} + \nu \frac{\partial}{\partial\zeta}\frac{\partial\Omega}{\partial\nu} = \omega' + \vec{a}_z\cdot\left(\mu \frac{d\vec{n}_o}{d\zeta} + \nu \frac{d\vec{b}_o}{d\zeta}\right) \quad .$$

Using Frenet's formulas we obtain

$$\vec{e}_\zeta = (1 - \mu\kappa)\,\vec{t}_o - \nu\tau\,\vec{n}_o + \mu\tau\,\vec{b}_o \quad . \tag{6.10c}$$

The covariant metric tensor components $g_{\sigma\lambda} = \vec{e}_\sigma \cdot \vec{e}_\lambda$ may then be evaluated. The resulting matrix is as follows:

$$(g_{\sigma\lambda}) = \begin{pmatrix} g_{\zeta\zeta} & g_{\zeta\mu} & g_{\zeta\nu} \\ g_{\mu\zeta} & g_{\mu\mu} & 0 \\ g_{\nu\zeta} & 0 & g_{\nu\nu} \end{pmatrix}$$

$$= \begin{pmatrix} (1-\mu\kappa)^2 + (\mu^2+\nu^2)\tau^2 & -\nu\tau & \mu\tau \\ -\nu\tau & 1 & 0 \\ \mu\tau & 0 & 1 \end{pmatrix} . \tag{6.17}$$

The determinant of this matrix is

$$g = \det\,(g_{\sigma\lambda}) = (1 - \mu\kappa)^2 \quad . \tag{6.18}$$

Now we use Eq. (4.13a) to evaluate the contravariant base vectors.

$$\vec{e}^{\zeta} = (1/\sqrt{g})\ \vec{e}_{\mu} \times \vec{e}_{\nu} = (1/\sqrt{g})\ \vec{t}_{o} \quad . \tag{6.19a}$$

$$\vec{e}^{\mu} = (1/\sqrt{g})\ \vec{e}_{\nu} \times \vec{e}_{\zeta} = \vec{n}_{o} + (\nu\tau/\sqrt{g})\ \vec{t}_{o} \quad . \tag{6.19b}$$

$$\vec{e}^{\nu} = (1/\sqrt{g})\ \vec{e}_{\zeta} \times \vec{e}_{\mu} = \vec{b}_{o} - (\mu\tau/\sqrt{g})\ \vec{t}_{o} \quad . \tag{6.19c}$$

Since $\vec{e}^{\zeta} = \vec{\nabla}\zeta$, this base vector is normal to the surface ζ = const. Similarly, $\vec{e}^{\mu}$ is normal to the surface μ = const. and $\vec{e}^{\nu}$ is normal to the surface ν = const. The surfaces of constant ζ are planes, while the other two sets of surfaces are not. Except when $\tau = 0$, these sets of surfaces are non-orthogonal, because $\vec{e}^{\zeta}$, $\vec{e}^{\mu}$, and $\vec{e}^{\nu}$ are not mutually perpendicular.

The contravariant metric tensor components are evaluated from the contravariant base vectors, also by taking scalar products.

$$(g^{\sigma\lambda}) = \begin{pmatrix} g^{\zeta\zeta} & g^{\zeta\mu} & g^{\zeta\nu} \\ g^{\mu\zeta} & g^{\mu\mu} & g^{\mu\nu} \\ g^{\nu\zeta} & g^{\nu\mu} & g^{\nu\nu} \end{pmatrix}$$

$$= \frac{1}{g}\begin{pmatrix} 1 & \nu\tau & -\mu\tau \\ \nu\tau & (1-\mu\kappa)^2+(\nu\tau)^2 & -\mu\nu\tau^2 \\ -\mu\tau & -\mu\nu\tau^2 & (1-\mu\kappa)^2+(\mu\tau)^2 \end{pmatrix} , \tag{6.20}$$

and

$$\det\ (g^{\sigma\lambda}) = g^{-1} = (1 - \mu\kappa)^{-2} \quad . \tag{6.21}$$

Here, we may note that, in general, none of the contravariant metric tensor components vanishes. On the reference curve, where $\mu = \nu = 0$, both $(g_{\sigma\lambda})$ and $(g^{\sigma\lambda})$ become equal to the unity diagonal matrix.

The differential of the position vector is

$$d\vec{r} = \vec{e}_\zeta \, d\zeta + \vec{e}_\mu \, d\mu + \vec{e}_\nu \, d\nu \tag{6.22a}$$

$$= \vec{t}_o \, (1 - \mu\kappa) \, d\zeta + \vec{n}_o \, (d\mu - \nu\tau d\zeta) + \vec{b}_o \, (d\nu + \mu\tau d\zeta), \tag{6.22b}$$

and the invariant (<u>fundamental</u>) quadratic form is

$$dr^2 = (1 - \mu\kappa)^2 \, d\zeta^2 + (d\mu - \nu\tau d\zeta)^2 + (d\nu + \mu\tau d\zeta)^2$$

$$= \left\{ (1 - \mu\kappa)^2 + (\mu\tau)^2 + (\nu\tau)^2 \right\} d\zeta^2 + d\mu^2 + d\nu^2$$

$$- 2\nu\tau \, d\zeta d\mu + 2\mu\tau \, d\nu d\zeta \quad . \tag{6.23}$$

This agrees with Eq. (6.17) as expected.

VII. THE GRADIENT OPERATOR

According to Eq. (4.18), the gradient operator is simply

$$\vec{\nabla} = \vec{e}^{\zeta} \frac{\partial}{\partial\zeta} + \vec{e}^{\mu} \frac{\partial}{\partial\mu} + \vec{e}^{\nu} \frac{\partial}{\partial\nu} \quad . \tag{7.1a}$$

When referred to the three mutually perpendicular unit vectors $(\vec{t}_o, \vec{n}_o, \vec{b}_o)$, it becomes

$$\vec{\nabla} = \vec{t}_o \frac{1}{\sqrt{g}} \left(\frac{\partial}{\partial\zeta} + \nu\tau \frac{\partial}{\partial\mu} - \mu\tau \frac{\partial}{\partial\nu} \right) + \vec{n}_o \frac{\partial}{\partial\mu} + \vec{b}_o \frac{\partial}{\partial\nu} \quad . \tag{7.1b}$$

Since $\vec{n}_o = \vec{e}_\mu$, $\vec{b}_o = \vec{e}_\nu$, and $\vec{t}_o = \sqrt{g} \, \vec{e}^{\zeta}$, these three unit vectors are neither a covariant nor a contravariant set of base vectors.

On the reference curve, $d\vec{r} = \vec{t}_o \, d\zeta$, the expression of the gradient operator is reduced to

$$\vec{\nabla} = \vec{t}_o \frac{\partial}{\partial\zeta} + \vec{n}_o \frac{\partial}{\partial\mu} + \vec{b}_o \frac{\partial}{\partial\nu} \quad . \quad (\mu = \nu = 0) \tag{7.2}$$

In studying the properties of vector fields, one may be more interested in the components of $\vec{\nabla}$ along the directions $(\vec{t}, \vec{n}, \vec{b})$ than along $(\vec{t}_o, \vec{n}_o, \vec{b}_o)$. As noted before, Bjorgum used the components $\vec{t} \cdot \vec{\nabla}$, $\vec{n} \cdot \vec{\nabla}$, and $\vec{b} \cdot \vec{\nabla}$ in his investigation of Beltrami fields. These components are obtained easily from Eq. (7.1a) or Eq. (7.1b). Let $\vec{A} = \vec{t}$, or $\vec{n}$, or $\vec{b}$, or any vector, we have

$$\vec{A} \cdot \vec{\nabla} = A^{\zeta} \frac{\partial}{\partial \zeta} + A^{\mu} \frac{\partial}{\partial \mu} + A^{\nu} \frac{\partial}{\partial \nu} \tag{7.3a}$$

$$= (\vec{A} \cdot \vec{t}_o) \frac{1}{\sqrt{g}} \left(\frac{\partial}{\partial \zeta} + \nu\tau \frac{\partial}{\partial \mu} - \mu\tau \frac{\partial}{\partial \nu} \right) + (\vec{A} \cdot \vec{n}_o) \frac{\partial}{\partial \mu} + (\vec{A} \cdot \vec{b}_o) \frac{\partial}{\partial \nu}$$

$$= A^{\zeta} \frac{\partial}{\partial \zeta} + (A_{\mu} + \nu\tau A^{\zeta}) \frac{\partial}{\partial \mu} + (A_{\nu} - \mu\tau A^{\zeta}) \frac{\partial}{\partial \nu} \quad . \tag{7.3b}$$

From these equations it follows that

$$A^{\mu} = A_{\mu} + \nu\tau A^{\zeta} \quad , \tag{7.4a}$$

$$A^{\nu} = A_{\nu} - \mu\tau A^{\zeta} \quad . \tag{7.4b}$$

Also, as can be obtained easily from Eq. (6.10c),

$$g A^{\zeta} = A_{\zeta} + \nu\tau A_{\mu} - \mu\tau A_{\nu} \quad . \tag{7.4c}$$

Of any vector $\vec{A}$, $(\sqrt{g} A^{\zeta}, A_{\mu} \; A_{\nu})$ are the three rectangular components referred to $(\vec{t}_o, \vec{n}_o, \vec{b}_o)$.

