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MISALIGNMENT AND QUADRUPOLE ERROR PROBLEMS AFFECTING THE CHOICE OF MULTIPLET TYPE FOR SECTOR FOCUSING OF THE TWO-MILE ACCELERATOR

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by

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I. INTRODUCTION AND SUMMARY

A. GENERAL REMARKS

In a previous report,[1] some misalignment and quadrupole error problems were considered with emphasis on a system consisting of quadrupoles of essentially equal strength, alternating sign, and equal spacing of nominally 40 feet along the accelerator.

The present report presents similar calculations for several quadrupole multiplet combinations spaced at sector intervals. Four representative cases - the "Spaced Doublet" (SD), "Close Doublet" (CD), "Spaced Triplet" (ST) and "Close Triplet" (CT) - are considered (see Fig. 1.1). The optical properties of these lenses in periodic systems are summarized in a previous report.[2]

The object of the present discussion principally is to compare these alternative systems in such a way as to provide a basis for deciding which would be more favorable for use in the two-mile accelerator. Consequently the emphasis is on errors whose effects depend on the structure of the multiplet, although some structure-independent effects are included for completeness.

Stray magnetic fields, gross misalignments of the accelerator axis and rf field asymmetries specifically are excluded from the discussion, since beam perturbations by such effects are expected to be nearly independent of the multiplet type.

B. ERROR COMPONENTS

The various sorts of mechanical errors are illustrated schematically in Fig. 1.2. Briefly, the errors which are considered in this report are:

1. Parallel displacement of the optic axis of a multiplet from a fixed reference axis.

2. Skew, or rotation about a transverse axis
   (a) of a multiplet as a whole or
   (b) of the individual quadrupoles.

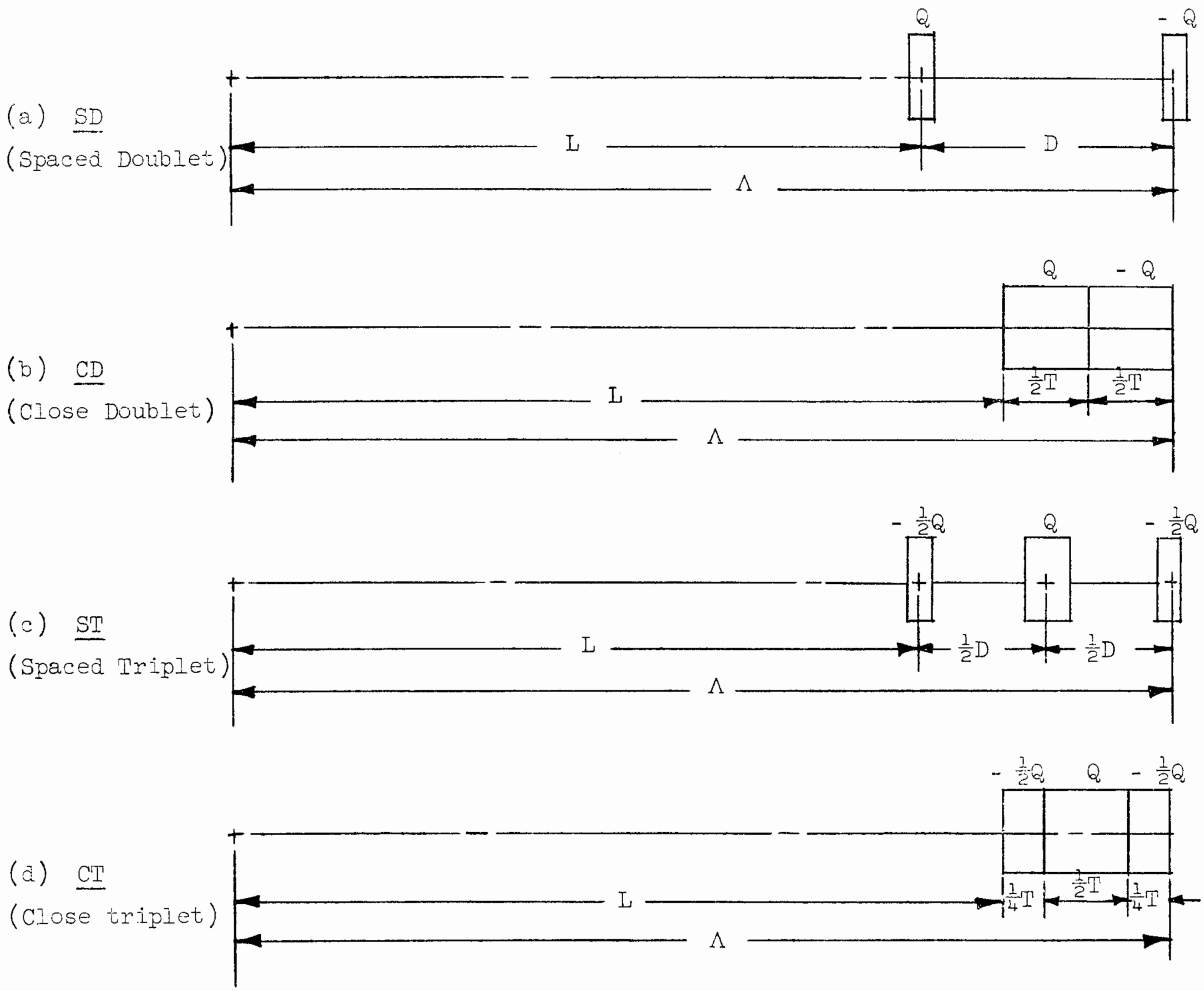
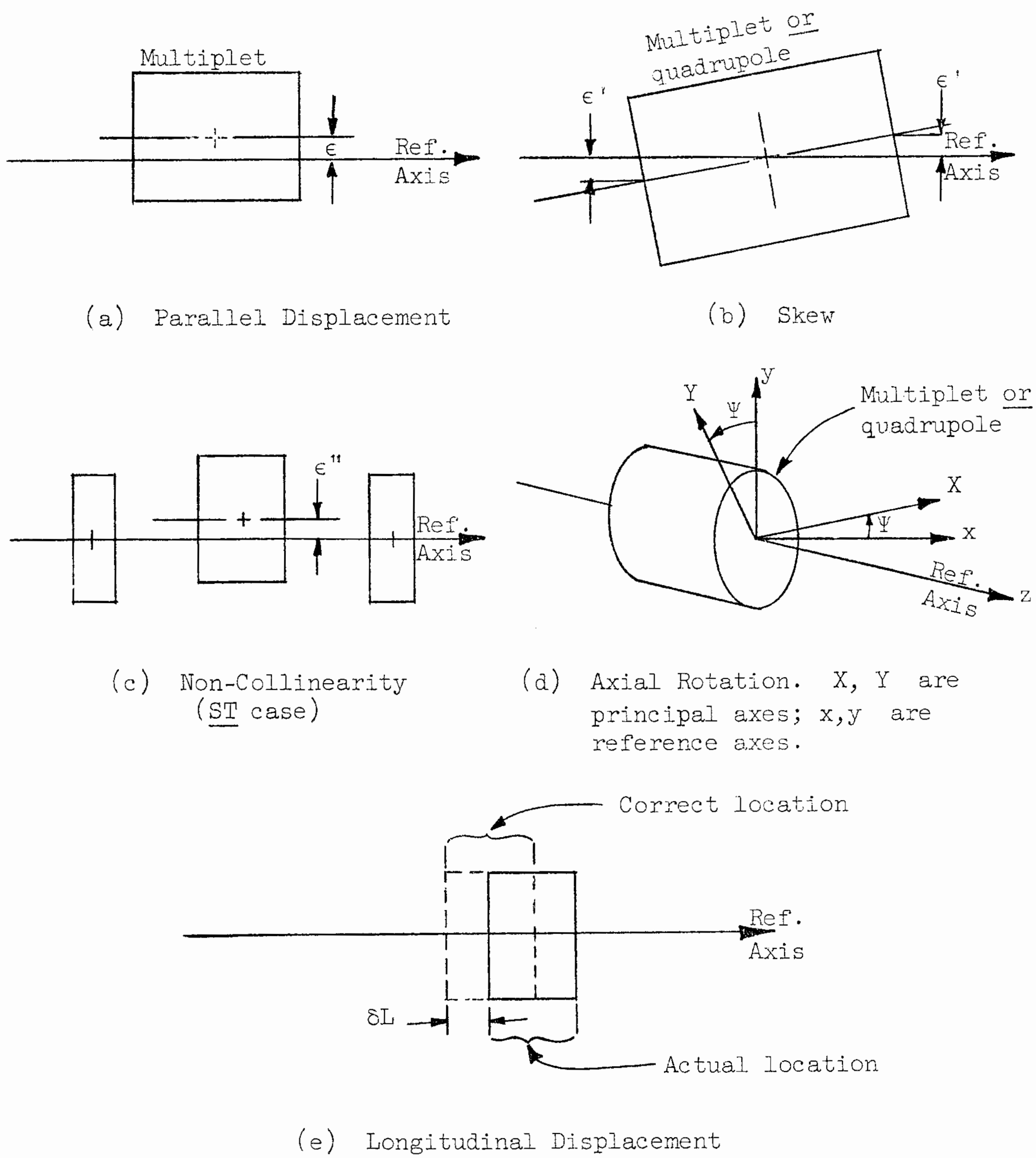


