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ON N-DIMENSIONAL TAYLOR AND LAGRANGE OPERATORS

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by

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I. INTRODUCTION

Taylor's theorem and the theorem of Lagrange both express the value of a function at one point in terms of its properties at another point. These two theorems have a reciprocal nature, and, in part, our purpose here is the exploration of many of the details of this reciprocity. Although Taylor's theorem is routinely employed in many branches of mathematics, the theorem of Lagrange is encountered much less frequently. Lagrange's[1] series for the one-dimensional development of functions was first described in 1770. Laplace[2] later devised a more general series expansion, demonstrating the one-dimensional series of Lagrange as a special case. Similar series expansions in many variables were also reported by Laplace in this work. By means of a simple transformation, Darboux[3] expressed Laplace's two-dimensional series in a particularly convenient form similar to that given by Lagrange. This form for the Lagrange theorem has now become classic, the Jacobian of the transformation taking a prominent role in the statement of the theorem. Stieltjes,[4] in correspondence with Hermite, reported that Darboux's result could be obtained by double integration over two complex variables, except for an uncertainty in the sign of the result. A short time later, Poincaré,[5] through a careful examination of the theory of double complex integration, removed the ambiguity noted by Stieltjes. In the meantime, Stieltjes[6] published an elegant extension of Lagrange's theorem to the N-dimensional case. In this latter work, Stieltjes did not employ complex integration; he remarked that the demonstration by Heine[7] of the one-dimensional Lagrange theorem (obtained by using the calculus of variations), could be generalized to the N-dimensional case.

The work of Stieltjes[6] has been overlooked in the recent papers by Good,[8] Sturrock,[9] and Dedrick & Wilson.[10] Through the use of complex integration, Good obtained the Lagrange theorem in N-dimensions. Sturrock, with the aid of Fourier analysis, generated the Lagrange series as the solution to a physical problem. Dedrick & Wilson reported an integral theorem in which both the N-dimensional Taylor and Lagrange series are involved. Lagrange's theorem is discussed in the works by Whittaker & Watson,[11] Pólya & Szegö,[12] Goursat,[13] Osgood,[14] etc. The article by

Osgood contains many references to the early work on this subject.

The Taylor and Lagrange theorems can be considered to be generated by application, respectively, of the Taylor differential operator Σ and the Lagrange differential operator Ω to the function which is to be developed. As described above, the statement of Lagrange's theorem requires, in addition, use of the Jacobian J of the transformation involved. The reciprocal nature of the Taylor and Lagrange theorems suggests that relations can be written between the Jacobian and the Taylor and Lagrange differential operators; it is shown below that this is indeed the case. Perhaps the most useful relation of this kind is the theorem that the operator $J\Sigma\Omega$ is the unity operator at all points where the operand is infinitely differentiable and the Jacobian is non-singular. Proof of the validity of this theorem does not require the use of either the Taylor or the Lagrange theorems; furthermore, through the use of this fundamental relation, Lagrange's theorem follows if Taylor's theorem is valid, and vice versa.

As used in the theories of elasticity, hydrodynamics, and continuum electrodynamics, the displacement vector is defined to be the difference between the perturbed position and the corresponding unperturbed position of a material point. The definitions of the Taylor and the Lagrange operators given here are based on the coordinate transformation defined by the displacement vector. Since the concept of the displacement vector is readily extended to N-dimensions, we shall consider the N-dimensional problem throughout this work.

The coordinate transformation and the Jacobian are treated in Section II. The Taylor operator and Taylor's theorem are described in Section III, and the theorem of Lagrange is discussed in Section IV. Section V is devoted to the proof of the theorem: $J\Sigma\Omega = 1$. In Section VI, other useful operator relations are derived and discussed, and in Section VII, some applications to various areas of mathematical physics are briefly described.

II. TRANSFORMATION OF COORDINATES AND THE JACOBIAN

Let $\vec{x} = (x^1, x^2, \ldots, x^N)$ and $\vec{x}_o = (x_o^1, x_o^2, \ldots, x_o^N)$ be two vectors in an N-dimensional manifold. These vectors can be expressed in terms of N base vectors $\vec{a}_k$ by: $\vec{x} = \vec{a}_k x^k$ and $\vec{x}_o = \vec{a}_k x_o^k$, where repeated indices are to be summed in accordance with the summation convention.*

Let $\vec{x}$ and $\vec{x}_o$ be related by the displacement vectors $\vec{\zeta}$ and $\vec{\zeta}'$ as follows:

$$\vec{x} = \vec{x}_o + \vec{\zeta}(\vec{x}_o) \ , \tag{2.1a}$$

$$\vec{x}_o = \vec{x} - \vec{\zeta}'(\vec{x}) \ , \tag{2.1b}$$

$$\vec{\zeta}(\vec{x}_o) = \vec{\zeta}'(\vec{x}) \ . \tag{2.1c}$$

The transformation given by (2.1a) is considered to be continuous and non-singular so that the inverse transformation (2.1b) exists, and vice versa. These equations are the same as those used in the theory of infinitesimal transformations, although here the displacement vector and its derivatives may not be sufficiently small to justify the usual linearization procedure.

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*The notation of vector analysis of Gibbs[15] is ordinarily used only if a Euclidean metric has been introduced. Here, a metric need not be defined. Nevertheless, we shall use the Gibbs notation because it may be readily extended to permit writing many of the equations discussed here more compactly than if the notation of tensor analysis is employed.

A vector $\vec{x}$ may either be given as above in terms of its contravariant components x^k or, alternatively, in terms of its covariant components x_ℓ. Thus we also have $\vec{x} = \vec{a}^\ell x_\ell$. The two sets of base vectors $\vec{a}_k$ and $\vec{a}^\ell$ are derivable from each other according to the relation $\vec{a}_k \cdot \vec{a}^\ell = \delta_k^\ell$, where δ_k^ℓ is the Kronecker delta. In this paper, it is permissible to consider the base vectors $\vec{a}_k$ and $\vec{a}^\ell$ to be the unit vectors in a Cartesian coordinate system so that $\vec{a}_k = \vec{a}^k$.

Differentiation of (2.1a) and (2.1b) yields, respectively,

$$\vec{dx} = \vec{dx}_o \cdot \overleftrightarrow{\Phi}_o \,, \tag{2.2a}$$

$$\vec{dx}_o = \vec{dx} \cdot \overleftrightarrow{\Phi}' \,, \tag{2.2b}$$

where the dyadics $\overleftrightarrow{\Phi}_o$ and $\overleftrightarrow{\Phi}'$ are defined by

$$\overleftrightarrow{\Phi}_o \equiv \vec{\nabla}_o \vec{x} = \overleftrightarrow{I} + \vec{\nabla}_o \vec{\zeta} \,, \tag{2.3a}$$

$$\overleftrightarrow{\Phi}' \equiv \vec{\nabla} \vec{x}_o = \overleftrightarrow{I} - \vec{\nabla} \vec{\zeta}' \,. \tag{2.3b}$$

In Eqs. (2.3), $\overleftrightarrow{I} = \vec{a}_i \vec{a}^i$ is the idemfactor (unity dyadic), and the gradient operators $\vec{\nabla}_o$ and $\vec{\nabla}$ are defined by

$$\vec{\nabla}_o \equiv \vec{a}^i \, \partial/\partial x_o^i \,, \tag{2.4a}$$

$$\vec{\nabla} \equiv \vec{a}^i \, \partial/\partial x^i \,. \tag{2.4b}$$

Substitution of (2.2b) into (2.2a), and vice versa, yields

$$\overleftrightarrow{\Phi}' \cdot \overleftrightarrow{\Phi}_o = \overleftrightarrow{I} = \overleftrightarrow{\Phi}_o \cdot \overleftrightarrow{\Phi}' \,,$$

i.e.,

$$\overleftrightarrow{\Phi}_o^{-1} = \overleftrightarrow{\Phi}' \;;\; \overleftrightarrow{\Phi}'^{-1} = \overleftrightarrow{\Phi}_o \,. \tag{2.5}$$