VIII. RIEMANN-CHRISTOFFEL SYMBOLS

These symbols are defined by Eq. (4.21); they appear in the differentiation of vector and tensor quantities. For example,

$$\frac{d}{ds}\vec{\vec{\Phi}} = \frac{d}{ds}\left(\vec{e}_i\vec{e}_k\ \Phi^{ik}\right) = \vec{e}_i\vec{e}_k\ \Phi^{ik}_{;m}\frac{dx^m}{ds} \tag{8.1}$$

where the covariant derivative $\Phi^{ik}_{;m}$ is given by

$$\Phi^{ik}_{;m} = \frac{\partial\Phi^{ik}}{\partial x^m} + \Gamma^i_{m\ell}\ \Phi^{\ell k} + \Gamma^k_{m\ell}\ \Phi^{i\ell} \tag{8.2}$$

The significance of these symbols becomes even clearer when one considers the partial derivatives of the base vectors. Thus,

$$\frac{\partial\vec{e}_i}{\partial x^\ell} = \vec{e}_m\left(\vec{e}^m \cdot \frac{\partial\vec{e}_i}{\partial x^\ell}\right) = \vec{e}_m\ \Gamma^m_{i\ell} \tag{8.3a}$$

$$\frac{\partial\vec{e}^i}{\partial x^\ell} = \vec{e}^m\left(\vec{e}_m \cdot \frac{\partial e^i}{\partial x^\ell}\right) = -\ \vec{e}^m\ \Gamma^i_{m\ell} \tag{8.3b}$$

Clearly, $\Gamma^i_{m\ell} = \Gamma^i_{\ell m}$ according to Eq. (4.3).

An alternative definition of $\Gamma^i_{m\ell}$ is through the metric tensor components.

$$\Gamma^i_{m\ell} = g^{ik}\ \frac{1}{2}\left(\frac{\partial g_{km}}{\partial x^\ell} + \frac{\partial g_{k\ell}}{\partial x^m} - \frac{\partial g_{m\ell}}{\partial x^k}\right) \tag{8.4}$$

This is proved by showing that

$$\vec{e}_k \cdot \frac{\partial\vec{e}_\ell}{\partial x^m} = \frac{1}{2}\left\{\frac{\partial}{\partial x^\ell}\left(\vec{e}_k \cdot \vec{e}_m\right) + \frac{\partial}{\partial x^m}\left(\vec{e}_k \cdot \vec{e}_\ell\right) - \frac{\partial}{\partial x^k}\left(\vec{e}_m \cdot \vec{e}_\ell\right)\right\}$$

We prefer to use the former definition to evaluate these symbols.

As noted earlier, $\vec{e}_\mu$ and $\vec{e}_\nu$ are functions of ζ only. Hence,

$$\frac{\partial \vec{e}_\mu}{\partial \mu} = \frac{\partial \vec{e}_\mu}{\partial \nu} = \frac{\partial \vec{e}_\nu}{\partial \mu} = \frac{\partial \vec{e}_\nu}{\partial \nu} = 0. \tag{8.5a}$$

The ζ-derivatives are obtained from Frenet's formulas. Thus,

$$\frac{\partial \vec{e}_\mu}{\partial \zeta} = \tau \vec{b}_o - \kappa \vec{t}_o = \frac{\partial \vec{e}_\zeta}{\partial \mu} \quad ; \tag{8.5b}$$

$$\frac{\partial \vec{e}_\nu}{\partial \zeta} = -\tau \vec{n}_o = \frac{\partial \vec{e}_\zeta}{\partial \nu} \quad ; \tag{8.5c}$$

$$\frac{\partial \vec{e}_\zeta}{\partial \zeta} = \vec{t}_o \left(\frac{\partial \sqrt{g}}{\partial \zeta} + \nu \tau \kappa \right) + \vec{n}_o \left(\sqrt{g}\kappa - \nu \tau' - \mu \tau^2 \right) + b_o \left(\mu \tau' - \nu \tau^2 \right) \tag{8.5d}$$

From these equations we obtain

$$\Gamma^i_{\mu\mu} = \Gamma^i_{\mu\nu} = \Gamma^i_{\nu\mu} = \Gamma^i_{\nu\nu} = 0. \quad (i = \zeta, \mu, \nu) \tag{8.6a}$$

$$\left.\begin{aligned} \Gamma^\zeta_{\zeta\mu} &= \Gamma^\zeta_{\mu\zeta} = -\kappa/\sqrt{g} \,, \\ \Gamma^\mu_{\zeta\mu} &= \Gamma^\mu_{\mu\zeta} = -\nu\tau\kappa/\sqrt{g} \,, \\ \Gamma^\nu_{\zeta\mu} &= \Gamma^\nu_{\mu\zeta} = \tau + \mu\tau\kappa/\sqrt{g} \,. \end{aligned}\right\} \tag{8.6b}$$

$$\left.\begin{aligned} \Gamma^\zeta_{\nu\zeta} &= \Gamma^\zeta_{\zeta\nu} = 0 \,, \\ \Gamma^\mu_{\nu\zeta} &= \Gamma^\mu_{\zeta\nu} = -\tau \,, \\ \Gamma^\nu_{\nu\zeta} &= \Gamma^\nu_{\zeta\nu} = 0 \,. \end{aligned}\right\} \tag{8.6c}$$

$$\left.\begin{aligned}
\Gamma^{\zeta}_{\zeta\zeta} &= \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\zeta} + \frac{\kappa}{\sqrt{g}}\,\nu\tau\ , \\
\Gamma^{\mu}_{\zeta\zeta} &= \left(\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\zeta} + \frac{\kappa}{\sqrt{g}}\,\nu\tau\right)\nu\tau + \left(\sqrt{g}\kappa - \nu\tau' - \mu\tau^2\right)\ , \\
\Gamma^{\nu}_{\zeta\zeta} &= -\left(\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\zeta} + \frac{\kappa}{\sqrt{g}}\,\nu\tau\right)\mu\tau + \left(\mu\tau' - \nu\tau^2\right)\ .
\end{aligned}\right\} \qquad (8.6d)$$

As required, the foregoing set of symbols satisfy the following identities:

$$\vec{\nabla}\cdot\vec{e}_{\ell} = \Gamma^{\zeta}_{\zeta\ell} + \Gamma^{\mu}_{\mu\ell} + \Gamma^{\nu}_{\nu\ell} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^{\ell}}\ . \quad (x^{\ell} = \zeta,\ \mu,\ \nu) \qquad (8.7)$$

$$\begin{aligned}
\vec{\nabla}\cdot\overleftrightarrow{I} &= \vec{\nabla}\cdot\left(\vec{e}_i\vec{e}_k\,g^{ik}\right) \\
&= \vec{e}_k\left\{\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\ell}}\left(\sqrt{g}\,g^{k\ell}\right) + \Gamma^{k}_{m\ell}\,g^{m\ell}\right\} = 0.
\end{aligned} \qquad (8.8)$$

They also satisfy

$$\vec{\nabla}\,\overleftrightarrow{I} = \vec{\nabla}\left(\vec{e}_i\vec{e}_k\,g^{ik}\right) = \vec{e}^{\,\ell}\vec{e}_i\vec{e}_k\,g^{ik}_{;\ell} = 0\ ,$$

i.e.,

$$g^{ik}_{;\ell} = \frac{\partial g^{ik}}{\partial x^{\ell}} + \Gamma^{i}_{m\ell}\,g^{km} + \Gamma^{k}_{m\ell}\,g^{im} = 0\ , \qquad (8.9a)$$

and

$$\vec{\nabla}\,\overleftrightarrow{I} = \vec{\nabla}\left(\vec{e}^{\,i}\vec{e}^{\,k}\,g_{ik}\right) = \vec{e}^{\,\ell}\vec{e}^{\,i}\vec{e}^{\,k}\,g_{ik;\ell} = 0\ ,$$

i.e.,

$$g_{ik;\ell} = \frac{\partial g_{ik}}{\partial x^{\ell}} - \Gamma^{m}_{k\ell}\,g_{im} - \Gamma^{m}_{i\ell}\,g_{km} = 0. \qquad (8.9b)$$

The foregoing set of symbols may be grouped into three 3×3 matrices, namely,

$$\left(\Gamma^{\zeta}_{\ell m}\right) = \begin{pmatrix} \Gamma^{\zeta}_{\zeta\zeta} & \Gamma^{\zeta}_{\zeta\mu} & 0 \\ \Gamma^{\zeta}_{\mu\zeta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ , \qquad \left(\Gamma^{\mu}_{\ell m}\right) = \begin{pmatrix} \Gamma^{\mu}_{\zeta\zeta} & \Gamma^{\mu}_{\zeta\mu} & \Gamma^{\mu}_{\zeta\nu} \\ \Gamma^{\mu}_{\mu\zeta} & 0 & 0 \\ \Gamma^{\mu}_{\nu\zeta} & 0 & 0 \end{pmatrix}\ ,$$

and
$$\left(\Gamma^{\nu}_{\ell m}\right) = \begin{pmatrix} \Gamma^{\nu}_{\zeta\zeta} & \Gamma^{\nu}_{\zeta\mu} & 0 \\ \Gamma^{\nu}_{\mu\zeta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These matrices correspond to the following gradient expressions:

$$\vec{\nabla}\vec{e}^{\,i} = \vec{\nabla}\vec{\nabla}\, x^i = -\vec{e}^{\,\ell}\vec{e}^{\,m}\,\Gamma^i_{\ell m}\ . \ (i = \zeta,\ \mu,\ \nu). \tag{8.10}$$

We may tentatively call these expressions "base dyadics."