FIG. 1.1--Schematic representation of multiplet types. Λ is the spacing period, or one sector.



(a) Parallel Displacement

(b) Skew

(c) Non-Collinearity (ST case)

(d) Axial Rotation. X, Y are principal axes; x,y are reference axes.

(e) Longitudinal Displacement

FIG. 1.2--Illustrating various components of misalignment.

3. Non-collinearity of the quadrupole centers, in the triplet combinations.

4. Axial rotation or rotation about the longitudinal reference axis
   (a) of the multiplet as a whole or
   (b) of the individual quadrupoles relative to one another.

5. Longitudinal displacement
   (a) of the multiplet as a whole or
   (b) of the individual quadrupoles relative to one another.

6. Quadrupole Strength Errors (not illustrated)
   (a) in a multiplet as a whole (e.g., regulation of a common power supply)
   (b) in the individual quadrupoles (e.g., mechanical uniformity).

Practically any quadrupole misalignment may be represented as a linear combination of effects (1) through (5).

For purposes of setting tolerances on the various listed error components, it is assumed

(1) that each of the components may be treated independently, as if the others were absent;

(2) that each of the components may be represented as an independent random variable;

(3) that one may choose a straight longitudinal reference axis which is essentially the accelerator axis.

C. NUMERICAL RESULTS

In order to assign tolerances to the various errors for the purpose of comparing the different multiplet types, a consistent set of numerical values of the parameters will be adopted. The following points are pertinent:

(1) As a result of the present study (Sections II through V, following), it is found that all of the error components considered here have their strongest orbit perturbing effect at the lowest beam energies.

(2) In a previous study of the optical properties of periodic multiplet systems[2] it was shown

(a) that all the systems considered here are essentially equivalent optically, in the sense of having nearly equal admittance, if they have the same value of γ_c, where γ_c is the low-energy cutoff of the periodic system;

(b) that a reasonable choice for the practical minimum energy is given by

$$\gamma_{min} = \gamma_c \sqrt{2}$$

(3) As a consequence of point (2), above, the parameters of the system must be essentially constant as a function of focusing period (sector) number, in order to maintain broad-band transmission.

(4) As a consequence of points (1) and (3), above, it is sufficient for purposes of setting tolerances, to consider the situation of constant parameters and constant (minimum) beam energy.

The "standard" set of numerical values now may be listed:

Minimum beam energy;*

$$\gamma_{min} = \gamma_c \sqrt{2}$$

Number of focusing sections;

$$n = 15$$

(corresponds to transmission of the beam at minimum energy through half the machine).