The inverse of the dyadic $\overleftrightarrow{\Phi}_o$, for example, can also be obtained by solving Eq. (2.2a) directly. However, if the elements of $\vec{\nabla}_o \vec{\zeta}$ are sufficiently small, the expansion

$$\overleftrightarrow{\Phi}_o^{-1} = \overleftrightarrow{I} - \vec{\nabla}_o\vec{\zeta} + (\vec{\nabla}_o\vec{\zeta})\cdot(\vec{\nabla}_o\vec{\zeta}) - \ldots = \sum_{m=0}^{\infty} (-\vec{\nabla}_o\vec{\zeta})^m , \tag{2.6}$$

is often useful.

Derivatives with respect to $\vec{x}_o$ and $\vec{x}$ can be expressed in terms of the gradient operators defined by Eqs. (2.4). These operators are related to one another by

$$\vec{\nabla} = \overleftrightarrow{\Phi}' \cdot \vec{\nabla}_o = \overleftrightarrow{\Phi}_o^{-1} \cdot \vec{\nabla}_o , \tag{2.7a}$$

$$\vec{\nabla}_o = \overleftrightarrow{\Phi}_o \cdot \vec{\nabla} = \overleftrightarrow{\Phi}'^{-1} \cdot \vec{\nabla} . \tag{2.7b}$$

In addition to the transformations between $\vec{x}_o$ and $\vec{x}$, we shall also consider successive transformations between $\vec{x}_{oo}$ and $\vec{x}_o$, and between $\vec{x}$ and $\vec{x}_1$. The equations of these transformations are extensions of Eqs. (2.1), i.e.,

$$\vec{x}_1 = \vec{x} + \vec{\zeta}(\vec{x}) , \qquad \vec{x} = \vec{x}_1 - \vec{\zeta}'(\vec{x}_1) ; \tag{2.8}$$

$$\vec{x}_o = \vec{x}_{oo} + \vec{\zeta}(\vec{x}_{oo}) , \qquad \vec{x}_{oo} = \vec{x}_o - \vec{\zeta}'(\vec{x}_o) . \tag{2.9}$$

Pertaining to each of these transformations is the dyadic defined by extending Eqs. (2.3). The determinants of these dyadics are the corresponding Jacobians. For convenience, we introduce the following definitions:

$$J \equiv \partial\vec{x}_1/\partial\vec{x} \equiv \det(\overleftrightarrow{\Phi}) , \tag{2.10a}$$

$$J_o \equiv \partial\vec{x}/\partial\vec{x}_o \equiv \det(\overleftrightarrow{\Phi}_o) , \tag{2.10b}$$

$$J' \equiv \partial\vec{x}_o/\partial\vec{x} \equiv \det(\overleftrightarrow{\Phi}') \ , \tag{2.10c}$$

$$J'_o \equiv \partial\vec{x}_{oo}/\partial\vec{x}_o \equiv \det(\overleftrightarrow{\Phi}'_o) \tag{2.10d}$$

Clearly, $J' = J_o^{-1}$ because $\overleftrightarrow{\Phi}' = \overleftrightarrow{\Phi}_o^{-1}$.

In the remainder of this section, we shall be mainly concerned with the evaluation of the Jacobian J defined by Eq. (2.10a). The other Jacobians defined by Eqs. (2.10b-d) may be readily calculated by using this result.

The Jacobian J can be written in terms of the characteristic polynomial $Q(\mu)$ associated with the dyadic $\vec{\nabla}\vec{\zeta}$ (or the matrix $\partial\zeta^k/\partial x^i$). This polynomial is given by

$$Q(\mu) = \det(\vec{\nabla}\vec{\zeta} - \mu\overleftrightarrow{I}) \tag{2.11a}$$

$$= (\mu_1 - \mu)(\mu_2 - \mu) \ \ldots \ (\mu_N - \mu) \tag{2.11b}$$

$$= (-\mu)^N + G_1(-\mu)^{N-1} + G_2(-\mu)^{N-2} + \ldots + G_N \ , \tag{2.11c}$$

where the quantities μ_i are the eigenvalues of $\vec{\nabla}\vec{\zeta}$. The Jacobian is equal to the characteristic polynomial evaluated at $\mu = -1$. That is,

$$J = Q(-1) = 1 + G_1 + G_2 + \ldots + G_N \ . \tag{2.12}$$

We shall evaluate the coefficients G_k in the characteristic polynomial (2.11c) according to the method of Leverrier[16] (see, e.g.: Gantmacher,[17] Bodewig[18]). In this method, we use Newton's formula which expresses the coefficients in a polynomial in terms of sums of powers of its roots. For the polynomial (2.11c), Newton's formula states:

$$- kG_k = \sum_{n=o}^{k-1} G_n D_{k-n} \quad ; \qquad (k = 1,2, \ldots ,N) \;, \tag{2.13}$$

where

$$D_m = (-)^m \sum_{i=1}^{N} \mu_i^m \;,$$

and $G_o = 1$. Leverrier shows that the sums of powers of the eigenvalues of a matrix are equal to the traces of the corresponding powers of the matrix in question. Hence in the case of the dyadic $\vec{\nabla}\vec{\zeta}$, we have

$$D_m = (-)^m \, tr(\vec{\nabla}\vec{\zeta})^m = (-\vec{\nabla}\vec{\zeta})^m : \overleftrightarrow{I} \;, \tag{2.14}$$

where, in the latter form, the double scalar product notation of Gibbs is employed.

The quantities G_k can be calculated successively with the aid of Eqs. (2.13-14), so that:

$$G_1 = tr(\vec{\nabla}\vec{\zeta}) = \vec{\nabla} \cdot \vec{\zeta} \;;$$

$$G_2 = \tfrac{1}{2}[(\vec{\nabla} \cdot \vec{\zeta})^2 - (\vec{\nabla}\vec{\zeta})^2 : \overleftrightarrow{I}] \;;$$

etc. This procedure can be used to show (by induction) that G_k is of order k in the displacement vector $\vec{\zeta}$. Furthermore, all the G_k are invariants (under the similarity transformation) of the dyadic $\vec{\nabla}\vec{\zeta}$. It is clear from (2.11b) that G_N is equal to the product of all the eigenvalues μ_i, and therefore: $G_N = \det(\vec{\nabla}\vec{\zeta})$. In the case of three dimensions ($N = 3$), the Jacobian may be written explicitly as

$$J = 1 + (\vec{\nabla}\vec{\zeta}) : \overleftrightarrow{I} + \frac{1}{2!}(\vec{\nabla}\vec{\zeta} \between \vec{\nabla}\vec{\zeta}) : \overleftrightarrow{I} + \frac{1}{3!}(\vec{\nabla}\vec{\zeta} \between \vec{\nabla}\vec{\zeta}) : \vec{\nabla}\vec{\zeta} \;, \tag{2.15}$$

where the double vector product of Gibbs is employed (see also Chu[19]).