By using the transformation laws of the base vectors, it may easily be shown that, under the coordinate transformation from (ζ, μ, ν) to (ζ', μ', ν'),

$$(\Gamma^i_{mn})' = \frac{\partial x'^i}{\partial x^k}\left[\frac{\partial x^p}{\partial x'^m}\,\frac{\partial x^q}{\partial x'^n}\,\Gamma^k_{pq} + \frac{\partial^2 x^k}{\partial x'^m\,\partial x'^n}\right]\ , \tag{8.11a}$$

i.e.,
$$\frac{\partial x^{\ell}}{\partial x'^i}\,(\Gamma^i_{mn})' = \frac{\partial x^p}{\partial x'^m}\,\frac{\partial x^q}{\partial x'^n}\,\Gamma^{\ell}_{pq} + \frac{\partial^2 x^{\ell}}{\partial x'^m\,\partial x'^n}\ . \tag{8.11b}$$

Christoffel symbols do not transform like tensor components, except when all the second derivatives $\partial^2 x^{\ell}/\partial x'^m\,\partial x'^n$ vanish. In other words, the "base dyadics" do not transform like dyadics. This should not disturb us, as we may recall that base vectors do not transform like vectors (they transform like vector components).

If desired, one may, however, avoid using these seemingly complicated symbols by considering , instead, the gradient expressions $\vec{\nabla}\vec{t}_o$, $\vec{\nabla}\vec{n}_o$, and $\vec{\nabla}\vec{b}_o$, tentatively called the "reference dyadics." All of these have quite simple expressions:

$$\vec{\nabla}\vec{t}_o = \frac{\kappa}{\sqrt{g}}\,\vec{t}_o\,\vec{n}_o \tag{8.12a}$$

$$\vec{\nabla}\vec{n}_o = \frac{1}{\sqrt{g}}\,(\tau\,\vec{t}_o\vec{b}_o - \kappa\,\vec{t}_o\vec{t}_o)\ . \tag{8.12b}$$

$$\vec{\nabla}\vec{b}_o = -\frac{\tau}{\sqrt{g}}\,\vec{t}_o\,\vec{n}_o\ . \tag{8.12c}$$

It is advantageous to use $(\vec{t}_o, \vec{n}_o, \vec{b}_o)$ as reference vectors, because this usually results in simpler expressions. On the other hand, the covariant and contravariant base vectors are preferred for general theoretical discussions.

IX. DIFFERENTIAL EXPRESSIONS OF VECTOR FIELDS

In previous sections we have already discussed the differential operators $\vec{\nabla}$ and $d/ds = \vec{a}^s \cdot \vec{\nabla} = (\vec{\nabla}s/|\nabla s|) \cdot \vec{\nabla}$, together with some differential expressions of the base and reference vectors, such as $\vec{\nabla} \cdot \vec{e}_i$ $\vec{\nabla}\vec{e}^i$, $\vec{\nabla}\vec{t}_o$, etc. Now, we shall present, for ready reference, the oft-used differential expressions of vector fields. We shall also discuss several characteristic quantities of infinitesimal vector tubes, used in Bjorgum's work, for the purpose of illustration.

The vector field will be denoted by

$$\vec{V} = \vec{e}_i V^i = \vec{e}^i V_i \tag{9.1a}$$

$$= \vec{t}_o \sqrt{g}\, V^\zeta + \vec{n}_o V_\mu + \vec{b}_o V_\nu \ . \tag{9.1b}$$

To obtain the differential expressions of $\vec{V}$ requires the corresponding expressions of the base vectors, $\vec{e}_i$ or $\vec{e}^i$, or of the referece vectors $(\vec{t}_o, \vec{n}_o, \vec{b}_o)$.

First we observe that, since $\vec{e}^i = \vec{\nabla}x^i$,

$$\vec{\nabla} \times \vec{e}^i = 0 \ . \tag{9.2}$$

On the other hand,

$$\vec{\nabla} \times \vec{e}_i = \vec{e}^\ell \times \frac{\partial \vec{e}_i}{\partial x^\ell} = \left(\vec{e}^\ell \times \vec{e}^m\right) \ \vec{e}_m \cdot \frac{\partial \vec{e}_i}{\partial x^\ell}$$

$$= \left(\vec{e}^\ell \times \vec{e}^m\right)\left\{ \frac{\partial}{\partial x^\ell}\left(\vec{e}_i \cdot \vec{e}_m\right) - \vec{e}_i \cdot \frac{\partial \vec{e}_m}{\partial x^\ell}\right\} .$$

Since the last term, $(\vec{e}^{\ell} \times \vec{e}^{m})(\vec{e}_i \cdot \partial\vec{e}_m/\partial x^{\ell})$, reverses its sign by interchanging ℓ and m and thus drops out, we obtain

$$\vec{\nabla} \times \vec{e}_i = (\vec{e}^{\ell} \times \vec{e}^{m}) \frac{\partial g_{im}}{\partial x^{\ell}} = \vec{e}_n \, \epsilon^{\ell mn} \frac{1}{\sqrt{g}} \frac{\partial g_{im}}{\partial x^{\ell}} \quad , \tag{9.3}$$

where

$$\epsilon^{\ell mn} = \begin{cases} 1, & (\ell, m, n) \text{ being even permutations of } (\zeta, \mu, \nu) \\ -1, & (\ell, m, n) \text{ being odd permutations of } (\zeta, \mu, \nu) \\ 0, & \text{otherwise.} \end{cases}$$

From Eq. (9.3) we obtain

$$\vec{\nabla} \times \vec{e}_{\zeta} = \frac{1}{\sqrt{g}} \Big\{ \vec{e}_{\zeta}(2\tau) + \vec{e}_{\mu}(2\nu\tau^2 - \mu\tau') + \vec{e}_{\nu}(2\kappa\sqrt{g} - 2\mu\tau^2 - \nu\tau') \Big\} \quad , \tag{9.4a}$$

$$\vec{\nabla} \times \vec{e}_{\mu} = - (\tau/\sqrt{g})\, \vec{e}_{\mu} \quad , \tag{9.4b}$$

$$\vec{\nabla} \times \vec{e}_{\nu} = - (\tau/\sqrt{g})\, \vec{e}_{\nu} \quad . \tag{9.4c}$$

Hence,

$$\left.\begin{aligned} \vec{e}_{\zeta} \cdot \vec{\nabla} \times \vec{e}_{\zeta} &= 2\tau \quad , \\ \vec{e}_{\mu} \cdot \vec{\nabla} \times \vec{e}_{\mu} = \vec{e}_{\nu} \cdot \vec{\nabla} \times \vec{e}_{\nu} &= - \tau/\sqrt{g} \ . \end{aligned}\right\} \tag{9.5}$$

Equations (9.2) and (9.5) tell us that, while the contravariant base vectors $\vec{e}^{i}$ are normal to a family of surfaces x^i = const., the covariant base vectors $\vec{e}_i$ are not. There exists no family of surfaces on which the so-called "covariant coordinate" x_i is constant. This is the same as to say that, contrary to the contravariant set of coordinates (ζ, μ, ν), the covariant coordinates do not actually exist. The covariant coordinates do not always have physical significance. For example,

$$\vec{\nabla} = \vec{e}^{i} \frac{\partial}{\partial x^i} = \vec{e}_k \, g^{ik} \frac{\partial}{\partial x^i} \equiv \vec{e}_k \frac{\partial}{\partial x_k} \quad .$$

The operator $\partial/\partial x_k$ does not mean differentiation with a variable x_k in real existence; it is merely a symbolic notation for $g^{ik}\,\partial/\partial x^i$.

The divergence of any vector is given by

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{\sqrt{g}}\,\frac{\partial(\sqrt{g}\,V^i)}{\partial x^i} \tag{9.6}$$

This is the same as Eq. (8.7) when $\vec{V} = \vec{e}_\ell$. Such specialization gives the following results:

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{e}_\zeta &= -\,\mu\kappa'/\sqrt{g} \\ \vec{\nabla}\cdot\vec{e}_\mu &= -\,\kappa/\sqrt{g} \\ \vec{\nabla}\cdot\vec{e}_\nu &= 0 \end{aligned}\right\} \,. \tag{9.7}$$

On the other hand,

$$\begin{aligned} \vec{\nabla}\cdot\vec{e}^{\,\zeta} &= \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x^i}\,(\sqrt{g}\,g^{\zeta i}) = -\,g^{\ell m}\,\Gamma^{\zeta}_{\ell m} \\ &= g^{-3/2}\,(\mu\kappa' + \nu\tau\kappa) \end{aligned} \tag{9.8a}$$

Similarly,

$$\vec{\nabla}\cdot\vec{e}^{\,\mu} = \frac{1}{\sqrt{g}}\left\{-\,\kappa + \frac{1}{\sqrt{g}}\,(\nu\tau' - \mu\tau^2) + \frac{\nu\tau}{g}\,(\mu\kappa' + \nu\tau\kappa)\right\}; \tag{9.8b}$$

$$\vec{\nabla}\cdot\vec{e}^{\,\nu} = -\,\frac{1}{\sqrt{g}}\left\{\frac{1}{\sqrt{g}}\,(\mu\tau' + \nu\tau^2) + \frac{\mu\tau}{g}\,(\mu\kappa' + \nu\tau\kappa)\right\}. \tag{9.8c}$$

The curl of any vector is given by

$$\vec{\nabla}\times\vec{V} = \vec{e}_n\,\epsilon^{\ell m n}\,\frac{1}{\sqrt{g}}\,\frac{\partial V_m}{\partial x^\ell}\quad. \tag{9.9}$$

This equation contains Eq. (9.2) and Eq. (9.3) as special cases.