Maximum allowable orbit perturbation (or beam deflection);

$$|\xi|_{max} = 0.1 \text{ cm} \approx 0.040 \text{ inches.}$$

Maximum possible orbit amplitude;

$$|X|_{max} = |Y|_{max} = \text{accelerator hole radius} \approx 1 \text{ cm.}$$

Length of standard focusing period;

$$\Lambda = 333\tfrac{1}{3} \text{ ft} = 4000 \text{ inches.}$$

*The important parameter turns out to be γ_c/γ in all cases.

Maximum multiplet spacing (Spaced-Doublet and Spaced Triplet combinations);

$$D = 80 \text{ inches}$$

(corresponds to approximate spacing available in sector drift spaces of nominal 108 inches length).

Maximum total length of quadrupole per sector;

$$T = 30 \text{ inches}$$

(arbitrarily imposed to provide adequate space for other instrumentation in drift spaces).

On the basis of these numbers and the formulae derived in Sections II through V, the various tolerances have been calculated and are listed in Table I.

D. EVALUATION AND CONCLUSIONS

The following comments apply to the tolerance figures listed in Table I:

1. The most critical errors are

   (a) Parallel displacement of the multiplet support (all cases);

   (b) Skew (rotation about a transverse axis) of the common support, in the Doublet combinations;

   (c) Collinearity of the optical centers of the three quadrupoles, in the Triplet combinations;

   (d) Relative axial rotations (about the longitudinal axis) particularly in the close-spaced configurations.

2. Because of such effects as earth movements, the parallel displacement tolerance ($\approx$ .0073 inch) probably cannot be held over long periods of time.

3. The skew tolerance ($<$ .002 inch relative transverse motion of the ends of the Doublet) probably cannot be met by mechanical alignment techniques.

4. For Triplets, the non-collinearity tolerances ($<$ .001 inch), probably cannot be attained by prealignment techniques.

5. The tolerances on relative axial rotation, although critical, probably can be attained in prealignment and should be expected to remain

TABLE I--Typical Error Tolerances for Periodic Multiplet Focusing for the Two-Mile Accelerator

Type of Error [a]	Designation	Eq. Ref.	Notes	RMS Tolerances [b,c] SD	CD	ST	CT
Parallel Displacement	ϵ	(2-12)		7.3 mils	7.3 mils	7.3 mils	7.3 mils
Skew: (1) Common	ϵ'	(2-16)	(d)	0.73 mils	0.52 mils	73 mils	138 mils
(2) Independent	e'	(2-21)	(e)	(large)	41 mils	(large)	83 mils
Collinearity	ϵ''	(2-25)				0.73 mils	0.26 mils
Axial Rotation: (1) Common	ψ	(3-10)	(d)	5.2°	(large)	(large)	(large)
(2) Independent	$\delta\psi$	(3-15)	(e)(f)	0.15°	0.052°	0.10°	0.037°
Longitudinal Displacement: (1) Common	δL	(4-7)	(d)	6 ft	6 ft	6 ft	6 ft
(2) Independent	δD	(4-11)	(e)	2.1 in.	0.26 in.	1.5 in.	0.18 in.
Quadrupole Strength: (1) Common	$\delta I/I$	(5-6)	(d)	1.3%	1.3%	1.3%	1.3%
(2) Independent	$\delta Q/Q$	(5-11)	(e)	0.37%	0.13%	0.21%	0.075%

Notes:

a. See Section I.B and Eq. References for definitions and other details.

b. Computed for typical numerical values of parameters as listed in Section I.C.

c. Units as stated (1 mil = .001 inch).

d. "Common" means errors which apply to the multiplet as a unit.

e. "Independent" means errors in individual quadrupoles, relative to one another.

f. For Doublets, $\delta\psi$ denotes rotations of one quadrupole relative to the other; for Triplets, $\delta\psi$ denotes (independent) rotations of the two outer quadrupoles relative to the center one.

satisfactory thereafter.

6. Because of points 2, 3, and 4, above, it is evident that two degrees of freedom of compensation are needed in both the **x** and **y** planes, for <u>either</u> Doublets or Triplets.* These corrections could be effected either by appropriate mechanical motions or by dipole biasing of the quadrupoles, but in either case four independent adjustments per multiplet are required.

7. The skew tolerances for Doublets imply, in the SD case, an angular misalignment tolerance of ±.73 mil in 80 inches, or 1.8×10^{-5} radian; in the CD case, the figure is ±.52 mil in 30 inches or 3.5×10^{-5} radian. Angular misalignments of this magnitude may arise from earth movements, or even from temperature differences between the support jacks at the ends of the 9-ft drift sections. Consequently, frequent readjustment of the skew compensation might be necessary with Doublets.

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*It should be mentioned that the orbit perturbation, ξ, has different energy dependence for the different types of errors; thus,

For parallel displacement; $$\overline{\xi^2} \propto \frac{(\gamma_c/\gamma)^2}{1 - (\gamma_c/\gamma)^2}$$ [Eq. (2-11)]

For Doublet skew and for Triplet non-collinearity; $$\overline{\xi^2} \propto \frac{1}{1 - (\gamma_c/\gamma)^2}$$ [Eq. (2-15 a,b)] [Eq. (2-24 a,b)]

Hence both of the critical error components must be compensated independently for broad-band transmission.

8. The collinearity tolerances for the Triplets require that the common support structure have high rigidity and dimensional stability. If these conditions can be met, the collinearity compensation, once achieved, should be stable and require readjustment very infrequently.

As a consequence of comments 7 and 8 above, it appears probable that Triplets are more favorable than Doublets as regards frequency of steering adjustment. It is not immediately obvious which of the Triplets is the better choice. The Spaced Triplet is less critical in some of the tolerances (see Table I) and requires considerably weaker quadrupoles.[2] On the other hand, the Close Triplet seems to offer the possibility of an extremely rigid and rugged structure — for instance, the outer yoke could be common to the three quadrupole elements. Probably the choice between the Close and Spaced configurations should await study of the mechanical problems.