According to the Cayley-Hamilton theorem,[17] the dyadic $\vec{\nabla}\vec{\zeta}$ satisfies

$$0 = (-\vec{\nabla}\vec{\zeta})^N + G_1(-\vec{\nabla}\vec{\zeta})^{N-1} + \dots + G_N\overleftrightarrow{I} . \qquad (2.16)$$

In Section V, we shall require the set of relations obtained by first multiplying (2.16) by $(-\vec{\nabla}\vec{\zeta})^q$, $(q = 0,1,2, \dots)$, and then taking the trace. This procedure yields:

$$0 = \sum_{r=0}^{N} G_r D_{N+q-r} \quad , (q = 0,1,2, \dots). \qquad (2.17)$$

We note that Eq. (2.17) for $q = 0$ is the same as Eq. (2.13) for $k = N$.

III. TAYLOR'S THEOREM AND THE TAYLOR OPERATOR

Cauchy[20] presented the N-dimensional Taylor theorem in a form that is particularly suitable for our discussion here. With the aid of the vector notation introduced in Section II, the result of Cauchy (see, e.g., Love[21]) can be written as

$$f(\vec{x}) = f(\vec{x}_o + \vec{\zeta}(\vec{x}_o)) = \Sigma_o\, f(\vec{x}_o) , \qquad (3.1)$$

where the N-dimensional Taylor differential operator Σ_o is defined by

$$\Sigma_o \equiv \exp(\vec{\zeta}(\vec{x}_o) \cdot \vec{\nabla}_o) \equiv \sum_{n=o}^{\infty} \frac{1}{n!}\left\{\vec{\zeta}^n(\vec{x}_o)\middle|\vec{\nabla}_o^n\right\} . \qquad (3.2)$$

The vertical bar in the differential operator $\left\{\vec{\zeta}^n(\vec{x}_o)\middle|\vec{\nabla}_o^n\right\}$

represents an extension of the notation used for the scalar product, and denotes that n vectors $\vec{\zeta}$ be completely contracted with n gradient operators $\vec{\nabla}_o$ in the following manner:

$$(\vec{\zeta}|\vec{\nabla}_o) \equiv \vec{\zeta} \cdot \vec{\nabla}_o \equiv \zeta^i(\partial/\partial x_o^i) \ ,$$

$$(\vec{\zeta}^2|\vec{\nabla}_o^2) \equiv \vec{\zeta}\,\vec{\zeta} : \vec{\nabla}_o\vec{\nabla}_o \equiv \zeta^i\zeta^j(\partial/\partial x_o^i)(\partial/\partial x_o^j) \tag{3.3}$$

and so on.

Taylor's theorem may also be stated in the reversed sense by using (2.1b) instead of (2.1a) to define the displacement. That is,

$$f(\vec{x}_o) = f\left(\vec{x} - \vec{\zeta}'(\vec{x})\right) = \Sigma' \ f(\vec{x}) \ , \tag{3.4}$$

where

$$\Sigma' \equiv \exp\left(- \vec{\zeta}'(\vec{x}) \cdot \vec{\nabla}\right) \equiv \sum_{n=o}^{\infty} \frac{(-)^n}{n!} \left\langle \vec{\zeta}'^n(\vec{x}) | \vec{\nabla}^n \right\rangle \ . \tag{3.5}$$

In addition to the operators Σ_o and Σ', we shall also need two other Taylor operators Σ and Σ_o'. These operators are pertinent to the transformations given by Eqs. (2.8-9) and are defined as follows:

$$\Sigma \equiv \exp\left(\vec{\zeta}(\vec{x}) \cdot \vec{\nabla}\right) \quad , \quad \Sigma_o' \equiv \exp\left(- \vec{\zeta}'(\vec{x}_o) \cdot \vec{\nabla}_o\right) \ . \tag{3.6}$$

A distinction should be pointed out here between Taylor's theorem [as exemplified by Eqs. (3.1) and (3.4)] on the one hand, and the result:

$$h(\vec{x}_o) = \Sigma_o \ f(\vec{x}_o)$$

on the other. If $h(\vec{x}_o)$ is equal to $f(\vec{x})$ as in Eq. (3.1), then $f(\vec{x}_o)$ is said to be a member of the class of functions $\mathcal{C}$ which can be developed by Taylor's theorem. A broader class $\mathcal{K}$ includes all functions that consist of piecewise segments of functions that are members of the class $\mathcal{C}$. If $f(\vec{x}_o)$ is not a member of $\mathcal{C}$, but is a member of $\mathcal{K}$, then Taylor's theorem is not always valid, although

$$h(\vec{x}_o) = \Sigma_o \, f(\vec{x}_o)$$

remains meaningful except at the discontinuities of $f(\vec{x}_o)$. Hereafter, unless explicitly stated otherwise, the operands and the displacement vector functions $\vec{\zeta}$ and $\vec{\zeta}'$ will not be assumed to be members of the class $\mathcal{C}$, but are restricted to the class $\mathcal{K}$, and are therefore piecewise differentiable.

The inverse Taylor operators, as well as the inverses of other scalar operators, are invariably defined by a pair of operator equations, e.g.,

$$\Sigma_o \, \Sigma_o^{-1} = 1 = \Sigma_o^{-1} \, \Sigma_o \quad . \tag{3.7}$$

The Taylor operators and their inverses obey the useful <u>product rule</u>

$$\Sigma \, f(\vec{x}) \, g(\vec{x}) = \left[\Sigma \, f(\vec{x})\right] \left[\Sigma \, g(\vec{x})\right] \tag{3.8}$$ *

In addition, the Taylor operators defined previously satisfy the following relations:

$$\Sigma_o = \Sigma'^{-1} \quad , \quad \Sigma_o^{-1} = \Sigma' \quad , \tag{3.9}$$

*In order to state results unambiguously, we employ the convention that differential operators enclosed by the square brackets: [...], operate only upon other quantities enclosed by the same pair of brackets.

$$\Sigma_o = \Sigma \quad , \quad (\vec{\xi} \in \mathcal{C}) \ , \tag{3.10}$$

$$\Sigma_o' = \Sigma' \quad , \quad (\vec{\xi}' \in \mathcal{C}) \ . \tag{3.11}$$

All the relations (3.8-11) can be readily verified through the use of Taylor's theorem. However, such proofs are satisfactory only if the operand is a member of the class $\mathcal{C}$. For this reason, we shall prove these relations as <u>point relations</u>, valid for operands that are members of the class $\mathcal{K}$.

Our proof of (3.9) requires the use of the operator relation

$$\vec{\nabla}^n \ \Sigma_o = \Sigma_o \vec{\nabla}_o^n \tag{3.12}$$

which, in turn, is based on the identity

$$\vec{\nabla}_o \Sigma_o = \overleftrightarrow{\Phi}_o \cdot \Sigma_o \vec{\nabla}_o \ . \tag{3.13}$$

Equation (3.10) can be proved by using the product rule and either (3.12) or the equivalent statement

$$\vec{\nabla}^n = \Sigma_o \vec{\nabla}_o^n \Sigma_o^{-1} \ , \tag{3.14}$$

while (3.11) can be obtained from (3.10) by means of a simple change in variables.

Let $f(\vec{x}_o) \in \mathcal{K}$ be an operand. Equation (3.13) is proved by noting that

$$\vec{\nabla}_o \Sigma_o f = \Sigma_o \vec{\nabla}_o f + \sum_{n=1}^{\infty} \frac{1}{n!} ([\vec{\nabla}_o \vec{\zeta}^n] | \vec{\nabla}_o^n) f \quad ,$$

$$([\vec{\nabla}_o \vec{\zeta}^n] | \vec{\nabla}_o^n) f = n [\vec{\nabla}_o \vec{\zeta}] \cdot (\vec{\zeta}^{n-1} | \vec{\nabla}_o^{n-1}) \vec{\nabla}_o f \quad .$$

Hence

$$\vec{\nabla}_o \Sigma_o f = [\overleftrightarrow{I} + \vec{\nabla}_o \vec{\zeta}] \cdot \Sigma_o \vec{\nabla}_o f = \overleftrightarrow{\Phi}_o \cdot \Sigma_o \vec{\nabla}_o f \quad ,$$

as required.