Similarly,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{V} = \vec{e}_n \, \epsilon^{\ell mn} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\ell} (\nabla \times V)_m$$

$$= \vec{e}_n \, \epsilon^{\ell mn} \, \epsilon^{ijk} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\ell} \left(\frac{g_{mk}}{\sqrt{g}} \frac{\partial V_j}{\partial x^i} \right) . \tag{9.10}$$

An alternative expression of curlcurl is the divergence of an antisymmetric tensor, namely,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{V} = - \vec{\nabla} \cdot (\vec{\nabla}\vec{V} - \vec{V}\vec{\nabla})$$

$$= \vec{e}_n \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^m} (\sqrt{g} \, g^{in} \, V^m_{\ ;i} - \sqrt{g} \, g^{im} \, V^n_{\ ;i}) . \tag{9.11}$$

The equality of these two expressions, (9.10) and (9.11), can be proved without much difficuly by straight-forward calculation in general curvilinear coordinates. The proof is trivial in Cartesian coordinates.

The Laplacian operator is variously denoted by Δ, or $\vec{\nabla} \cdot \vec{\nabla}$, or ∇^2.

$$\Delta \equiv \vec{\nabla} \cdot \vec{\nabla} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} \, g^{ik} \frac{\partial}{\partial x^k} \right) \equiv \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} \frac{\partial}{\partial x_i} \right) . \tag{9.12}$$

$$\Delta\vec{V} \equiv \vec{\nabla} \cdot (\vec{\nabla}\vec{V}) = (\vec{\nabla} \cdot \vec{\nabla})\vec{V} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} \, g^{ik} \frac{\partial \vec{V}}{\partial x^k} \right)$$

$$= \vec{e}_\ell \left\{ \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} \, g^{ik} \, V^\ell_{\ ;k} \right) + \Gamma^\ell_{mi} \left(g^{ik} \, V^m_{\ ;k} \right) \right\} . \tag{9.13}$$

Alternatively, we have according to Eq. (9.11)

$$\Delta\vec{V} = \vec{\nabla} \cdot (\vec{V}\vec{\nabla}) - \vec{\nabla} \times \vec{\nabla} \times \vec{V} . \tag{9.14}$$

This equation is usually written as

$$\Delta\vec{V} = \vec{\nabla} (\vec{\nabla} \cdot \vec{V}) - \vec{\nabla} \times \vec{\nabla} \times \vec{V} , \tag{9.15}$$

because $\vec{\nabla}\,(\vec{\nabla}\cdot\vec{V}) = \vec{\nabla}\cdot(\vec{\nabla}\vec{V})$ in Cartesian cocrainates and, therefore, in every system of coordinates which can be transformed into Cartesian. Here, we may note that, in general,

$$\Delta\vec{V} = \vec{\nabla}\,(\vec{\nabla}\cdot\vec{V}) - \vec{\nabla}\times\vec{\nabla}\times\vec{V} + \vec{e}^{\ell}\, V^{k}\, R_{k\ell} \tag{9.16}$$

In this equation,

$$R_{k\ell} = R^{n}_{kn\ell} \tag{9.17a}$$

are the second-rank tensor components obtained by contraction from the Riemann-Christoffel curvature tensor which has components

$$\begin{aligned} R^{m}_{kn\ell} &= \frac{\partial}{\partial x^{n}}\Gamma^{m}_{k\ell} - \frac{\partial}{\partial x^{\ell}}\Gamma^{m}_{kn} + \Gamma^{m}_{in}\Gamma^{i}_{k\ell} - \Gamma^{m}_{i\ell}\Gamma^{i}_{km} \\ &= \vec{e}^{m}\cdot\left(\frac{\partial}{\partial x^{n}}\frac{\partial\vec{e}_{k}}{\partial x^{\ell}} - \frac{\partial}{\partial x^{\ell}}\frac{\partial\vec{e}_{k}}{\partial x^{n}}\right) \end{aligned} \tag{9.17b}$$

In Euclidean space every component of this curvature tensor vanishes, so $R_{k\ell} = 0$ and the last term in Eq. (9.16) drops out.

All the expressions, discussed so far in this section, refer to either the covariant or the contravariant base vectors. These expressions are discussed in detail in most treatises of tensor analysis and of relativity.[12] Now we shall present the corresponding forms by referring to the orthogonal set of unit vectors $(\vec{t}_{o}, \vec{n}_{o}, \vec{b}_{o})$.

The divergence and curl of the reference unit vectors may be obtained readily from Eqs. (8.12a, b, and c). Thus,

$$\vec{\nabla}\cdot\vec{t}_{o} = 0, \qquad \vec{\nabla}\cdot\vec{n}_{o} = -\kappa/\sqrt{g}\,, \qquad \vec{\nabla}\cdot\vec{b}_{o} = 0\,. \tag{9.18}$$

$$\vec{\nabla}\times\vec{t}_{o} = \frac{\kappa}{\sqrt{g}}\,\vec{b}_{o}\,, \qquad \vec{\nabla}\times\vec{n}_{o} = -\frac{\tau}{\sqrt{g}}\,\vec{n}_{o}\,, \qquad \vec{\nabla}\times\vec{b}_{o} = -\frac{\tau}{\sqrt{g}}\,\vec{b}_{o}. \tag{9.19}$$

Hence,

$$\vec{t}_{o}\cdot\vec{\nabla}\times\vec{t}_{o} = 0\,. \tag{9.20a}$$

while

$$\vec{n}_o \cdot \vec{\nabla} \times \vec{n}_o = \vec{b}_o \cdot \vec{\nabla} \times \vec{b}_o = -\tau/\sqrt{g} \ . \qquad (9.20b)$$

Let us denote

$$\vec{t}_o \cdot \vec{\nabla} = \sqrt{g}\left(g^{\zeta\zeta}\frac{\partial}{\partial\zeta} + g^{\zeta\mu}\frac{\partial}{\partial\mu} + g^{\zeta\nu}\frac{\partial}{\partial\nu}\right) \equiv \sqrt{g}\,\frac{\partial}{\partial x_\zeta} \ , \qquad (9.21)$$

where $\partial/\partial x_\zeta$ is the contravariant ζ-component of $\vec{\nabla}$. Then, according to Eq. (7.1b),

$$\vec{\nabla} = \vec{t}_o\,\sqrt{g}\,\frac{\partial}{\partial x_\zeta} + \vec{n}_o\,\frac{\partial}{\partial\mu} + \vec{b}_o\,\frac{\partial}{\partial\nu} \ . \qquad (9.22)$$

Using Eqs. (8.12), (9.1b), and (9.22) we obtain

$$\begin{aligned}\overleftrightarrow{\nabla V} = {} & \vec{t}_o\vec{t}_o\left\{\sqrt{g}\,\frac{\partial}{\partial x_\zeta}(\sqrt{g}V^\zeta) - \frac{\kappa}{\sqrt{g}}\,V_\mu\right\} + \vec{t}_o\vec{n}_o\left\{\sqrt{g}\,\frac{\partial V_\mu}{\partial x_\zeta} + \kappa V^\zeta - \frac{\tau}{\sqrt{g}}\,V_\nu\right\} + \vec{t}_o\vec{b}_o\left\{\sqrt{g}\,\frac{\partial V_\nu}{\partial x_\zeta} + \frac{\tau}{\sqrt{g}}\,V_\mu\right\} \\ & + \vec{n}_o\vec{t}_o\,\frac{\partial}{\partial\mu}\,(\sqrt{g}\,V^\zeta) \qquad + \vec{n}_o\vec{n}_o\,\frac{\partial V_\mu}{\partial\mu} \qquad + \vec{n}_o\vec{b}_o\,\frac{\partial V_\nu}{\partial\mu} \\ & + \vec{b}_o\vec{t}_o\,\frac{\partial}{\partial\nu}\,(\sqrt{g}\,V^\zeta) \qquad + \vec{b}_o\vec{n}_o\,\frac{\partial V_\mu}{\partial\nu} \qquad + \vec{b}_o\vec{b}_o\,\frac{\partial V_\nu}{\partial\nu} \ . \qquad (9.23)\end{aligned}$$

Thus,

$$\begin{aligned}\vec{\nabla} \cdot \vec{V} &= \overleftrightarrow{I} : \overleftrightarrow{\nabla V} \\ &= \left\{\sqrt{g}\,\frac{\partial}{\partial x_\zeta}\,(\sqrt{g}\,V^\zeta) - \frac{\kappa}{\sqrt{g}}\,V_\mu\right\} + \frac{\partial V_\mu}{\partial\mu} + \frac{\partial V_\nu}{\partial\nu} \ . \qquad (9.24)\end{aligned}$$

$$\begin{aligned}\vec{\nabla} \times \vec{V} &= \overleftrightarrow{I} \dot{\times} \overleftrightarrow{\nabla V} = \tfrac{1}{2}\,\overleftrightarrow{I} \dot{\times} (\overleftrightarrow{\nabla V} - \overleftrightarrow{V\nabla}) \\ &= \vec{t}_o\sqrt{g}\,(\nabla \times V)^\zeta + \vec{n}_o\,(\nabla \times V)_\mu + \vec{b}_o\,(\nabla \times V)_\nu \ , \qquad (9.25)\end{aligned}$$

where

$$\sqrt{g}\ (\nabla \times V)^{\zeta} = \frac{\partial V_{\nu}}{\partial \mu} - \frac{\partial V_{\mu}}{\partial \nu} \quad , \tag{9.26a}$$

$$(\nabla \times V)_{\mu} = \frac{\partial}{\partial \nu} (\sqrt{g}\ V^{\zeta}) - \sqrt{g}\ \frac{\partial V_{\nu}}{\partial x_{\zeta}} - \frac{\tau}{\sqrt{g}} V_{\mu} \quad , \tag{9.26b}$$

$$(\nabla \times V)_{\nu} = \sqrt{g}\ \frac{\partial V_{\mu}}{\partial x_{\zeta}} + \kappa V^{\zeta} - \frac{\tau}{\sqrt{g}} V_{\nu} - \frac{\partial}{\partial \mu} (\sqrt{g}\ V^{\zeta}) \quad . \tag{9.26c}$$