In the event that future studies of site movements indicate that angular motions are $\lesssim 10^{-5}$ radian, rms, over periods on the order of 90 days, then perhaps Doublets should be reconsidered.

II. TRANSVERSE MISALIGNMENTS

A. DEFINITION AND FORMULATION

The present section will concern misalignments in which the principal planes of the various quadrupoles remain parallel to a fixed reference system. The quadrupoles may have accidental parallel displacements from the correct reference axis, or they may have skew rotations -- i.e., rotations about a transverse axis. It will be assumed that the quadrupoles are all correctly programmed as to strength and spacing. Other types of misalignments and errors are considered in Sections III through V.

1. Misaligned Optic Element

Consider a generalized optic element (e.g., a lens) which has the following properties:

1) A well-defined neutral axis--i.e., a straight reference axis along which a ray (particle) of any energy will suffer no transverse deflection.

2) Two mutually orthogonal principal systems [the (x,z) or $\mathbf{x}$ plane and the (y,z) or $\mathbf{y}$ plane] such that there is no coupling between motions in the $\mathbf{x}$ and $\mathbf{y}$ planes.

3) In a system in which the reference axis is the neutral axis, the effect of the optic element is described by a linear, homogeneous transformation of the transverse dynamical coordinates; for example, in one of the principal coordinate systems,

$$\mathbf{X}_2 = \mathbf{M}\mathbf{X}_1 \qquad (2\text{-}1)$$

where

$$\mathbf{X} = \begin{pmatrix} X \\ P_x \end{pmatrix}$$

and

$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

The transformation has the usual property optic matrices,[3]

$$\mathrm{Det}\,(\mathbf{M}) \equiv m_{11} m_{22} - m_{12} m_{21} = 1$$

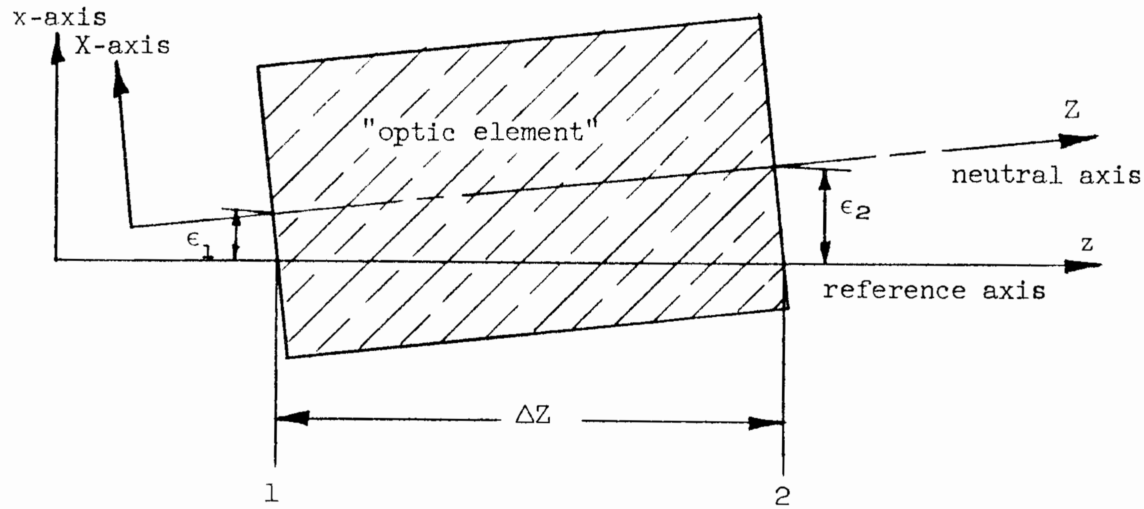

FIG. 2.1--Illustrating misaligned optic element.

Suppose that the neutral axis is misaligned from some other reference axis (see Fig. 2.1). Then to first order in the misalignments ϵ_1 and ϵ_2, the transformation from the (x, p) system to the (X, P_X) system is

$$\mathbf{X}_{1,2} = x_{1,2} - \epsilon_{1,2}$$

$$P_{X1,2} = p_{1,2} - \frac{\epsilon_2 - \epsilon_1}{\Delta Z} P$$

where P is the scalar momentum of the particle, and ΔZ is the length of the optic element. Substitution in (2-1) gives

$$\mathbf{x}_2 = \mathbf{M}\,\mathbf{x}_1 + \begin{bmatrix} -m_{11} + \frac{P}{\Delta Z} m_{12} \\ -m_{21} - \frac{P}{\Delta Z}\left(1 - m_{22}\right) \end{bmatrix} \epsilon_1 + \begin{bmatrix} 1 - \frac{P}{\Delta Z} m_{12} \\ \frac{P}{\Delta Z}\left(1 - m_{22}\right) \end{bmatrix} \epsilon_2 \qquad (2\text{-}2a)$$

It will sometimes be convenient to make the substitution

$$\epsilon \equiv \tfrac{1}{2}\left(\epsilon_1 + \epsilon_2\right) = \underline{\text{mean}}\ \underline{\text{displacement}}\ \text{of optic element}$$

$$\epsilon' \equiv \tfrac{1}{2}\left(\epsilon_2 - \epsilon_1\right) = \underline{\text{skew}}\ \underline{\text{misalignment}}\ \text{of optic element;}$$

then Eq. (2-2a) becomes

$$\mathbf{x}_2 = \mathbf{M}\,\mathbf{x}_1 + \begin{bmatrix} 1 - m_{11} \\ \\ -m_{21} \end{bmatrix}\epsilon + \begin{bmatrix} 1 + m_{11} - \frac{2P}{\Delta Z}\,m_{12} \\ \\ m_{21} + \frac{2P}{\Delta Z}\left(1 - m_{22}\right) \end{bmatrix}\epsilon' \qquad (2\text{-}2b)$$

In general, then, the effect of the misaligned element is expressed by the linear inhomogeneous transformation

$$\mathbf{x}_2 = \mathbf{M}\,\mathbf{x}_1 + \mathbf{m} \qquad (2\text{-}2c)$$

where

$$\mathbf{m} \equiv \begin{pmatrix} \delta x \\ \\ \delta p \end{pmatrix}$$

and δx and δp are the perturbations of the orbit displacement and momentum, respectively, caused by the misalignments.