Equation (3.12) is readily proved by induction. Under the assumption that (3.12) is valid for $n = k$, we obtain

$$\begin{aligned} \vec{\nabla}^{k+1} \Sigma_o f &= \vec{\nabla} \Sigma_o \vec{\nabla}_o^k f \\ &= \overleftrightarrow{\Phi}_o^{-1} \cdot \vec{\nabla}_o \Sigma_o \vec{\nabla}_o^k f = \overleftrightarrow{\Phi}_o^{-1} \cdot \overleftrightarrow{\Phi}_o \cdot \Sigma_o \vec{\nabla}_o^{k+1} f \\ &= \Sigma_o \vec{\nabla}_o^{k+1} f \quad , \end{aligned}$$

using Eqs. (2.7a) and (3.13). The proof of (3.12) is completed by noting that the case $n = 0$ is trivial.

The product rule (3.8) can be proved through the use of an extension of the Leibnitz formula, viz:

$$(\vec{\zeta}^n | \vec{\nabla}^n) fg = \sum_{m=0}^{n} \binom{n}{m} [(\vec{\zeta}^m | \vec{\nabla}^m) f][(\vec{\zeta}^{n-m} | \vec{\nabla}^{n-m}) g] \quad , \qquad (3.15)$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ are the binomial coefficients. This expression is also proved by induction. The proof is straightforward, but is too lengthy to be included here. The product rule is then readily proved by forming the series Σfg and using the Leibnitz formula (3.15).

Equations (3.9) are proved by showing that $\Sigma'\Sigma_o$ is the unit operator. Using (2.1c) and (3.12), we obtain

$$\Sigma' \Sigma_o f = \sum_{n=o}^{\infty} \frac{(-)^n}{n!} \left\{ \vec{\zeta}^n(\vec{x}_o) \middle| \vec{\nabla}^n \right\} \Sigma_o f$$

$$= \sum_{n=o}^{\infty} \frac{(-)^n}{n!} \left\{ \vec{\zeta}^n(\vec{x}_o) \middle| \Sigma_o \vec{\nabla}_c^n \right\} f = \sum_{s=o}^{\infty} \sum_{n=o}^{s} \frac{(-)^n}{n!(s-n)!} \left\{ \vec{\zeta}^s(\vec{x}_o) \middle| \vec{\nabla}_o^s \right\} f$$

$$= \sum_{s=o}^{\infty} \frac{1}{s!} \left\{ \sum_{n=o}^{s} \binom{s}{n} (-)^n \right\} \left\{ \vec{\zeta}^s(\vec{x}_o) \middle| \vec{\nabla}_o^s \right\} f = f \; ,$$

as required. In the final step in this proof, we note that the binomial series

$$\sum_{n=o}^{s} \binom{s}{n} (-)^n$$

is equal to unity if $s = 0$, and vanishes otherwise.

Our proof of Eq. (3.10) requires the use of (3.8) and (3.14). In addition, we must assume that the displacement vector $\vec{\zeta}$ can be developed according to Taylor's theorem. That is, if

$$\vec{\zeta}(\vec{x}) = \Sigma_o \vec{\zeta}(\vec{x}_o) \; ,$$

then the quantity $\Sigma f(\vec{x}_o)$ becomes

$$\Sigma f = \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \vec{\zeta}^n(\vec{x}) \middle| \Sigma_o \vec{\nabla}_o^n \Sigma_o^{-1} \right\} f$$

$$= \Sigma_o \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \vec{\zeta}^n(\vec{x}_o) \middle| \vec{\nabla}_o^n \right\} \Sigma_o^{-1} f$$

$$= \Sigma_o f \ ,$$

as stated in (3.10).

If the displacement vector can be developed according to Taylor's theorem, then the Jacobian can be developed as well to give

$$J = \Sigma_o J_o \quad , \quad (\vec{\zeta} \in \mathcal{C}) \ ; \tag{3.16a}$$

$$J_o' = \Sigma' J' \quad , \quad (\vec{\zeta}' \in \mathcal{C}) \ . \tag{3.16b}$$

The former expression is readily obtained by writing the Jacobian J_o as the determinant of the dyadic $\overleftrightarrow{\Phi}_o$ as in (2.10b), and then using (3.8) and (3.12). Equation (3.16b) follows immediately from (3.16a) after a simple change in variables.

IV. LAGRANGE'S THEOREM AND THE LAGRANGE OPERATOR

The extension by Stieltjes[6] of Lagrange's theorem to the N-dimensional case, is based on properties of the derivatives of the Jacobian. The result of Stieltjes can be written

$$f(\vec{x}_o)/J_o = \Omega f(\vec{x}) \ , \tag{4.1}$$

where $\vec{x}$ and $\vec{x}_o$ are related by Eq. (2.1a), the Jacobian J_o is given by Eq. (2.10b), and the Lagrange differential operator Ω is defined by

$$\Omega \equiv \exp\left(- \vec{\nabla} \cdot \vec{\zeta}(\vec{x})\right) \equiv \sum_{n=o}^{\infty} \frac{(-)^n}{n!} \left\{ \vec{\nabla}^n \middle| \vec{\zeta}^n(\vec{x}) \right\} . \tag{4.2}$$

Here, the gradient operators $\vec{\nabla}^n$ differentiate both $\vec{\zeta}^n$ and the operand. The vector notation used in (4.2) is to be interpreted according to the prescription given by Eq. (3.3), so that, for example, the $n = 2$ term in $\Omega f(\vec{x})$ is

$$\tfrac{1}{2}(\partial/\partial x^i)(\partial/\partial x^j) \ \zeta^i \zeta^j \ f(\vec{x}) \ .$$

An interchange in the independent variables transforms the Lagrange theorem (4.1) as follows:

$$f(\vec{x})/J' = \Omega_o' \ f(\vec{x}_o) \ , \tag{4.3}$$

$$\Omega_o' \equiv \exp\left(\vec{\nabla}_o \cdot \vec{\zeta}'(\vec{x}_o)\right) \ , \tag{4.4}$$

where J' is given by (2.10c). Replacement of $f(\vec{x})$ by $J \ f(\vec{x})$

in (4.1), and of $f(\vec{x}_o)$ by $J'_o\, f(\vec{x}_o)$ in (4.3), yields, respectively,

$$f(\vec{x}_o) = \Omega J\, f(\vec{x}) \ , \tag{4.5}$$

$$f(\vec{x}) = \Omega'_o J'_o\, f(\vec{x}_o) \ . \tag{4.6}$$

These equations express the Lagrange theorem in the same form as reported by Good.[8]

We shall also have need of the Lagrange operators

$$\Omega_o \equiv \exp\left(- \vec{\nabla}_o \cdot \vec{\zeta}(\vec{x}_o)\right) \quad , \qquad \Omega' \equiv \exp\left(\vec{\nabla} \cdot \vec{\zeta}'(\vec{x})\right) . \tag{4.7}$$

These operators, together with Ω and Ω'_o , satisfy operator identities analogous to those given in the case of the Taylor operator by Eqs. (3.8 - 3.11). Proof of these identities as point relations is given in Section VI, where more convenient methods are available