Similarly,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{V} = - \vec{\nabla} \cdot (\vec{\nabla}\vec{V} - \vec{V}\vec{\nabla})$$

$$= \vec{t}_o \sqrt{g}(\nabla \times \nabla \times V)^{\zeta} + \vec{n}_o(\nabla \times \nabla \times V)_{\mu} + \vec{b}_o(\nabla \times \nabla \times V)_{\nu}, \tag{9.27}$$

where

$$\sqrt{g}(\nabla \times \nabla \times V)^{\zeta} = \frac{\partial}{\partial \mu} (\nabla \times V)_{\nu} - \frac{\partial}{\partial \nu} (\nabla \times V)_{\mu} \quad , \tag{9.28a}$$

$$(\nabla \times \nabla \times V)_{\mu} = \frac{\partial}{\partial \nu}\left\{\sqrt{g}(\nabla \times V)^{\zeta}\right\} - \sqrt{g}\ \frac{\partial}{\partial x_{\zeta}} (\nabla \times V)_{\nu} - \frac{\tau}{\sqrt{g}} (\nabla \times V)_{\mu}, \tag{9.28b}$$

$$(\nabla \times \nabla \times V)_{\nu} = \sqrt{g}\ \frac{\partial}{\partial x_{\zeta}} (\nabla \times V)_{\mu} + \kappa(\nabla \times V)^{\zeta} - \frac{\tau}{\sqrt{g}} (\nabla \times V)_{\nu} - \frac{\partial}{\partial \mu}\left\{\sqrt{g}(\nabla \times V)^{\zeta}\right\} . \tag{9.28c}$$

The Laplacian operator in the (ζ, μ, ν) system may be written as

$$\Delta = \vec{\nabla} \cdot \vec{\nabla} = \sqrt{g}\ \frac{\partial}{\partial x_{\zeta}}\left(\sqrt{g}\ \frac{\partial}{\partial x_{\zeta}}\right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial \mu}\left(\sqrt{g}\ \frac{\partial}{\partial \mu}\right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial \nu}\left(\sqrt{g}\ \frac{\partial}{\partial \nu}\right)$$

$$= \sqrt{g}\ \frac{\partial}{\partial x_{\zeta}}\left(\sqrt{g}\ \frac{\partial}{\partial x_{\zeta}}\right) + \frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \nu^2} - \frac{\kappa}{\sqrt{g}} \frac{\partial}{\partial \mu} . \tag{9.29}$$

$$\begin{aligned}
\Delta\vec{V} &= \vec{\nabla}\cdot(\vec{\nabla}\vec{V}) - \vec{\nabla}\times\vec{\nabla}\times\vec{V} \\
&= \vec{t}_o\,\Delta\,(\sqrt{g}\;V^{\zeta}) + \vec{n}_o\,\Delta\,V_\mu + \vec{b}_o\,\Delta\,V_\nu \\
&\quad + 2\left\{\vec{\nabla}\,(\sqrt{g}\;V^{\zeta})\cdot\vec{\nabla}\vec{t}_o + \vec{\nabla}V_\mu\cdot\vec{\nabla}\vec{n}_o + \vec{\nabla}V_\nu\cdot\vec{\nabla}\vec{b}_o\right\} \\
&\quad + \sqrt{g}\;V^{\zeta}\,\Delta\vec{t}_o + V_\mu\,\Delta\,\vec{n}_o + V_\nu\,\Delta\,\vec{b}_o \quad .
\end{aligned} \tag{9.30}$$

where the Laplacian expressions of the unit vectors are obtained from Eqs. (8.12) and (9.18).

$$\Delta\vec{t}_o = -\,\vec{t}_o\left(\frac{\kappa}{\sqrt{g}}\right)^2 + \vec{n}_o\left(\sqrt{g}\,\frac{\partial}{\partial x_\zeta}\,\frac{\kappa}{\sqrt{g}}\right) + \vec{b}_o\,\frac{\kappa}{\sqrt{g}}\cdot\frac{\tau}{\sqrt{g}} \quad . \tag{9.31a}$$

$$\Delta\vec{n}_o = -\,\vec{t}_o\left(\sqrt{g}\,\frac{\partial}{\partial x_\zeta}\,\frac{\kappa}{\sqrt{g}}\right) - \vec{n}_o\left\{\left(\frac{\kappa}{\sqrt{g}}\right)^2 + \left(\frac{\tau}{\sqrt{g}}\right)^2\right\} + \vec{b}_o\left(\sqrt{g}\,\frac{\partial}{\partial x_\zeta}\,\frac{\tau}{\sqrt{g}}\right) . \tag{9.31b}$$

$$\Delta\vec{b}_o = \vec{t}_o\,\frac{\kappa}{\sqrt{g}}\cdot\frac{\tau}{\sqrt{g}} - \vec{n}_o\left(\sqrt{g}\,\frac{\partial}{\partial x_\zeta}\,\frac{\tau}{\sqrt{g}}\right) - \vec{b}_o\left(\frac{\tau}{\sqrt{g}}\right)^2 \quad . \tag{9.31c}$$

Equation (9.30) and Eqs. (9.23) to (9.28) inclusive are the general expressions for any vector field $\vec{V}$. It is often very instructive to specialize by considering the field of the orthogonal unit vectors $(\vec{t}, \vec{n}, \vec{b})$. These unit vectors become the reference vectors $(\vec{t}_o, \vec{n}_o, \vec{b}_o)$ by putting $\mu = \nu = 0$. To make such specializations we simply replace, respectively, the components $(\sqrt{g}\;V^{\zeta}, V_\mu, V_\nu)$ by $(\sqrt{g}\;t^{\zeta}, t_\mu, t_\nu)$, or by $(\sqrt{g}\;n^{\zeta}, n_\mu, n_\nu)$, or by $(\sqrt{g}\;b^{\zeta}, b_\mu, b_\nu)$.

For example, we obtain from Eq. (9.24)

$$\vec{\nabla}\cdot\vec{t} = \sqrt{g}\,\frac{\partial}{\partial x_\zeta}\,(\sqrt{g}\;t^{\zeta}) - \frac{\kappa}{\sqrt{g}}\,t_\mu + \frac{\partial t_\mu}{\partial\mu} + \frac{\partial t_\nu}{\partial\nu} \quad . \tag{9.32}$$

If the particular vector line $\vec{V} = v\vec{t}$ is taken to be the reference axis, then

$$\left.\begin{aligned} & t_\mu = t_\nu = 0, \\ & \sqrt{g}\ t^\zeta = 1, \\ & g = 1, \\ \text{and} \qquad & \sqrt{g}\ \frac{\partial}{\partial x_\zeta} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \zeta}\ , \end{aligned}\right\} \qquad (\mu = \nu = 0) \tag{9.33}$$

if not acted upon by differential operators. Thus, Eq. (9.32) becomes

$$\vec{\nabla} \cdot \vec{t} = \frac{\partial t_\mu}{\partial \mu} + \frac{\partial t_\nu}{\partial \nu} \equiv \Theta \qquad (\mu = \nu = 0), \tag{9.34}$$

and

$$\begin{aligned} \vec{\nabla} \cdot v\vec{t} &= v\vec{\nabla} \cdot \vec{t} + \vec{t} \cdot \vec{\nabla} v \\ &= v\ \Theta + \partial v/\partial \zeta\ , \qquad (\mu = \nu = 0). \end{aligned} \tag{9.35}$$

In passing, we may note that

$$(\sqrt{g}\ t^\zeta)^2 + t_\mu^2 + t_\nu^2 = 1\ , \tag{9.36}$$

so

$$\sqrt{g}\ t^\zeta \frac{\partial}{\partial x^i}(\sqrt{g}\ t^\zeta) + t_\mu \frac{\partial t_\mu}{\partial x^i} + t_\nu \frac{\partial t_\nu}{\partial x^i} = 0\ . \qquad (x^i = \zeta,\ \mu,\ \nu) \tag{9.37}$$

Hence,

$$\frac{\partial}{\partial \zeta}(\sqrt{g}\ t^\zeta) = \frac{\partial}{\partial \mu}(\sqrt{g}\ t^\zeta) = \frac{\partial}{\partial \nu}(\sqrt{g}\ t^\zeta) = 0\ . \qquad (\mu = \nu = 0) \tag{9.38}$$

Using these relations we obtain from Eqs. (9.25) and (9.26)

$$\vec{\nabla} \times \vec{t} = \vec{t}_o \, \Omega + \vec{b}_o \, \kappa \,, \qquad (\mu = \nu = 0) \,, \tag{9.39}$$

where

$$\Omega = \frac{\partial t_\nu}{\partial \mu} - \frac{\partial t_\mu}{\partial \nu} \;. \tag{9.40}$$

Thus,

$$\begin{aligned} \vec{\nabla} \times v\vec{t} &= v\vec{\nabla} \times \vec{t} + (\vec{\nabla} v) \times \vec{t} \\ &= \vec{t}_o \, v\Omega + \vec{n}_o \frac{\partial v}{\partial \nu} + \vec{b}_o \left(v\kappa - \frac{\partial v}{\partial \mu} \right) , \\ &\quad (\mu = \nu = 0) \end{aligned} \tag{9.41}$$

Here, we may further note that, while $\vec{t} = \vec{t}_o$,

$$\vec{\nabla} \cdot \vec{t} = \vec{\nabla} \cdot \vec{t}_o + \Theta \,, \qquad (\mu = \nu = 0) \,, \tag{9.42a}$$

and

$$\vec{\nabla} \times \vec{t} = \vec{\nabla} \times \vec{t}_o + \vec{t}_o \, \Omega \,, \qquad (\mu = \nu = 0) \,. \tag{9.42b}$$

The scalar quantity Θ is called by Bjorgum the <u>divergence of a vector tube</u> or the <u>divergence of neighbouring vector lines</u>, and Ω the <u>torsion of neighbouring vector lines</u>. Both Θ and Ω are independent of the magnitude v of the vector field $\vec{V} = v\vec{t}$. These quantities determine the geometric properties of infinitesimal vector tubes; $\Omega = \vec{t} \cdot \vec{\nabla} \times \vec{t}$ plays a unique role in the study of Beltrami fields.