In a system of ganged optical elements, successive application of Eq. (2-2c) gives

$$\mathbf{x}_j = \mathbf{X}_j + \boldsymbol{\xi}_j \qquad (2\text{-}3a)$$

where

$$\mathbf{X}_j \equiv \begin{pmatrix} X \\ \\ P_x \end{pmatrix}_j = \mathbf{M}(j|o)\,\mathbf{X}_o \qquad (2\text{-}3b)$$

$$\boldsymbol{\xi}_j \equiv \begin{pmatrix} \xi \\ \\ \rho \end{pmatrix}_j = \sum_{k=1}^{j} \mathbf{M}(j|k)\,\mathbf{m}_k \qquad (2\text{-}3c)$$

and

$$\left.\begin{aligned} \mathbf{M}(j|k) &\equiv \mathbf{M}_j \mathbf{M}_{j-1} \cdots \mathbf{M}_{k+1} \qquad (k < j) \\ \mathbf{M}(j|j) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}\right\} \qquad (2\text{-}3d)$$

Note that $\mathbf{X}_j$ is the solution of the homogeneous system; $\boldsymbol{\xi}_j$, which represents the orbit perturbations, is a particular solution of the inhomogeneous system; and $\mathbf{M}(j|k)$ plays the part of the "Green's function" of the system.

2. Periodic Focusing System with Independent Random Errors

A focusing system for the accelerator might consist of periodically recurring groups of focal elements. The transformation for one of the basic groups or focusing sections will be denoted as

$$\mathbf{x}_n = \mathbf{A}_n \mathbf{x}_{n-1} + \mathbf{a}_n \qquad (2\text{-}4)$$

where

$$\mathbf{A}_n \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{22} & a_{11} \end{pmatrix},$$

the homogeneous transformation over the n-th section, is constant, or adiabatically varying as a function of n, and

$$\mathbf{a}_n \equiv \begin{pmatrix} \delta x \\ \delta p \end{pmatrix}_n$$

represents the orbit perturbation due to the various misalignments in the n-th section.

For the present discussion it will be sufficient to describe the gross properties of the orbits in terms of the adiabatic invariant function[3] defined by

$$u = \frac{1}{\sin\theta_n}\left\{-a_{21}X^2 + \left(a_{11} - a_{22}\right) X P_x + a_{12} P_x^2\right\}_n \qquad (2\text{-}5)$$

$\cong$ constant for a given orbit,

where the characteristic phase angle θ is defined by

$$\cos\theta_n = \tfrac{1}{2}\left(a_{11} + a_{22}\right)_n \qquad (2\text{-}6)$$

The maximum orbit amplitude for a given value of u is[3]

$$\left.\begin{aligned} \left(X_n\right)^2_{max} &= \left(\frac{a_{12}}{\sin\theta}\right)_n u \\ \left(P_{xn}\right)^2_{max} &= \left(\frac{-a_{21}}{\sin\theta}\right)_n u \end{aligned}\right\} \qquad (2\text{-}7)$$

If the errors are random and independent, then the expectation value of the orbit perturbation is given by[1]

$$\overline{\xi_n^2} \cong \tfrac{1}{2}\left(\frac{a_{12}}{\sin\theta}\right)_n \sum_{m=1}^{n} \delta\bar{u}_m \qquad (2\text{-}8)$$

where

$$\delta\bar{u}_m \equiv \frac{1}{\sin\theta_m}\left\{-a_{21}\overline{(\delta x)^2} + \left(a_{11} - a_{22}\right)\overline{\delta x\,\delta p} + a_{12}\overline{(\delta p)^2}\right\}_m \qquad (2\text{-}9)$$

[δx, δp are the components of the perturbation vector $\mathbf{a}$ as defined previously.]

B. APPLICATION TO MULTIPLET SYSTEMS

1. Parallel Displacement of Multiplet Support

The assumption here is that the quadrupoles of a given multiplet are correctly aligned with respect to one another on a common support, but that the various supports have random parallel displacements from the correct beam axis. With reference to Fig. 2.1, considering the "optic element" to be the multiplet, we have

$$\epsilon_1 = \epsilon_2 = \epsilon$$

Equation (2-2b) then may be written

$$\mathbf{x}_n = \mathbf{M}_n \hat{x}_n + \begin{pmatrix} 1 - m_{11} \\ \\ - m_{21} \end{pmatrix}_n \epsilon_n$$

where $\hat{\mathbf{x}}_n$, $\mathbf{x}_n$ are respectively the coordinate vectors just ahead of and just beyond the multiplet, and $\mathbf{M}$ is the transformation for the (unperturbed) multiplet.