V. THE FUNDAMENTAL RELATION BETWEEN THE LAGRANGE AND THE TAYLOR OPERATORS

The Jacobian J, the Taylor operator Σ, and the Lagrange operator Ω [defined respectively by Eqs. (2.10a), (3.6), and (4.2)] satisfy the operator identity:

$$J\Sigma\Omega = 1 \ . \tag{5.1}$$

This is a <u>point relationship</u>. At any point $\vec{x}$ where $f(\vec{x})$ is differentiable, the identity (5.1) indicates that

$$J\Sigma\Omega\, f(\vec{x}) = f(\vec{x}) \ . \tag{5.2}$$

Equation (5.2) is suggested by operating upon the Lagrange theorem (4.1) with the Taylor operator Σ, and interpreting the results according to Taylor's theorem; that is,

$$f(\vec{x}) = \Sigma\, f(\vec{x}_o) = \Sigma J_o \Omega\, f(\vec{x})$$

$$= [\Sigma J_o]\, \Sigma\Omega\, f(\vec{x}) = J\Sigma\Omega\, f(\vec{x}) \;,$$

where we make use of Eqs. (3.8), (3.10), and (3.16a).

The objective of this section is to prove Eq. (5.2) directly without the use of either the Lagrange or the Taylor theorems. Such a direct proof is necessary in order that (5.2) be a point relationship. The proof will be carried out in two steps. First we shall show that

$$\Lambda\, f(\vec{x}) = \lambda\, f(\vec{x}) \;, \tag{5.3}$$

where the operator Λ is defined by

$$\Lambda = \Sigma\Omega \;, \tag{5.4}$$

and λ is the result obtained by operating with Λ upon unity (the unit function). In the second step in the proof of (5.2), we shall show that λ is equal to the inverse of the Jacobian, i.e.,

$$J\lambda = 1 \;. \tag{5.5}$$

The operator Λ can be written as a sum of operators Λ_p, each of order p in the vector $\vec{\zeta}$, as follows:

$$\Lambda = \left\{ \exp(\vec{\zeta}\cdot\vec{\nabla}) \right\} \left\{ \exp(-\,\vec{\nabla}\cdot\vec{\zeta}) \right\} = \sum_{p=o}^{\infty} \Lambda_p \;, \tag{5.6}$$

$$\Lambda_p = \sum_{n=o}^{p} \frac{(-)^n}{(p-n)!\,n!} \left(\vec{\zeta}^{\,p-n} \middle| \vec{\nabla}^{p-n}\right)\left(\vec{\nabla}^{n} \middle| \vec{\zeta}^{\,n}\right) \;. \tag{5.7}$$

Hence, in order to prove Eq. (5.3), it is sufficient to show that

$$\Lambda_p f(\vec{x}) = \lambda_p f(\vec{x}) , \qquad (5.8)$$

where λ_p is equal to $\Lambda_p(1)$. This relation is verified by induction. Under the assumption that (5.8) is valid for $p = 0,1, \ldots , q$, it is shown to be valid for $p = q + 1$. We begin by writing $\Lambda_{q+1} f(\vec{x})$ according to (5.7). By adjusting the coefficients and the summation indices in the resulting series, we obtain

$$\Lambda_{q+1} f = - \frac{1}{q+1} \sum_{n=0}^{q} \frac{(-)^n}{(q-n)!n!} (\vec{\zeta}^{q-n}|\vec{\nabla}^{q-n})(\vec{\nabla}^{n+1}|\vec{\zeta}^{n+1}) f$$

$$+ \frac{1}{q+1} \sum_{n=0}^{q} \frac{(-)^n}{(q-n)!n!} (\vec{\zeta}^{q+1-n}|\vec{\nabla}^{q+1-n})(\vec{\nabla}^{n}|\vec{\zeta}^{n}) f . \qquad (5.9)$$

These two series can be transformed with the help of

$$(\vec{\nabla}^{n+1}|\vec{\zeta}^{n+1}) f = \sum_{m=0}^{n} \frac{n!}{(n-m)!} (\vec{\nabla}^{n-m}|\vec{\zeta}^{n-m}) \vec{\nabla} \cdot (f\vec{\zeta} \cdot [\vec{\nabla}\vec{\zeta}]^m) , \qquad (5.10)$$

$$\vec{\nabla}(\vec{\nabla}^{n}|\vec{\zeta}^{n}) f = \sum_{m=0}^{n} \frac{n!}{(n-m)!} (\vec{\nabla}^{n-m}|\vec{\zeta}^{n-m}) \vec{\nabla} \cdot (f[\vec{\zeta}\vec{\nabla}]^m) . \qquad (5.11)$$

The two latter expressions are derived by performing successive differentiations so as to form the $(m = 0)$ terms in the series on the right, together with remainder terms which are then differentiated to form the $(m = 1)$ terms, etc. In Eq. (5.11), the dyadic $\vec{\zeta}\vec{\nabla}$ is the transpose of $\vec{\nabla}\vec{\zeta}$.

Substitution of (5.10 -11) into (5.9) yields

$$\Lambda_{q+1} f = - \frac{1}{q+1} \sum_{m=o}^{q} (-)^m \Lambda_{q-m} \vec{\nabla} \cdot (f\vec{\zeta} \cdot [\vec{\nabla} \zeta]^m)$$

$$+ \frac{1}{q+1} \sum_{m=o}^{q} (-)^m \vec{\zeta} \cdot \left\{ \Lambda_{q-m} \vec{\nabla} \cdot (f[\vec{\zeta} \vec{\nabla}]^m) \right\}$$

Equation (5.8) can be used in this result to give

$$\Lambda_{q+1} f = - \frac{1}{q+1} \sum_{m=o}^{q} (-)^m \lambda_{q-m} \left[\vec{\nabla} \cdot (f\vec{\zeta} \cdot [\vec{\nabla} \zeta]^m) - \vec{\zeta} \cdot \left\{ \vec{\nabla} \cdot (f[\vec{\zeta} \vec{\nabla}]^m) \right\} \right] .$$

The factor enclosed by the brackets [...] may be simplified by differentiation, so that

$$\Lambda_{q+1} f = \frac{1}{q+1} f \sum_{m=o}^{q} D_{m+1} \lambda_{q-m} , \qquad (5.12)$$

where D_{m+1} is defined in accordance with Eq. (2.14).

In the special case where the function f is chosen equal to unity, Eq. (5.12) becomes

$$(q+1) \lambda_{q+1} = \sum_{m=o}^{q} D_{m+1} \lambda_{q-m} . \qquad (5.13)$$

This result can be combined with (5.12) to give Eq. (5.8) for the case $p = q + 1$ as required.

Equation (5.5) will also be proved valid order by order in the vector $\vec{\zeta}$. Through the use of Eq. (2.12) for the Jacobian and the expansion for λ derived by operating upon the unit function with Λ

according to Eq. (5.6), we obtain

$$J\lambda = \sum_{k=o}^{N} G_k \sum_{s=o}^{\infty} \lambda_s$$

$$= \sum_{m=o}^{N} \sum_{k=o}^{m} G_k \lambda_{m-k} + \sum_{m=N+1}^{\infty} \sum_{k=o}^{N} G_k \lambda_{m-k} \tag{5.14}$$

where $G_o \equiv 1$, and the terms in the sums over the index m are of m^{th} order in the vector $\vec{\zeta}$. Hence Eq. (5.5) is proved if it can be shown that

$$G_o \lambda_o = 1 \ , \tag{5.15}$$

$$\sum_{k=o}^{m} G_k \lambda_{m-k} = 0 \quad , \quad (m = 1,2, \ldots , N) \ , \tag{5.16}$$

$$\sum_{k=o}^{N} G_k \lambda_{p+N-k} = 0 \quad , \quad (p = 1,2, \ldots , \infty) \ . \tag{5.17}$$

Equation (5.15) is satisfied identically, while (5.16 - 17) can be proved by using Eqs. (2.13 - 14) and (5.13). In our proof of (5.17), we shall also use the Cayley-Hamilton theorem.