The best way to illustrate the usefulness of Θ and Ω is, perhaps, to consider the variation of the normal cross sectional area of an infinitesimal vector tube. Consider a small segemnt of a vector line passing through the point $(\zeta, \delta\mu, 0)$. This segment is defined by

$$d\vec{s} = \vec{t} \, ds = (\vec{t}_o \sqrt{g} \, t^\zeta + \vec{n}_o \, t_\mu + \vec{b}_o \, t_\nu) \, ds \,, \tag{9.43}$$

where

$$t_\mu = \frac{\partial t_\mu}{\partial \mu} \delta\mu + O(\delta\mu^2) \quad , \tag{9.44a}$$

$$t_\nu = \frac{\partial t_\nu}{\partial \mu} \delta\mu + O(\delta\mu^2) \quad , \tag{9.44b}$$

and $\sqrt{g} = 1 - \kappa\delta\mu$. Let this segment begin from the point $(\zeta, \delta\mu, 0)$ and end at the point $(\zeta + d\zeta, \delta\mu + d(\delta\mu), d\nu)$. Then, according to Eq. (6.22b), we also have

$$d\vec{s} = \vec{t}_o \sqrt{g}\, d\zeta + \vec{n}_o\, d(\delta\mu) + \vec{b}_o\, (d\nu + \tau\delta\mu d\zeta) \ . \tag{9.45}$$

From Eqs. (9.43), (9.44), and (9.45) follow:

$$t_\mu\, ds = d(\delta\mu) \qquad \text{and} \qquad ds = \frac{d\zeta}{t^\zeta} = d\zeta \left\{ 1 + O(\delta\mu) \right\} \ .$$

Hence,

$$\frac{\partial t_\mu}{\partial \mu} \delta\mu = \frac{d(\delta\mu)}{ds} = \frac{d(\delta\mu)}{d\zeta} \cdot \left\{ 1 + O(\delta\mu) \right\} \quad ,$$

and, when $\delta\mu \to 0$,

$$\frac{\partial t_\mu}{\partial \mu} = \frac{d}{d\zeta} \log \delta\mu \equiv \Theta_\mu \quad . \tag{9.46a}$$

Similarly, if we consider the small segment of a vector line connecting the two points $(\zeta, 0, \delta\nu)$ and $(\zeta + d\zeta, d\mu, \delta\nu + d(\delta\nu))$, we obtain

$$\frac{\partial t_\nu}{\partial \nu} = \frac{d}{d\zeta} \log \delta\nu \equiv \Theta_\nu \quad . \tag{9.46b}$$

Since $\delta\mu\delta\nu = \delta\sigma$ is the area of the small rectangular section normal to $\vec{t}_o$, Eqs. (9.46a and 9.46b) yield immediately

$$\Theta = \Theta_\mu + \Theta_\nu = \frac{d}{d\zeta} \log \delta\sigma = \frac{1}{\delta\sigma} \frac{d}{d\zeta} \delta\sigma \quad . \tag{9.47}$$

This equation states that Θ is the rate of change of the normal sectional area per unit area along the tangential direction of an infinitesimal vector tube. However, Θ does not tell us in what manner the normal cross-sectional area varies. As discussed by Bjorgum, the latter question leads to further illuminating results. It seems advisable to describe his discussion here for the purpose of illustrating the usefulness of natural coordinates.

Following Bjorgum, we consider a small segment of a small vector tube, surrounding the reference axis between the two points $(\zeta, 0, 0)$ and $(\zeta + d\zeta, 0, 0)$. This segment of the vector tube has a circular cross section of radius $d\rho$ at $(\zeta, 0, 0)$.

$$\vec{d\rho} = \vec{n}_o \, d\mu + \vec{b}_o \, d\nu \quad . \tag{9.48}$$

The problem at hand is to determine how the cross section varies as ζ changes.

Let the radius vector $\vec{d\rho}$ be changed to $\vec{d\rho}'$ when ζ is changed to $\zeta + d\zeta$. The reference unit vectors will also change with ζ, from $(\vec{t}_o, \vec{n}_o, \vec{b}_o)$ to $(\vec{t}_o', \vec{n}_o', \vec{b}_o')$.

$$\vec{d\rho}' = \vec{n}_o' \, d\mu' + \vec{b}_o' \, d\nu' \quad . \tag{9.49}$$

$$\vec{t}_o' \cong \vec{t}_o + \frac{\partial \vec{t}_o}{\partial \zeta} d\zeta = \vec{t}_o + \vec{n}_o \, \kappa d\zeta \quad . \tag{9.50a}$$

$$\vec{n}_o' \cong \vec{n}_o + \frac{\partial \vec{n}_o}{\partial \zeta} d\zeta = \vec{n}_o + (\vec{b}_o \tau - \vec{t}_o \kappa) \, d\zeta \quad . \tag{9.50b}$$

$$\vec{b}_o' \cong \vec{b}_o + \frac{\partial \vec{b}_o}{\partial \zeta} d\zeta = \vec{b}_o - \vec{n}_o \, \tau d\zeta \quad . \tag{9.50c}$$

A representative vector line segment on the boundary surface of this infinitesimal vector tube has the following expression:

$$\begin{aligned} \vec{ds} = \vec{t}_o \sqrt{g} \, d\zeta &+ \vec{n}_o \, (d\mu' - d\mu - \tau d\nu d\zeta) \\ &+ \vec{b}_o \, (d\nu' - d\nu + \tau d\mu d\zeta) \quad . \end{aligned} \tag{9.51}$$

Comparing this equation with Eq. (9.43) we obtain

$$(t_\mu/t^\zeta)\ d\zeta = d\mu' - d\mu - \tau d\nu d\zeta \cong t_\mu d\zeta \quad , \tag{9.52a}$$

$$(t_\nu/t^\zeta)\ d\zeta = d\nu' - d\nu + \tau d\mu d\zeta \cong t_\nu\ d\zeta \quad . \tag{9.52b}$$

Here,

$$t_\mu\ (\zeta,\ d\mu,\ d\nu) \cong \frac{\partial t_\mu}{\partial \mu}\, d\mu + \frac{\partial t_\mu}{\partial \nu}\, d\nu \quad , \tag{9.53a}$$

$$t_\nu\ (\zeta,\ d\mu,\ d\nu) \cong \frac{\partial t_\nu}{\partial \mu}\, d\mu + \frac{\partial t_\nu}{\partial \nu}\, d\nu \quad . \tag{9.53b}$$

We then calculate $(\vec{d\rho}' - \vec{d\rho})$, using Eqs. (9.50), (9.52), and (9.53).

$$\begin{aligned}
\vec{d\rho}' - \vec{d\rho} &= (\vec{n}_o'\ d\mu' + \vec{b}_o'\ d\nu') - (\vec{n}_o\ d\mu + \vec{b}_o\ d\nu) \\
&= (-\ \vec{t}_o\ \kappa d\mu + \vec{n}_o\ t_\mu + \vec{b}_o\ t_\nu)\ d\zeta \\
&\quad + (\vec{n}_o' - \vec{n}_o)(d\mu' - d\mu) + (\vec{b}_o' - \vec{b}_o)(d\nu' - d\nu) \\
&\cong (-\ \vec{t}_o\ \kappa d\mu + \vec{n}_o\ t_\mu + \vec{b}_o\ t_\nu)\ d\zeta \quad .
\end{aligned} \tag{9.54}$$

The terms neglected are one order smaller than the terms retained.

