APPLYING QCD TO
INCLUSIVE DECAY RATES OF THE $\tau$ LEPTON

ERIC BRAATEN
Department of Physics and Astronomy
Northwestern University
Evanston, IL 60201

ABSTRACT

The application of QCD to the inclusive semihadronic decay rates of the $\tau$ are described. It is argued that the QCD corrections, both perturbative and nonperturbative, can be systematically calculated within the framework of the short-distance expansion. Phenomenological evidence is presented that indicates that $M_\tau$ is safely within the region of validity of the QCD predictions.

The inclusive semihadronic decay rate of the $\tau$ lepton is conveniently expressed in terms of the ratio

$$R = \frac{\Gamma(\tau^- \to \nu_\tau + \text{hadrons})}{\Gamma(\tau^- \to \nu_\tau + e^- \bar{\nu}_e)} . \tag{1}$$

A naive estimate [1] of R can be obtained by ignoring the strong QCD interaction and approximating the numerator by the decay rate into $\nu_\tau$ plus a free quark-antiquark pair, either $d\bar{u}$ or $s\bar{u}$. The result is

$$R = N_c(|V_{ud}|^2 + |V_{us}|^2) \simeq 3 . \tag{2}$$

The simplest way to determine this ratio experimentally is to measure the branching fraction for the $\tau$ to decay into electrons ($B_e$) or muons

($B_\mu$ = .973 $B_e$). The ratio R is then given by

$$R = \frac{1 - 1.973\ B_e}{B_e} . \tag{3}$$

The direct measurement of $B_e$ yields R = 3.68 ± .13 [2]. Given the universality of lepton couplings, R can also be measured indirectly by measuring the $\tau$ lifetime and this indirect measurement yields R = 3.32 ± .16 [2]. While they differ by two standard deviations, both of these measurements are reasonably close to the naive prediction (2). Since the accuracy of both determinations would be improved dramatically at a Tau-Charm Factory, this presents the challenge of understanding the corrections to (2) and predicting the ratio R.

The corrections to the naive prediction (2) can be classified into electroweak, perturbative QCD, and nonperturbative QCD. The electroweak corrections are small but not negligible, because they contain a large logarithm $\log(M_Z/M_\tau)$. The coefficient of this logarithm has been calculated by Marciano and Sirlin [3], and a calculation of the constant under the logarithm is in progress [4]. Using only the logarithm, the electroweak corrections have been estimated to be +2% [3].

The QCD corrections to R may be classified into perturbative and nonperturbative. For the perturbative corrections, one simply aprroximates the numerator of (2) by the decay into $d\bar{u}$ or $s\bar{u}$ plus gluons and additional $q\bar{q}$ pairs, ignoring the fact that the QCD interaction binds these quanta into color singlet hadrons. The correction has been calculated by Narison and Pich and by Braaten to order $\alpha_s{}^3$ [5,6], with the result

$$R_{pert} = 3 \left\{ 1 + \frac{\alpha_s(M_\tau)}{\pi} + 5.20\left( \frac{\alpha_s(M_\tau)}{\pi} \right)^2 + 104.\left( \frac{\alpha_s(M_\tau)}{\pi} \right)^3 \right\}. \tag{4}$$

When the running coupling constant of QCD as determined from $e^+e^-$ annihilation [7] is evolved down to the scale $M_\tau$, we find $\alpha_s(M_\tau) \simeq .33 \pm .08$. The $\alpha_s{}^3$ correction is comparable to the $\alpha_s$ correction, which is around 10%. This gives us reason to worry that higher order corrections may be even larger, so that the perturbation expansion breaks down altogether. There are two points that should be made about the large $\alpha_s{}^3$