To prove Eq. (5.16), we consider the series

$$m \sum_{k=o}^{m} G_k \lambda_{m-k} = \sum_{k=o}^{m-1} G_k (m - k) \lambda_{m-k} + \sum_{k=1}^{m} kG_k \lambda_{m-k} \ . \tag{5.18}$$

Substitution of (5.13) into the first series on the right-hand side of

(5.18) yields

$$\sum_{k=0}^{m-1} G_k \sum_{r=0}^{m-k-1} D_{r+1} \lambda_{m-k-1-r} = \sum_{k=0}^{m-1} G_k \sum_{s=k+1}^{m} D_{s-k} \lambda_{m-s}$$

$$= \sum_{s=1}^{m} \left(\sum_{k=0}^{s-1} G_k D_{s-k} \right) \lambda_{m-s} = - \sum_{s=1}^{m} s \, G_s \lambda_{m-s} \, ,$$

so that the first series on the right-hand side of (5.18) is the negative of the second series. Hence Eq. (5.16) is proved.

Our proof of (5.17) proceeds similarly. Corresponding to (5.18), we shall show that the right-hand side of

$$(p + N) \sum_{k=0}^{N} G_k \lambda_{p+N-k} = \sum_{k=0}^{N} G_k (p + N - k) \lambda_{p+N-k} + \sum_{k=1}^{N} kG_k \lambda_{p+N-k} \qquad (5.19)$$

vanishes. The first series on the right is also transformed by using (5.13) as follows:

$$\sum_{k=0}^{N} G_k \sum_{r=0}^{p+N-k-1} D_{r+1} \lambda_{p+N-k-1-r} = \sum_{k=0}^{N} G_k \sum_{s=k+1}^{p+N} D_{s-k} \lambda_{p+N-s}$$

$$= \sum_{s=1}^{N} \left(\sum_{k=0}^{s-1} G_k D_{s-k} \right) \lambda_{p+N-s} + \sum_{s=N+1}^{p+N} \left(\sum_{k=0}^{N} G_k D_{s-k} \right) \lambda_{p+N-s} \, .$$

In the final form, the second series vanishes since each term in the sum over the index s contains the null factor given by (2.17), while the first series is simplified by using (2.13 - 14) and becomes equal to the negative of the second series on the right-hand side of (5.19). This completes our proof of (5.1 - 2).

It is clear at this point that the corresponding identities

$$J_o \, \Sigma_o \, \Omega_o = 1 \quad , \quad J' \, \Sigma' \, \Omega' = 1 \quad , \quad J_o' \, \Sigma_o' \, \Omega_o' = 1 \ , \tag{5.20}$$

are also valid.

VI. OTHER RELATIONS SATISFIED BY THE TAYLOR AND LAGRANGE OPERATORS

To the fundamental relation discussed in the foregoing section we may add two similar ones. These are obtained by cyclic permutation of J, Σ, and Ω as follows:

$$J\Sigma\Omega = \Sigma\Omega J = \Omega J\Sigma = 1 \ . \tag{6.1}$$

In the permutation group of three objects, there are altogether six elements. Three elements have appeared in Eqs. (6.1). Of the remaining three, the operators $\Omega\Sigma J$ and $J\Omega\Sigma$ are equal to each other. This can be shown by considering the operator $\Omega\Sigma$. Since

$$\Omega\Sigma = \Omega J J^{-1}\Sigma = \Sigma^{-1}J^{-1}\Sigma \ ,$$

according to Eq. (6.1), and since

$$\Sigma^{-1}J^{-1}\Sigma = [\Sigma^{-1}J^{-1}] \ \Sigma^{-1}\Sigma = [\Sigma^{-1}J^{-1}]$$

according to the product rule for the Taylor operator, it follows that

$$\Omega\Sigma = [\Sigma^{-1}J^{-1}] \ .$$

Hence

$$\Omega\Sigma J = J\Omega\Sigma = [J\Sigma^{-1}J^{-1}] \ . \tag{6.2}$$

An operator enclosed by the square brackets [...] denotes the functional value of the operator, i.e., the result obtained when the operand is chosen equal to unity.

The remaining element $\Sigma J\Omega$, which incidentally may be taken to be the identity element of the permutation group, is equal to a scalar function different from all the other elements:

$$\Sigma J\Omega = [\Sigma J]\ \Sigma\Omega = [\Sigma J]\ J^{-1}$$

$$= [J^{-1}\Sigma J] \ . \tag{6.3}$$

All six elements are functional multipliers when they operate upon any operand. Let X_i be any of these six elements and let $f(\vec{x})$ be any permissible operand. The result obtained by operating upon $f(\vec{x})$ with x_i is equal to the product of $f(\vec{x})$ and the functional value $[X_i]$, i.e.,

$$X_i\ f(\vec{x}) = f(\vec{x})[X_i] \ .$$

Because of this, the Taylor and the Lagrange operators may be said to be reciprocal operators within a functional factor.

Each of the relations (6.1 - 3) has its inverse. The inverse relations are

$$\Omega^{-1}\Sigma^{-1}J^{-1} = J^{-1}\Omega^{-1}\Sigma^{-1} = \Sigma^{-1}J^{-1}\Omega^{-1} = 1 \ , \tag{6.1a}$$

$$J^{-1}\Sigma^{-1}\Omega^{-1} = \Sigma^{-1}\Omega^{-1}J^{-1} = [J^{-1}\Sigma^{-1}J] \ , \tag{6.2a}$$

$$\Omega^{-1}J^{-1}\Sigma^{-1} = [J\Sigma J^{-1}] \ . \tag{6.3a}$$

The Jacobian J may be expressed exclusively as the functional value of the inverse Lagrange operator. From $J = \Omega^{-1}\Sigma^{-1} = [\Omega^{-1}\Sigma^{-1}]$ and $[\Sigma^{-1}] = 1 = [\Sigma]$, we obtain

$$J = [\Omega^{-1}] \ . \tag{6.4}$$

The inverse of Eq. (6.4) is

$$J^{-1} = [\Omega^{-1}]^{-1} \; . \tag{6.4a}$$

Here, it should be noted that $[\Omega] \neq [\Omega^{-1}]^{-1}$, and $[\Omega]^{-1} \neq [\Omega^{-1}]$, but

$$[\Sigma^{-1}J^{-1}] = [\Omega] \; , \tag{6.5}$$

$$[\Sigma^{-1}J] = [\Omega]^{-1} . \tag{6.5a}$$

Many operator relations attain a more symmetrical form when $[\Omega]$ and $[\Omega^{-1}]$ are introduced. Consider the inverse Taylor operator Σ^{-1}. Since

$$\Sigma^{-1} = \Sigma^{-1}JJ^{-1} = [\Sigma^{-1}J] \, \Sigma^{-1}J^{-1} \; ,$$

we have, according to Eqs. (6.1a) and (6.5a),

$$\Sigma^{-1} = \frac{\Omega}{[\Omega]} \; . \tag{6.6}$$

Similarly,

$$\Sigma = \frac{\Omega^{-1}}{[\Omega^{-1}]} \; . \tag{6.6a}$$

These two equations exhibit the reciprocity between Σ and Ω most clearly. The product rule for the Lagrange operator is obtained by direct substitution of Eq. (6.6) into the product rule for the inverse Taylor operator to give

$$\frac{\Omega fg}{[\Omega]} = \frac{\Omega f}{[\Omega]} \cdot \frac{\Omega g}{[\Omega]} \; . \tag{6.7}$$