The transformation over one focusing section, equivalent to Eq. (2-4), now is

$$\mathbf{x}_n = (\mathbf{M} \ \ \mathbf{L})_n \mathbf{x}_{n-1} + \begin{pmatrix} 1 - m_{11} \\ \\ - m_{21} \end{pmatrix}_n \epsilon_n$$

$$\equiv \mathbf{A}_n \mathbf{x}_{n-1} + \mathbf{a}_n$$

where

$$\mathbf{L}_n \equiv \begin{pmatrix} 1 & \ell_n \\ & \\ 0 & 1 \end{pmatrix}$$

and

$$\ell_n \equiv \int_{z_{n-1}}^{z_{n-1}+L_n} \frac{dz}{\gamma} \qquad \text{(See Fig. 2.2)}$$

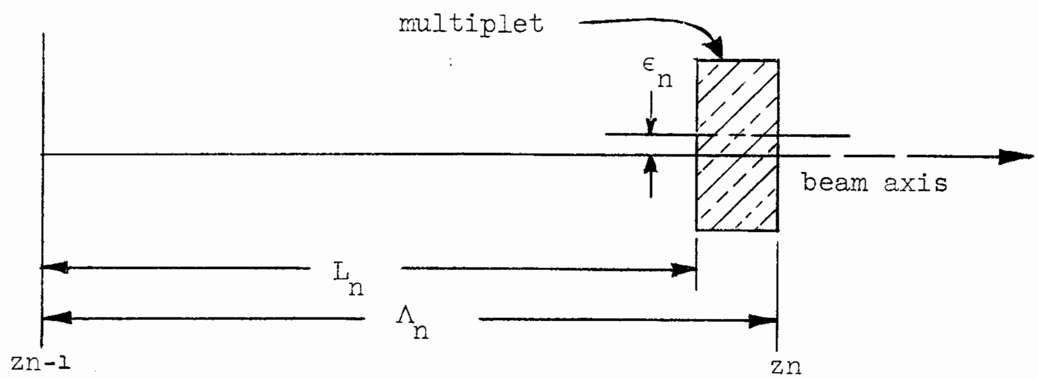

FIG. 2.2--Basic section of periodic multiplet system with parallel displacement of support.

Application of Eq. (2-9) gives after appropriate manipulation

$$\delta\bar{u}_n = 2\left(-\ a_{21}\ \frac{1 - \cos\theta}{\sin\theta}\right)_n \overline{\epsilon^2} \tag{2-10}$$

where $\overline{\epsilon^2}$ is the mean square expectation value of the ϵ_n, which are assumed random and independent.

To the extent that the singlet approximation of Ref. (2) is valid, the effect of this type of misalignment is seen to be independent of the structure of the multiplet. If we use the singlet parameters from Ref. (2), Eq. (2-10) becomes

$$\delta\bar{u}_n \cong 8\ \overline{\epsilon^2}\left(\frac{\gamma_c^3}{\Lambda\gamma^2}\ \frac{1}{\sqrt{1 - \gamma_c^2/\gamma^2}}\right)_n \tag{2-10a}$$

where γ_c is the low-energy cutoff of the periodic system. [The assumptions are that $L_n \cong \Lambda_n$ and $\gamma_{n-1} \cong \gamma_n$.] If the parameters ($\gamma_c$ and Λ) are essentially constant, the effect is strongest at low energy; hence it will suffice to calculate the beam deflection at constant (minimum) beam energy. The result is, by use of Eq. (2-8),

$$\overline{\xi_n^2} \cong 2n\overline{\epsilon^2} \frac{(\gamma_c/\gamma)^2}{1 - (\gamma_c/\gamma)^2} \qquad (2\text{-}11)$$

A reasonable low-energy band limit is given by[2]

$$\gamma_o \cong \gamma_c\sqrt{2}$$

whence the tolerance on ϵ is given by

$$< \epsilon >_{rms} \lesssim \frac{|\xi|_{max}}{\sqrt{2n}} \qquad (2\text{-}12)$$

where $|\xi|_{max}$ is the tolerable beam deflection.

2. Skew Rotation of Support

Figure 2.3 illustrates the situation where the common support structure is rotated about the geometric center of the multiplet.

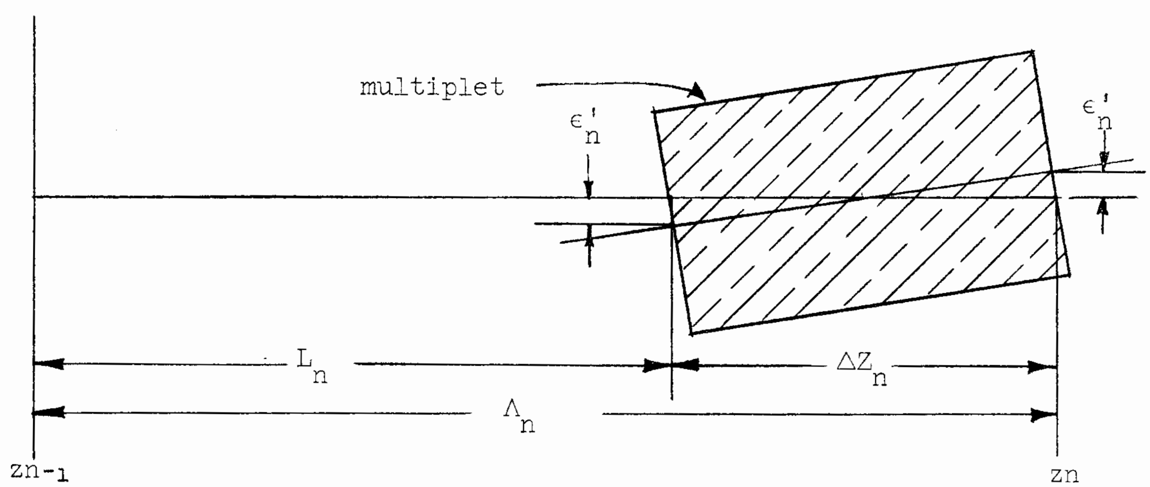

FIG. 2.3--Skew misalignment of multiplet supporting system.

Application of Eq. (2-2b) gives*

$$\mathbf{a}_n = \begin{bmatrix} 1 + m_{11} - \frac{2\gamma}{\Delta Z} m_{12} \\ \\ m_{21} + \frac{2\gamma}{\Delta Z}(1 - m_{22}) \end{bmatrix}_n \epsilon'_n \qquad (2\text{-}13)$$

In this case the orbit perturbation turns out to depend strongly on the particular multiplet structure. Hence each of the combinations must be considered separately.

a) Spaced Doublet (SD) (See Fig. 1.1a)

In this case we make the identification

$$\Delta Z \equiv D = \text{doublet spacing.}$$

By use of the SD matrix elements from Ref. (2),** Eq. (2-13) becomes

$$\mathbf{a}_n = \begin{bmatrix} Qd \\ \\ 2Q(1 - \frac{1}{2}Qd) \end{bmatrix}_n \epsilon'_n$$

where $d = D/\gamma$ and Q = strength of the individual quadrupoles.[1,2]

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*The approximation

$$P = \sqrt{\gamma^2 - 1} \cong \gamma$$

where P is total (scalar) momentum in units of mc and γ is relativistic energy in units of mc^2, will be used throughout.