correction in (4). First, if the perturbation expansion is indeed breaking down, it is not because the scale $M_\tau$ is too small. The running coupling constant is still reasonably small at the scale $M_\tau$; the large $\alpha_S{}^3$ correction is due to the large coefficient and would still be uncomfortably large if $M_\tau$ had been much larger. Second, the calculation of the $\alpha_S{}^3$ correction was based on a calculation to order $\alpha_S{}^3$ of the ratio R for $e^+e^-$ annihilation [8] which has not yet been confimed by an independent calculation. However if we take the calculation (4) at face value, the perturbative prediction for R is 3.85 (+.50/-.35), with most of the uncertainty arising from the order $\alpha_S{}^3$ term. This is much larger than the experimental uncertainty in R, so a much more precise determination of $\alpha_S$ is required before (4) would have any predictive power. Alternatively, we can exploit the sensitivity of (4) to the value of $\alpha_S$ to obtain a precise low energy determination of $\alpha_S$.

The nonperturbative QCD corrections can be organized into an expansion in powers of $1/M_\tau{}^2$. We first express R as an integral over the hadronic invariant mass s from $s = 0$ to $s = M_\tau{}^2$. The analytic structure of the integrand is then exploited to express R as an integral around the contour $|s| = M_\tau{}^2$ in the complex s plane [9]. In this form, we can apply the short-distance operator product expansion to expand the integrand in powers of 1/s and to systematically separate the perturbative short-distance effects from the long distance effects, which have nonperturbative contributions. The long-distance effects are factorized into vacuum expectation values of local operators such as $\langle\bar{\psi}\psi\rangle$, $\langle G^{\mu\nu}G_{\mu\nu}\rangle$, and $\langle\bar{\psi}\psi\bar{\psi}\psi\rangle$ [10], which can either be determined phenomenologically or calculated using a nonperturbative method such as lattice gauge theory. The coefficients of these matrix elements can be calculated perturbatively in terms of $\alpha_S(s)$. After applying this expansion to the integrand and evaluating the contour integrals over the invariant mass s, the final result is an expansion in powers of $1/M_\tau{}^2$ whose coefficients are matrix elements multiplied by functions of $\alpha_S(M_\tau)$. Incidentally, this justifies the choice of $M_\tau$ for the scale of $\alpha_S$ in (4), because the perturbative QCD corrections are simply the coefficient of the unit operator in this expansion.

This method was first applied to $\tau$ decay by Schilcher and Tran [11]. In addition to making predictions for inclusive decays of the $\tau$ [12,5], this method can be used to test the Weinberg sum rules [13] and to determine the values of nonperturbative matrix elements such as $\langle G^{\mu\nu}G_{\mu\nu}\rangle$ and $\langle\bar{\psi}\psi\bar{\psi}\psi\rangle$ [14]. These specific applications are discussed in more detail by Pich [15]. I will simply describe the quantities that can be calculated using this method and discuss some objections that have been raised concerning the reliability of these QCD predictions.

The short-distance expansion method can only be used to predict inclusive quantities that are summed over all final states with a definite set of quantum numbers and are suitably smeared over the hadronic invariant mass s. For example we can calculate R itself and the first few moments of the invariant mass distribution,

$$\langle s^n\rangle = \frac{1}{R}\int_0^{M_\tau^2} ds\ s^n \frac{dR}{ds} . \tag{5}$$

We can not calculate the invariant mass distribution dR/ds itself because it is sensitive to nonperturbative resonance effects which can not be treated systematically using the short-distance expansion.

We can also resolve the predictions for R and $\langle s^n\rangle$ into the contributions from different currents, namely into vector (V) versus axial vector (A), and into nonstrange ($d\bar{u}$) versus strange ($s\bar{u}$). For example, the predictions for the vector and axial components of the nonstrange contribution to R are

$$R^V_{d\bar{u}} = |V_{ud}|^2 \left( \frac{1}{2} R_{pert} - \frac{55\pi}{8} \left(\frac{\alpha_s}{\pi}\right)^2 \frac{\alpha_s\langle GG\rangle}{M_\tau^4} + 7\,\frac{128\pi^3}{9}\,\frac{\alpha_s\langle\bar{\psi}\psi\bar{\psi}\psi\rangle}{M_\tau^6} \right), \tag{6}$$

$$R^A_{d\bar{u}} = |V_{ud}|^2 \left( \frac{1}{2} R_{pert} + 48\pi^2 \frac{(m_u+m_d)\langle\bar{\psi}\psi\rangle}{M_\tau^4} - 11\,\frac{128\pi^3}{9}\,\frac{\alpha_s\langle\bar{\psi}\psi\bar{\psi}\psi\rangle}{M_\tau^6} \right). \tag{7}$$