In addition to the reciprocal nature of Σ and Ω, we can show that Σ and Ω are adjoint operators.* The adjoint property is established as follows:

$$f\Omega g = f\Sigma^{-1}J^{-1}g = [\Omega]\ f\Sigma^{-1}g$$

$$= [\Omega]\ \Sigma^{-1}g\Sigma f = \Omega g\Sigma f.$$

Hence,

$$f\Omega g - g\Sigma f = (\Omega - 1)\ g\Sigma f \tag{6.8}$$

$$= \vec{\nabla}\cdot\left[\sum_{n=1}^{\infty}\frac{(-)^n}{n!}\vec{\nabla}^{n-1}\Big|\vec{\zeta}^{n}g\Sigma f\right] \tag{6.9}$$

Since the right-hand side of (6.8) can be expressed as the divergence of a vector, the adjoint property is verified. The inverse operators Σ^{-1} and Ω^{-1} are shown to be adjoint operators in a similar manner, i.e.,

$$f\Omega^{-1}g - g\Sigma^{-1}f = -(\Omega - 1)\ f\Omega^{-1}g \tag{6.8a}$$

$$= -\vec{\nabla}\cdot\left[\sum_{n=1}^{\infty}\frac{(-)^n}{n!}\vec{\nabla}^{n-1}\Big|\vec{\zeta}^{n}f\Omega^{-1}g\right]. \tag{6.9a}$$

It is clear that all of the foregoing formulas in this section with the exception of (6.9) and (6.9a) are valid if J, Σ, and Ω are replaced respectively by J_o, Σ_o, and Ω_o, or by J', Σ', and Ω', etc. Furthermore, all of these expressions are point relations derived from operator identities, including the theorem $J\Sigma\Omega = 1$. The proof of this latter identity, as given in Section V, depends neither on the Taylor theorem nor on the Lagrange theorem. We shall now use the fundamental theorem (6.1) to show that the Lagrange theorem follows from the assumption that Taylor's theorem is valid, and vice versa.

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*See, for example: Courant & Hilbert,[22] Vol. II, p. 235.

Taylor's theorem can be written:

$$f(\vec{x}_o) = \Sigma_o^{-1} f(\vec{x}) = \Sigma^{-1} f(\vec{x}) ,$$

or:

$$f(\vec{x}) = \Sigma_o f(\vec{x}_o) = \Sigma f(\vec{x}_o) .$$

Equations (6.1 - 2) may be used in these two expressions to give, respectively:

$$f(\vec{x}_o) = \Sigma^{-1} f(\vec{x}) = \Omega J f(\vec{x}) ,$$

$$\Omega f(\vec{x}) = \Omega\Sigma f(\vec{x}_o) = [\Sigma^{-1}J^{-1}] f(\vec{x}_o) = J_o^{-1} f(\vec{x}_o) .$$

In turn, these relations are identified as the Lagrange theorems given respectively by Eqs. (4.5) and (4.1). Proof of the converse assertion proceeds in a similar manner.

There are also operator identities connecting Ω_o, Ω, and Ω', that are similar to those connecting the corresponding Taylor operators. Corresponding to Eq. (3.9), we have

$$\Omega_o\Omega' = [\Omega_o][\Omega'^{-1}]^{-1} = [\Omega_o][\Omega_o^{-1}] , \qquad (6.10a)$$

$$\Omega'\Omega_o = [\Omega'][\Omega_o^{-1}]^{-1} = [\Omega'][\Omega'^{-1}] ; \qquad (6.10b)$$

and corresponding to Eqs. (3.10 - 11), because of (6.1) and (6.6), we have

$$\Omega J = \Omega_o J_o \quad , \quad \Omega' J' = \Omega_o' J_o' \quad , \quad (\zeta, \zeta' \in \mathcal{C}) ; \qquad (6.11)$$

$$\frac{\Omega}{[\Omega]} = \frac{\Omega_o}{[\Omega_o]} \quad , \quad \frac{\Omega'}{[\Omega']} = \frac{\Omega_o'}{[\Omega_o']} \quad , \quad \frac{\Omega^{-1}}{[\Omega^{-1}]} = \frac{\Omega_o^{-1}}{[\Omega_o^{-1}]} \quad , \quad (\zeta, \zeta' \in \mathcal{C}) . \qquad (6.12)$$

In addition, we have

$$\Omega' = \Sigma'^{-1} J'^{-1} = \Sigma_o J_o = [\Sigma_o J_o] \Sigma_o = J\Sigma_o = J\Sigma .$$

Hence,

$$\Omega'\Omega = 1 = \Omega\Omega' \ , \ (\zeta,\zeta' \in \mathcal{C}) \ . \qquad (6.13)$$

Equations (6.12) can also be written

$$\Omega = \Sigma_o \Omega_o \Sigma_o^{-1} \quad , \quad \Omega^{-1} = \Sigma_o \Omega_o^{-1} \Sigma_o^{-1} \quad , \quad (\zeta \in \mathcal{C}) \ . \qquad (6.14)$$

Thus the rule for transforming Ω is the same as the rule for transforming $\vec{\nabla}^n$ given by Eq. (3.14).

VII. APPLICATIONS

Lagrange's theorem has been applied by Laplace to problems in celestial mechanics, and by Jacobi to the theory of Legendre polynomials. References to these works are listed in the article by Osgood.[14] More recently, Good[8] has used Lagrange's theorem in the theory of stochastic processes.

In many areas of applied mathematics, the Jacobian and Taylor's theorem are used together. Problems in these areas can often be discussed more conveniently upon introduction of the Lagrange operator Ω defined in Section IV. The fundamental operator relation $J\Sigma\Omega = 1$, given in Section V, provides the required connection between the Lagrange operator, the Taylor operator, and the Jacobian. Two topics in which this procedure is particularly useful are discussed below.

The density of a strained elastic medium has been calculated by Cauchy[23] in terms of the displacement vector and the density of the medium in the unstrained state. In this work, use was made of Taylor's theorem and the Jacobian; the result is given through first-order terms in the displacement.

With the help of the $J\Sigma\Omega$ theorem, this problem can be solved in terms of the Lagrange operator. The result is correct to all orders in the displacement vector. Let $\rho_o(\vec{x}_o)$ be the density of matter in the unstrained or unperturbed state, and $\rho(\vec{x}_o)$ be the density in the corresponding perturbed state. The small volume of matter initially located at $\vec{x}_o$ is, in the perturbed configuration, located at $\vec{x}$, which is considered to be related to $\vec{x}_o$ by the displacement vector $\vec{\zeta}$ as in Eq. (2.1a). At the perturbed position $\vec{x}$, the density in the perturbed configuration is given by $\rho(\vec{x}) = J_o^{-1} \rho_o(\vec{x}_o)$, according to the law of conservation of matter. With the help of Taylor's theorem and the fundamental relation (6.1), we transform this conservation law as follows:

$$\rho_o(\vec{x}_o) = J_o \Sigma_o \rho(\vec{x}_o) = \Omega_o^{-1} \rho(\vec{x}_o) \ .$$

Hence

$$\rho(\vec{x}_o) = \Omega_o \rho_o(\vec{x}_o) \ . \tag{7.1}$$

Through first-order terms in the displacement, this is the result given by Cauchy.[23] The second-order terms of (7.1) have been calculated by Chu[19] and by Sturrock;[9] it is also clear in both these works how to proceed to any desired order. Equation (7.1) is derived by Dedrick & Wilson[10] with the aid of an integral theorem that can be shown to be equivalent to the adjoint property given by Eqs. (6.8 -9). Fourier analysis is used by Sturrock[9] and by Dedrick & Wilson[10] in their treatments of this problem. An advantage gained in the method given here is that Fourier analysis is unnecessary.