Since

$$\begin{aligned}
(\vec{\nabla}\vec{t})_{\mu=\nu=o} &\equiv (\vec{\nabla}\vec{t})_o \\
&= 0 + \vec{t}_o\ \vec{n}_o\ \kappa + 0 \\
&\quad + 0 + \vec{n}_o\ \vec{n}_o\ (\partial t_\mu/\partial\mu) + \vec{n}_o\ \vec{b}_o(\partial t_\nu/\partial\mu) \\
&\quad + 0 + \vec{b}_o\ \vec{n}_o\ (\partial t_\mu/\partial\nu) + \vec{b}_o\ \vec{b}_o(\partial t_\nu/\partial\nu) \quad ,
\end{aligned} \tag{9.55}$$

as may readily be shown from Eq. (9.23),

$$\vec{d\rho} \cdot (\vec{\nabla}\vec{t})_o = \vec{n}_o\, t_\mu + \vec{b}_o\, t_\nu \quad . \tag{9.56}$$

Thus, Eq. (9.54) becomes

$$\begin{aligned}\vec{d\rho}' + \vec{t}_o\, \kappa\, d\mu\, d\zeta &= \vec{d\rho} + \vec{d\rho} \cdot (\vec{\nabla}\vec{t})_o\, d\zeta \\ &= \vec{d\rho} \cdot \left\{ (\vec{n}_o\, \vec{n}_o + \vec{b}_o\, \vec{b}_o + (\vec{\nabla}\vec{t})_o\, d\zeta \right\}\end{aligned}$$

Multiplying both sides of this equation on the right by

$$\left\{ \vec{n}_o\, \vec{n}_o + \vec{b}_o\, \vec{b}_o - (\vec{\nabla}\vec{t})_o\, d\zeta \right\}$$

we obtain

$$\vec{d\rho}' \cdot \left\{ \vec{n}_o\, \vec{n}_o + \vec{b}_o\, \vec{b}_o - (\vec{\nabla}\vec{t})_o\, d\zeta \right\} - \vec{n}_o\, d\mu (\kappa d\zeta)^2 = \vec{d\rho} - \vec{d\rho} \cdot (\vec{\nabla}\vec{t})_o \cdot (\vec{\nabla}\vec{t})_o (d\zeta)^2 .$$

When higher-order small terms are again omitted, this equation is reduced to

$$\vec{d\rho} = \vec{d\rho}' \cdot \left\{ \vec{n}_o\, \vec{n}_o + \vec{b}_o\, \vec{b}_o - (\vec{\nabla}\vec{t})_o\, d\zeta \right\} \tag{9.57a}$$

$$= \left\{ \vec{n}_o\, \vec{n}_o + \vec{b}_o\, \vec{b}_o - (\vec{t}\vec{\nabla})_o\, d\zeta \right\} \cdot \vec{d\rho}' . \tag{9.57b}$$

Therefore,

$$d\rho^2 = \vec{d\rho}' \cdot \left\{ \vec{n}_o\, \vec{n}_o + \vec{b}_o\, \vec{b}_o - (\vec{\nabla}\vec{t} + \vec{t}\vec{\nabla})_o\, d\zeta \right\} \cdot \vec{d\rho}' , \tag{9.58}$$

Now we further introduce Bjorgum's notation.

$$\Psi_{\mu\nu} \equiv \frac{\partial t_\nu}{\partial \mu} + \frac{\partial t_\mu}{\partial \nu} \quad . \tag{9.59}$$

From Eq. (9.55) follows

$$(\vec{\nabla}\vec{t} + \vec{t}\vec{\nabla})_o = 0 \qquad + \vec{t}_o\,\vec{n}_o\,\kappa \qquad + 0$$
$$+ \vec{n}_o\,\vec{t}_o\,\kappa + \vec{n}_o\,\vec{n}_o\,2\,\Theta_\mu + \vec{n}_o\,\vec{b}_o\,\Psi_{\mu\nu}$$
$$+ 0 \qquad + \vec{b}_o\,\vec{n}_o\,\Psi_{\mu\nu} + \vec{b}_o\,\vec{b}_o\,2\,\Theta_\nu\;. \qquad (9.60)$$

Substituting Eqs. (9.49), (9.50), and (9.60) into Eq. (9.58), we obtain

$$d\rho^2 = (1 - 2\,\Theta_\mu\,d\zeta)\,d\mu'^2 + (1 - 2\,\Theta_\nu d\zeta)\,d\nu'^2 - 2\Psi_{\mu\nu}\,d\zeta\,d\mu'\,d\nu'\;, \qquad (9.61)$$

again neglecting small terms of higher orders. This equation tells us that the normal cross section of an infinitesimal vector tube changes with ζ from a circular area at the point $(\zeta, 0, 0)$ to an elliptic at $(\zeta + d\zeta, 0, 0)$. The change of its boundary from section to section, $d\zeta$ apart, is determined by the three quantities Θ_μ, Θ_ν, and $\Psi_{\mu\nu}$.

The curlcurl and the Laplacian expression of $\vec{V} = v\vec{t}$ under the specialization $\mu = \nu = 0$ remain to be given. It must now be noted that we cannot differentiate the specialized equations, such as (9.39), (9.41), and (9.55). We must always specialize from the corresponding general expression. Thus, from Eqs. (9.27) and (9.28), we obtain

$$(\vec{\nabla}\times\vec{\nabla}\times v\vec{t})_o = \vec{t}_o\left[\frac{\partial}{\partial\zeta}(v\,\Theta) + \left(\frac{\partial^2}{\partial\zeta^2} - \Delta\right)v - v\left\{\Delta(\sqrt{g}\;t^\zeta) - \kappa^2\right\}\right]$$
$$+ \vec{n}_o\left[\frac{\partial}{\partial\zeta}\left(\frac{\partial v}{\partial\mu} - \kappa v\right) + \frac{\partial v}{\partial\mu}\,\Theta_\nu - \frac{\partial v}{\partial\nu}\,\frac{\partial t_\mu}{\partial\nu}\right.$$
$$\left. + \frac{\partial}{\partial\nu}\left\{v\left(\frac{\partial t_\nu}{\partial\mu} - \frac{\partial t_\mu}{\partial\nu}\right)\right\} - \tau\,\frac{\partial v}{\partial\nu}\right]$$
$$+ \vec{b}_o\left[\frac{\partial}{\partial\zeta}\frac{\partial v}{\partial\nu} + \frac{\partial v}{\partial\nu}\,\Theta_\mu - \frac{\partial v}{\partial\mu}\,\frac{\partial t_\nu}{\partial\mu} - \frac{\partial}{\partial\mu}\left\{v\left(\frac{\partial t_\nu}{\partial\mu} - \frac{\partial t_\mu}{\partial\nu}\right)\right\}\right.$$
$$\left. + \kappa v\Omega + \tau\left(\frac{\partial v}{\partial\mu} - \kappa v\right)\right]\;; \qquad (\mu = \nu = 0). \qquad (9.62)$$

In this equation,

$$\Delta v = (\Delta v)_o = \left(\frac{\partial^2}{\partial\zeta^2} + \frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2} - \kappa\frac{\partial}{\partial\mu}\right) v \tag{9.63}$$

and, as may easily be derived from Eqs. (9.37) and (9.38),

$$\Delta(\sqrt{g}\ t^\zeta) = \left\{\Delta\ (\sqrt{g}\ t^\zeta)\right\}_o = \left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right)(\sqrt{g}\ t^\zeta)$$

$$= -\left\{\ \Theta_\mu^2 + \ \Theta_\nu^2 + \frac{1}{2}\left(\Omega^2 + \Psi_{\mu\nu}^2\right)\right\} \quad . \tag{9.64}$$

Similarly, from Eq. (9.30),

$$\begin{aligned}\left\{\Delta(v\vec{t})\right\}_o &= \vec{t}_o\left[\Delta v + v\left\{\Delta(\ g\ t^\zeta) - \kappa^2\right\}\right] \\ &+ \vec{n}_o\left[2\kappa\frac{\partial v}{\partial\zeta} + 2\frac{\partial v}{\partial\mu}\quad\Theta_\mu + \frac{\partial v}{\partial\nu}\left(\Psi_{\mu\nu} - \Omega\right) + v\left(\Delta t_\mu + \frac{\partial\kappa}{\partial\zeta}\right)\right] \\ &+ \vec{b}_o\left[\frac{\partial v}{\partial\mu}\left(\Psi_{\mu\nu} + \Omega\right) + 2\frac{\partial v}{\partial\nu}\quad\Theta_\nu + v\left(\Delta t_\nu + \kappa\tau\right)\right] \quad .\end{aligned} \tag{9.65}$$

Here,

$$\Delta t_i = (\Delta t_i)_o = \left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2} - \kappa\frac{\partial}{\partial\mu}\right) t_i \ ,\ (i = \mu \text{ or } \nu),$$

because $(\partial^2 t_i/\partial\zeta^2)_o = 0$.

X. CONCLUDING REMARKS

In Section VI we have shown from fundamental principles how a simple non-orthogonal curvilinear coordinate system may be set up by referring to one given vector line or space curve. The metric tensor components contain the coordinates μ and ν explicitly and the coordinate ζ implicitly through the curvature $\kappa(\zeta)$ and the torsion $\tau(\zeta)$, (ζ, μ, ν) being measured, respectively, along the tangential $(\vec{t}_o)$, the principal normal $(\vec{n}_o)$, and the binormal $(\vec{b}_o)$ direction of the reference curve.

Analyses may be carried out with reference to either the non-orthogonal sets of base vectors, covariant and contravariant, or the orthogonal set of unit vectors $(\vec{t}_o, \vec{n}_o, \vec{b}_o)$.

Prior to Section VI, introductory and preparatory materials were discussed. Many important formulas concerning the metric properties of non-orthogonal curvilinear coordinate systems were included. Special emphases were given to the concepts of covariant and contravariant base vectors. Frenet's formulas were discussed in detail. These formulas play an important role in using the natural coordinate system; they enable us to introduce, in a simple manner, Riemann-Christoffel symbols of the coordinate system and the covariant derivatives of vector components. The transformation of coordinates and of vector and tensor components from one system to another, and the transformation of vector and tensor components in the same system from covariant to contravariant or to a mixed kind and vice versa, have also been discussed. These subjects become very simple when base vectors are explicitly used. Our discussion is based on the sole fact that not only scalars but also tensors of different ranks (vectors, dyadics, triadics, etc.) are invariant quantities under all the aforementioned transformations.

The all important gradient operator was discussed in Section VII. The next section was solely concerned with Christoffel symbols. We have developed specific formulas for these symbols in the natural coordinate system, and have collected some general ones concerning the basic properties of these symbols in any curvilinear coordinate system, including the law of transformation from one system to another.