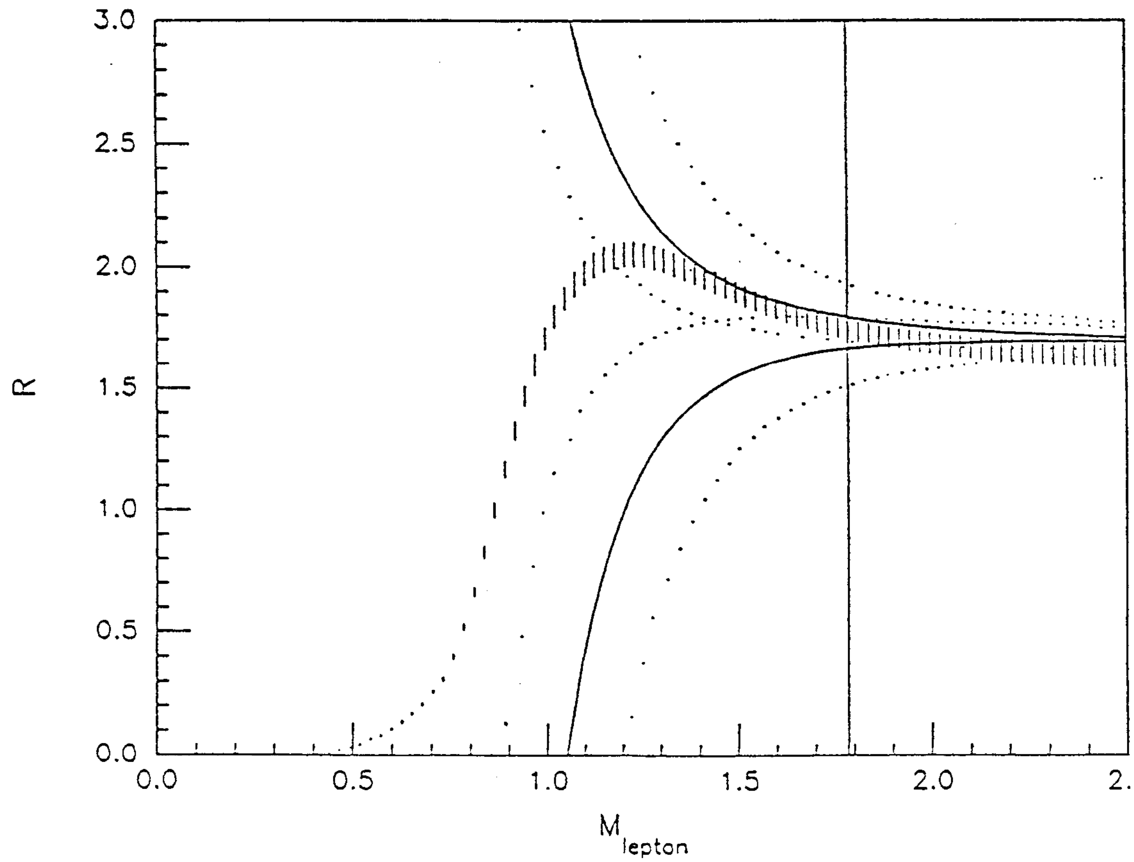
The $1/M_\tau^8$ and higher order terms are assumed to be negligible. Note that the matrix element $\alpha_s\langle GG\rangle$, which in many applications provides the largest nonperturbative correction, is suppressed by two powers of $\alpha_s$. The contribution from the matrix element $\langle\bar{\psi}\psi\rangle$ is also small, because it is suppressed by the light quark masses. Therefore the only significant source of nonperturbative corrections is the 4-quark matrix element $<\bar{\psi}\psi\bar{\psi}\psi>$. The difference $R^V - R^A$ is particularly sensitive to the value of this matrix element.