**The matrix $\mathbf{M}$ (transformation through the multiplet) is recovered readily from the matrix

$$\mathbf{A} \equiv \mathbf{M}\ \mathbf{L} = \mathbf{M}\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \qquad \text{[given in Ref. (2)]}$$

by setting $\ell = 0$.

As in Ref. (2), it is assumed that

$$D/L \ll 1$$

and

$$Qd = QD/\gamma \ll 1$$

Then Eq. (2-9), if we retain only the most significant terms, becomes

$$\delta\overline{u_n} \cong \left(\frac{4Q^2\lambda\ \overline{(\epsilon')^2}}{\sin\theta}\right)_n \tag{2-14a}$$

where

$$\lambda_n = \int_{z_{n-1}}^{z_n} \frac{dz}{\gamma} \approx \frac{\Lambda_n}{\gamma_n}$$

and[2]

$$\sin\theta \approx \sqrt{Q^2\lambda d\ (1 - \tfrac{1}{4}Q^2\lambda d)}$$

If the parameters Q, D, and Λ are essentially constant the effect is strongest at the lowest energy; hence it will suffice to calculate the orbit perturbation at constant energy. Then substitution of Eq. (2-14) in (2-8) gives

$$\overline{\xi_n^2} \approx 2n\,\frac{\Lambda}{D}\,\frac{\overline{(\epsilon')^2}}{1 - (\gamma_c/\gamma)^2} \tag{2-15a}$$

where $\gamma_c \approx \frac{1}{2}Q\sqrt{\Lambda D}$ is the cutoff energy in the SD case, and n is the number of focusing sections.

It is reasonable to base the skew-misalignment tolerance on the condition $\gamma = \gamma_c\sqrt{2}$ which is the half-transmission energy.[2] The tolerance then is

$$< \epsilon' >_{rms} \lesssim \frac{1}{2} |\xi|_{max} \sqrt{\frac{D}{n\Lambda}} \tag{2-16a}$$

where $|\xi|_{max}$ is the tolerable orbit perturbation.

b) Close Doublet (CD) (See Fig. 1.1b)

In this case we make the definitions

$\Delta Z = T$ = doublet thickness;

$t = T/\gamma$;

$g = \left| \frac{\partial B_y}{\partial x} \right|$ = magnetic gradient (B in units of $mc^2/e/cm$ or 1703 gauss);

$Q = \frac{1}{2} gT$ = strength of the quadrupoles;

$k = \sqrt{g/\gamma}$;

$\varphi = \frac{1}{2} kT = \sqrt{\frac{1}{2} Qt}$;

$c = \cos \varphi$; $C = \cosh \varphi$;

$s = \sin \varphi$; and $S = \sinh \varphi$.

Then the results of Ref. (2) combined with Eq. (2-13) give

$$\mathbf{a}_n = \begin{pmatrix} 1 + cC - sS - \frac{1}{\varphi}(sC + cS) \\ -k\gamma \left[sC - cS + \frac{1}{\varphi}(1 - cC + sS) \right] \end{pmatrix}_n \epsilon'_n$$

Series expansion under the assumptions

$$T/L \ll 1,$$

$$Qt \ll 1$$

gives

$$\mathbf{a}_n \cong \begin{bmatrix} \frac{1}{2} Qt \\ Q(1 - \frac{1}{4} Qt) \end{bmatrix}_n \epsilon'_n$$

The phase-space increment is found by Eq. (2-9) to be

$$\delta\bar{u} \approx \left(\frac{Q^2\lambda}{\sin\theta}\right)_n \overline{(\epsilon')^2} \tag{2-14b}$$

where in the CD case

$$\sin\theta \cong \sqrt{\frac{1}{3}\,Q^2\lambda t\left(1 - \frac{1}{12}\,Q^2\lambda t\right)}$$

For constant parameters (Q, Λ, and T) and constant energy, the orbit perturbation then is found to be

$$\overline{\xi_n^2} \approx \frac{3}{2}n\,\frac{\Lambda}{T}\,\frac{\overline{(\epsilon')^2}}{1 - (\gamma_c/\gamma)^2} \tag{2-15b}$$

where

$$\gamma_c = \tfrac{1}{2}Q\sqrt{\tfrac{1}{3}\,\Lambda t}$$

is the CD cutoff energy. Hence the skew misalignment tolerance in this case is (again taking $\gamma = \gamma_c\sqrt{2}$ as the practical low energy band limit)

$$<\epsilon'>_{rms} \lesssim |\xi|_{max}\sqrt{\frac{T}{3n\Lambda}} \tag{2-16b}$$

c) Spaced Triplet (ST) (See Fig. 1.1c)

The treatment is analogous to the SD case [Part (a), above]. The results are:

Perturbation vector;

$$\mathbf{a}_n = \begin{bmatrix} -\tfrac{1}{2}Qd\,(1 + \tfrac{1}{4}Qd) \\ \\ \tfrac{1}{16}\,Q^3d^2 \end{bmatrix}_n \epsilon'_n$$

Phase-space increment;

$$\delta\overline{u_n} \cong \frac{1}{4}\left(\frac{Q^2d^2}{\lambda}\,\overline{(\epsilon')^2}\,\sin\theta\right)_n \qquad (2\text{-}14c)$$

where $\sin\theta \cong \sqrt{\tfrac{1}{4}Q^2\lambda d\left(1 - \tfrac{1}{16}Q^2\lambda d\right)}$ in the ST case.[2]