Christoffel symbols do not transform like tensor components except under linear coordinate transformations. According to Eq. (8.11a), we note that

$$\vec{e}\,'^m \, \vec{e}\,'^n \, \vec{e}\,'_i \, (\Gamma')^i_{mn} = \vec{e}^p \, \vec{e}^q \, \vec{e}_k \, \Gamma^k_{pq} + \vec{e}\,'^m \, \vec{e}\,'^n \, \vec{e}_k \, \frac{\partial^2 x^k}{\partial x'^m \, \partial x'^n} \quad .$$

i.e.,

$$\vec{e}\,'^m \, (\vec{\nabla}' \vec{e}\,'_m) = \vec{e}^p \, (\vec{\nabla} \, \vec{e}_p) + \vec{e}\,'^m \left(\vec{\nabla}' \, \frac{\partial x^k}{\partial x'^m} \right) \vec{e}_k \quad ,$$

Since

$$\vec{e}'^m \left(\vec{\nabla}' \frac{\partial x^k}{\partial x'^m} \right) \vec{e}_k = (\vec{\nabla}'\vec{\nabla}' x^k)\ \vec{e}_k - (\vec{\nabla}'\vec{e}'^n)\ \vec{e}'_n \quad ,$$

and

$$\vec{\nabla}'\vec{\nabla}' x^k = \vec{\nabla}\ \vec{\nabla}\ x^k = \vec{\nabla}\ \vec{e}^k \quad ,$$

we obtain

$$\vec{e}'^m\ (\vec{\nabla}'\vec{e}'_m) + (\vec{\nabla}'\vec{e}'^m)\ \vec{e}'_m = \vec{e}^p\ (\vec{\nabla}\ \vec{e}_p) + (\vec{\nabla}\ \vec{e}^p)\ \vec{e}_p \ . \qquad (10.1)$$

Such quantities as $\vec{e}^p\ (\vec{\nabla}\ \vec{e}_p) \neq \vec{e}'^m\ (\vec{\nabla}'\vec{e}'_m)$ and $(\vec{\nabla}\ \vec{e}^p)\ \vec{e}_p \neq (\vec{\nabla}'\vec{e}'^m)\ \vec{e}'_m$ may tentatively be called "<u>base</u> <u>triadics</u>." They are not tensors of the third rank, but the sum $\left\{ \vec{e}^p\ (\vec{\nabla}\ \vec{e}_p) + (\vec{\nabla}\ \vec{e}^p)\ \vec{e}_p \right\}$ is. In fact, the latter quantity is a null tensor which vanishes identically. Quantities obtained by applying successively the gradient operator on the metric tensor $\overleftrightarrow{g} = \vec{e}^i\vec{e}^k\ g_{ik} = \overleftrightarrow{I}$ are all null tensors, namely,

$$\left.\begin{aligned} \vec{\nabla}\ \overleftrightarrow{g} &= \vec{\nabla}\ (\vec{e}^p\vec{e}_p) = \vec{\nabla}\ (\vec{e}_p\vec{e}^p) = 0 \ , \\ \vec{\nabla}\ \vec{\nabla}\ \overleftrightarrow{g} &= 0 \ , \text{ etc.} \end{aligned}\right\} \ . \qquad (10.2)$$

In Section IX, many oft-used differential expressions of vector fields were compiled. Corresponding formulas referring to the orthogonal unit vectors $(\vec{t}_o, \vec{n}_o, \vec{b}_o)$ in the natural coordinate system are also given. In particular, we have discussed the curl of the covariant base vectors $\vec{e}_\zeta$, $\vec{e}_\mu$, and $\vec{e}_\nu$. Not only $\vec{\nabla} \times \vec{e}_i$ but also $\vec{e}_i \cdot \vec{\nabla} \times \vec{e}_i$, $i = \zeta, \mu$, and ν, do not vanish, because the torsion τ is non-zero. This implies that there exists no family of surfaces x_i = const. to which the covariant base vectors are orthogonal. In other words, the differential equation $dx_i = g_{ik}\ dx^k = 0$ is not integrable amd the covariant coordinates x_k do not actually exist. One may, however, still use the covariant coordinates symbolically, for the sake of brevity, as we did in Eq. (9.22) and elsewhere.

In orthogonal coordinate systems, the two corresponding families of coordinate surfaces x_i = const. and x^i = const. are the same; the two corresponding base vectors $\vec{e}_i$ and $\vec{e}^i$ can only differ in magnitude by a scale factor.

The Laplacian expression of a vector field has been discussed in detail. The usual definition of $\Delta\vec{V}$, as given by Eq. (9.15), is the same as either Eq. (9.13) or Eq. (9.14) except in non-Euclidean spaces, where Eq. (9.15) should be replaced by Eq. (9.16).

From the general expressions which are valid for any vector field, e.g., $\vec{V} = v\vec{t}$, in the whole domain, where the natural coordinate system is defined, we may specialize to obtain local relations by putting $\mu = \nu = 0$, i.e., $\vec{t} = \vec{t}_o$ $(t_\mu = t_\nu = 0;\ \sqrt{g}\ t^\zeta = 1)$ and $g = 1$. The specialized expressions are the ones Bjorgum derived and used in his work on Beltrami fields. As exemplified by the derivation of various equations in Section IX, including Eq. (9.47) and Eq. (9.61), using the reference vectors $(\vec{t}_o, \vec{n}_o, \vec{b}_o)$ defined by one vector line, enables one to carry out easier and clearer analyses than using $(\vec{t}, \vec{n}, \vec{b})$ defined by the vector field. The former set of unit vectors are functions of ζ only, while the latter are functions of all three coordinates. We believe that the mathematical tools discussed in this note can be quite useful in studying vector problems especially by the method of perturbation.

FOOTNOTES

1. R. Herzog, Z. Physik, 89, 447 (1934).
   R. Herzog and V. Hauk, Ann. Physik, 33,89 (1938).
   R. Herzog and V. Hauk, Z. Physik, 108, 609 (1938).
2. A. Klemm, Z. Naturforsch., 1, 137 (1946).
3. J. F. Streib, "Design Considerations for Magnetic Spectrometers," Stanford University, HEPL 104 (1960).
   K. L. Brown, R. Belbeoch and P. Bounin, Rev. Sci. Instr., 35, 481-485 (1964).
4. O. Bjorgum, "On Beltrami Vector Fields and Flows," Part I, Universitet i Bergen, Årb. 1951, Naturv. rekke Nr. 1, pp.1-86.
   O. Bjorgum and T. Godal, "On Beltrami Vectcr Fields and Flows," Part II, Universitet i Bergen, Årb. 1952, Naturv. rekke Nr. 13, pp.1-64.
   T. Godal, "On Beltrami Vector Fields and Flows," Part III, Universitet i Bergen, Årb. 1957, Naturv. rekke Nr. 12, pp. 1-28
5. V. Bjerknes, Th. Hesselberg and O. Devik, "Dynamic Meteorology and Hydrography," Part II, pp. 111-121, Washington, 1910.
6. A. S. Ramsay, "A Treatise on Hydromechanics," Part II, pp. 57-58, London, 1935.
7. L. M, Milne-Thomson, "Theortical Hydrodynamics," pp. 99-102, London, 1949.
8. We are following Gibbs' terminology in calling the *unity dyadic* an idemfactor. (J. W. Gibbs and E. B. Wilson, "Vector Analysis," p. 288 et seg., Dover Publications, Inc., New York.) In vector and tensor analyses the idemfactor plays the same role as the *unity matrix* in matrix algebra.
9. Frenet's formulas are discussed in most treatises of tensor analysis and differential geometry. See, e.g., A. J. McConnel, "Applications of Tensor Analysis," 1931, Dover Publications, Inc. New York; L. P. Eisenhart, "An Introduction to Differential Geometry," 1940, Princeton Univ. Press, Princeton; L. Brand, "Vector and Tensor Analysis," 1947, John Wiley and Sons, Inc., New York.

10. There are two kinds of Riemann-Christoffel symbols. The other kind is defined by

$$\Gamma_{i,km} = \vec{e}_i \cdot \frac{\partial \vec{e}_m}{\partial x^k} .$$

Evidently, $\Gamma_{q,km} = g_{qi}\, \Gamma^i_{km}$ and $\Gamma^q_{km} = g^{qi}\, \Gamma_{i,km}$.

In earlier publications on relativity and in many treatises of differential geometry, the symbols Γ^q_{km} are variedly denoted by

$\left\{ km,q \right\}$ or $\left\{ \begin{matrix} k\ m \\ q \end{matrix} \right\}$ or $\left\{ \begin{matrix} q \\ k\ m \end{matrix} \right\}$ and $\Gamma_{q,km}$ by $\left[km,q \right]$ or $\left[\begin{matrix} k\ m \\ q \end{matrix} \right]$.

11. L. P. Eisenhart, loc. cit. footnote 9. E. P. Adams and R.L. Hippisley, "Smithsonian Mathematical Formulae and Tables of Elliptic Functions," pp. 57-60, Smithsonian Institute, City of Washington (1922).

12. A. J. McConnel, loc. cit. footnote 9.
L. Brand, loc. cit. footnote 9.
A. S. Eddington, "The Mathematical Theory of Relativity," Chapter II, Cambridge Univ. Press (1923).
W. Pauli, "Theory of Relativity," Part II, Pergamon Press, New York (1958).
See also: E. Madelung, "Die Mathematischen Hilfsmittel des Physikers," pp. 131-140, Dover Publications, New York (1943).