The predictions for $R<s^n>$ can also be decomposed into the two possible values $J = 0$ and $J = 1$ for the total angular momentum in the rest frame of the hadrons. Unfortunately this can not be done for R itself, because it is sensitive to long distance effects associated with the single pion decay mode which can not be resolved into $J = 0$ and $J = 1$ within the short distance expansion.

One common objection to applying QCD to $\tau$ decay is that the mass scale is to small to allow reliable predictions. However, the derivation of the short-distance expansion for R indicates that the appropriate mass scale is $M_\tau \approx 1.8$ GeV. For some applications of QCD like deep inelastic scattering, this scale would be uncomfortably low. But this is because the $1/Q^2$ (higher twist) corrections to deep inelastic scattering can not be estimated reliably since they depend on 2-parton correlation distributions which are not easily extracted from experimental data. For $\tau$ decay, the corrections up to $1/M_\tau^6$ can be computed systematically in terms of a few phenomenologically determined matrix elements.

Another common objection is that the $\tau$ decay spectrum is dominated by resonances, which can not be treated in a short-distance expansion. We certainly could not use such an approach to calculate dR/ds as a function of s for individual decay modes. Summing over decay modes to get inclusive decay rates tends to smooth out all but the lowest and most dramatic resonances, and it is an experimental observation that QCD predictions seem to be valid above these resonances. By constructing weighted integrals of dR/ds, such as R and $<s^n>$, the resonances are smoothed out even further and the QCD predictions can be extended to lower energy. Of course the QCD predictions would certainly break down if $M_\tau$ was too small. The question we must answer is whether the physical value $M_\tau = 1.784$ Gev is in the region of validity of the predictions.

This question can be answered by comparing the QCD predictions with the data as a function of $M_\tau$. While we cannot actually vary the mass of the $\tau$, we can determine the dependence on $M_\tau$ of the $d\bar{u}$ vector current component of R by using isospin symmetry to extract dR/ds for this channel from data on $e^+e^-$ annihilation into 2, 4, and 6 pions [16]. The ratio R for this channel can then be calculated as a function of the lepton mass and the result is shown in the figure below. The curve plotted in the form of data points is the experimental prediction, with the size of the error bars reflecting the statistical error in the data only. There is also a systematic error on the order of 10%. The upper solid curve is the prediction (6), with the nearby dotted curves representing upper and lower error bars due to the uncertainties in $\alpha_s$ and in the matrix element $\langle\bar{\psi}\psi\bar{\psi}\psi\rangle$. Near the $\tau$ mass, the uncertainty is due almost entirely to $\alpha_s$. We have not included the controversial $\alpha_s{}^3$ term given in (4). For comparison, the lower solid curve is the prediction (7) for the axial vector current. The shape of the theoretical prediction for the vector current agrees very well with the experimental data for lepton masses above 1.3 GeV. The corrections to the asymptotic prediction R = 3/2 are



also reasonably small in this region. We conclude that the physical value $M_\tau = 1.784$ GeV is well within the region of validity of the QCD predictions.

Further progress in the applications of QCD to decays of the $\tau$ lepton is certainly possible. On the experimental front, it requires accurate measurements of the invariant mass spectrum $dR/ds$ for all possible hadronic final states. A Tau-Charm Factory would be ideal for such measurements. On the theoretical front, the most urgent problem is to repeat the calculation of R for $e^+e^-$ annihilation to see if the large order $\alpha_s{}^3$ correction is correct. Improved methods for calculating the matrix elements such as $\langle\bar{\psi}\psi\bar{\psi}\psi\rangle$ that appear in the nonperturbative QCD corrections would also be of great value.

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