The theory of infinitesimal transformations has been applied successfully in many fields. Of particular note is the use of this method in deriving conservation laws and the equations of motion in the theory of relativity (see, e.g., Pauli,[24] Weyl[25]). For example, the equations of motion are obtained by setting the variation of the action integral equal to zero in accordance with Hamilton's principle. The variation is expressed in terms of the infinitesimal transformation of coordinates, which in turn is written in the form of the transformation (2.1a) for the case of four dimensions. In certain perturbation calculations however, the displacement vector may not be considered infinitesimal. The $J\Sigma\Omega$ theorem enables us to examine the effects of a perturbation order by order in a convenient manner. Consider, for example, the volume integral of a function $F(\vec{x})$, which may be a scalar, vector, or a tensor. Through the use of (6.1), we find

$$\int_V d\tau\, F(\vec{x}) = \int_{V_o} d\tau_o J_o \Sigma_o\, F(\vec{x}_o) = \int_{V_o} d\tau_o \Omega_o^{-1}\, F(\vec{x}_o)$$

$$= \int_{V_o} d\tau_o\, F(\vec{x}_o) + \int_{V_o} d\tau_o\, (\Omega_o^{-1} - 1)\, F(\vec{x}_o)\ . \qquad (7.2a)$$

The second integral on the right-hand side of (7.2a) may be transformed into a surface integral over the closed surface bounding the volume V_o because the integrand is the divergence of a vector or a tensor.

Similarly,

$$\int_{V_o} d\tau_o\, F(\vec{x}_o) = \int_V d\tau\, J'\Sigma'\, F(\vec{x}) = \int_V d\tau\, \Omega\, F(\vec{x})$$

$$= \int_V d\tau\, F(\vec{x}) + \int_V d\tau\, (\Omega - 1)\, F(\vec{x})\ . \qquad (7.2b)$$

We conclude from Eqs. (7.2) that finite (rather than infinitesimal) perturbation in a volume integral can be written as a surface integral. This important property of the operator $J\Sigma = \Omega^{-1}$ has been used by Chu[26] in the perturbation theory of classical electrodynamics.

LIST OF REFERENCES

1. J. L. Lagrange, "Nouvelle méthode pour résoudre les equations littérales par moyen des séries," Mém. de l'Acad. de Berlin 24, 251 (1770) = Oeuvres 3, 5-73 (see p. 25).

2. P. S. Laplace, "Mémoire sur l'usage du calcul aux différences partielles dans la théorie des suites," Mém. de l'Acad. roy. de Sci. de Paris, ann. 1777-1780 = Oeuvres 9, 313-335 (see p. 330).

3. M. G. Darboux, "Sur la série de Laplace," C. R. Acad. Sci. Paris 68, 324-327 (1869).

4. B. Baillaud & H. Bourget, Ed., Correspondance d'Hermité et de Stieltjes (Gauthier-Villars, Paris, 1905). (See letters 68-70, pp. 142-146.)

5. H. Poincaré,"Sur les résidus des intégrales doubles,"Acta Math. 9, 321-380 (1887); (see pp. 323, 357-362) = Oeuvres 3, 440.

6. T. J. Stieltjes, "Sur une généralisation de la série de Lagrange," Ann. Sci. Éc. norm., Paris (3) 2, 93-98 (1885) = Oeuvres 1, 445-450.

7. E. Heine,"Lagrange's Umkehrungsformel," J. Reine Angew. Math. 54, 388 (1857).

8. I. J. Good, "Generalizations to Several Variables of Lagrange's Expansion, with Applications to Stochastic Processes," Proc. Cambridge Phil. Soc. 56, 367-380 (1960).

9. P. A. Sturrock, "Generalization of the Lagrange Expansion with Application to Physical Problems," J. Math. Phys. 1, 405-408 (1960).

10. K. G. Dedrick & R. N. Wilson, "Use of the Displacement Vector in Electromagnetic Theory," J. Math. Phys. 3, 97-105 (1962).

11. E. T. Whittaker & G. N. Watson, A Course of Modern Analysis, 4th Edn. (Cambridge University Press, 1958) [see p. 133].

12. G. Pólya & G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. I. (J. Springer, Berlin, 1925) [see pp. 125, 300].

13. E. Goursat, A Course in Mathematical Analysis, Vols. I & II, trans. by E. R. Hendrick and O. Dunkel (Ginn and Company, Boston, 1904-1917). [The one-dimensional Lagrange theorem is given in Vol. I, Sec. 189, p. 404, and in Vol. II, Part I, Sec. 51, p. 106. The two-dimensional theory is given in Vol. II, Part I, Sec. 104, p. 250.]

14. W. F. Osgood, "Analytische Funktionen komplexer Grössen," in Encyklopädie der Mathematischen Wissenschaften, Band II, Teil 2 (Teubner, Leipzig, 1901-1921) [see pp. 44-48].

15. J. W. Gibbs & E. B. Wilson, Vector Analysis (Yale University Press, New Haven, 1901).

16. U.J.J. Leverrier, a) "Sur les variations séculaires des éléments elliptiques des sept planètes principales: Mercure, Vénus, la Terre, Mars, Jupiter, Saturne et Uranus," J. de Math. (1) 5, 220-254 (1840). b) Enlarged in Connaissance des Temps pour l'an 1843, Additions 1-66 (Paris, 1840).

17. F. R. Gantmacher, The Theory of Matrices, Vols. I & II, trans. from the Russian by K.A. Hirsch (Chelsea Publishing Company, New York, 1960). [See Vol. I, p. 87.]

18. E. Bodewig, Matrix Calculus (North-Holland Publishing Company, Amsterdam, 1956).

19. E. L. Chu, "On the Concept of Ficticious Surface Charges of an Electron Beam," J. Appl. Phys. 31, 381-388 (1960).

20. A. L. Cauchy, Le Calcul Infinitesimal 1 (Paris, 1823) = Oeuvres (2) 4, 82-87.

21. A.E.H. Love, art. "Infinitesimal Calculus," in Enc. Brit., 11th Ed. (New York, 1910). [See Vol. 14, pp. 547-548.]

22. R. Courant & D. Hilbert, Methods of Mathematical Physics, Vols. I & II (Interscience Publishers, New York, 1962).

23. A. L. Cauchy, "Sur les équations qui expriment les conditions d'équilibre ou les lois du mouvement intérieur d'un corps solide, élastique ou non élastique," Ex. Math. 3 = Oeuvres (2) 8, 195-226. [See Eq. (47), p. 210.]

24. W. Pauli, Theory of Relativity (Pergamon Press, Ltd., London, 1958). [See pp. 64-70.]

25. H. Weyl, Space-Time-Matter, 4th Ed. (Methuen & Co., London, 1922); Chap. IV.

26. E. L. Chu, "The Lagrangian and the Energy-Momentum Tensors in the Perturbation Theory of Classical Electrodynamics," Ann. Phys. 9, 76-92 (1960).