Orbit perturbation with constant parameters and constant energy;

$$\overline{\xi_n^2} \cong 2n\,\frac{D}{\Lambda}\,\frac{\gamma_c^2}{\gamma^2}\,\overline{(\epsilon')^2} \qquad (2\text{-}15c)$$

where $\gamma_c \approx \frac{1}{4}Q\sqrt{\Lambda D}$ is the <u>ST</u> cutoff energy.[2]

Skew misalignment tolerance (again taking $\gamma = \gamma_c\sqrt{2}$ as the practical low-energy limit);

$$< \epsilon' >_{rms} \lessapprox |\xi|_{max}\sqrt{\frac{1}{n}\frac{\Lambda}{D}} \qquad (2\text{-}16c)$$

d) <u>Close Triplet (CT)</u> (See Fig. 1.1d)

The definitions and treatment are analogous to the <u>CD</u> case [Part (b), above]. The results are:

Perturbation vector;

$$\mathbf{a}_n = \begin{pmatrix} 1 + cC - \frac{1}{\varphi}(sC + S) \\ k\gamma\left[- sC + S + \frac{1}{\varphi}(1 - cC)\right] \end{pmatrix}_n \epsilon'_n$$

$$\cong \begin{pmatrix} -\frac{1}{4}Qt \\ \frac{1}{96}Q^3t^2 \end{pmatrix}_n \epsilon'_n$$

Phase-space increment;

$$\delta\bar{u}_n \cong \frac{1}{16}\left(\frac{Q^2t^2}{\lambda}\,\overline{(\epsilon')^2}\,\sin\theta\right)_n \qquad (2\text{-}14d)$$

where $\sin\theta \cong \sqrt{\frac{1}{12}Q^2\lambda t\left(1 - \frac{1}{48}Q^2\lambda t\right)}$ in the CT case.[2]

Orbit perturbation for constant parameters and constant energy;

$$\overline{\xi_n^2} \cong \frac{3n}{2} \frac{T}{\Lambda} \frac{\gamma_c^2}{\gamma^2} \overline{(\epsilon')^2} \tag{2-15d}$$

where $\gamma_c \cong \frac{1}{4} Q \sqrt{\frac{1}{3} \Lambda T}$ is the CT cutoff energy.[2]
Skew-misalignment tolerance (again setting $\gamma = \gamma_c \sqrt{2}$ at the practical low-energy band limit);

$$< \epsilon' >_{rms} \lesssim |\xi|_{max} \sqrt{\frac{4}{3n} \frac{\Lambda}{T}} \tag{2-16d}$$

3. Independent Skew Rotations of the Quadrupoles

Consider a quadrupole which has a skew-rotation about the y axis, as illustrated in Fig. 2.4. If the quadrupole is focusing in the x plane, the perturbation vector is given with the help of Eq. (2-2b) as

$$\mathbf{m} = \begin{pmatrix} 1 + c - \frac{2}{\varphi} s \\ k\gamma \left[- s + \frac{2}{\varphi} (1 - c) \right] \end{pmatrix} e' \tag{2-17}$$

where

$$c \equiv \cos \varphi, \ s \equiv \sin \varphi, \ \varphi \equiv k \Delta Z$$

and k is as defined in Section II.B.2, above.

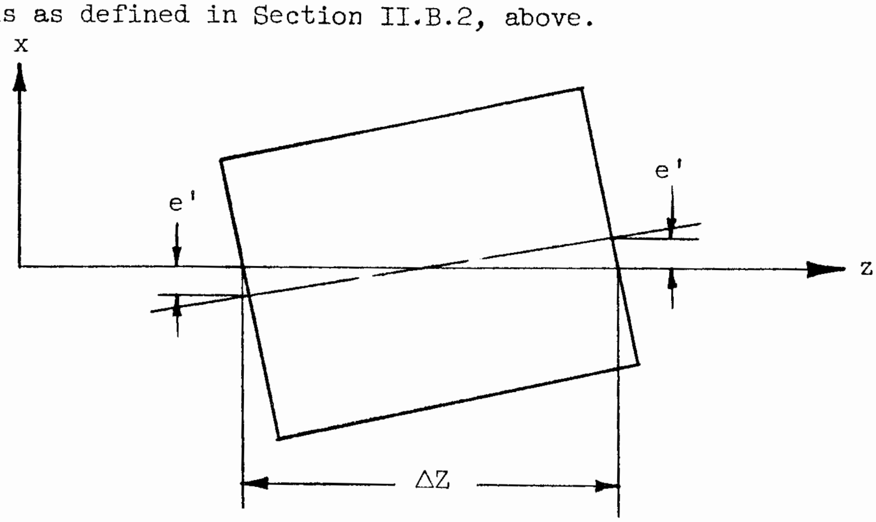

FIG. 2.4--Skew rotation in a quadrupole.

Expansion in power series gives to first order in ΔZ

$$\mathbf{m} \cong \frac{Q\Delta Z}{12\gamma}\, e' \begin{pmatrix} -2 \\ Q \end{pmatrix} \tag{2-17a}$$

where $Q \equiv g\Delta Z$.

The effect becomes vanishingly small in the thin-lens approximation $(Q\frac{\Delta Z}{\gamma} \to 0)$; hence it will be sufficient to calculate the effect only for the contiguous multiplets CD and CT.

a) Close Doublet (CD) (See Fig. 1.1b)

In this case we make the identifications

$$\Delta Z_1 = \Delta Z_2 = \tfrac{1}{2}T$$

$$-Q_1 = Q_2 = Q$$

and define $t \equiv T/\gamma$. Then the perturbation vector for the n-th focusing section is

$$\mathbf{a}_n = \begin{pmatrix} c & \frac{s}{k\gamma} \\ -k\gamma s & c \end{pmatrix}_n \mathbf{m}_{1,n} + \mathbf{m}_{2,n}$$

where

$$\mathbf{m}_{2,n} \cong \frac{(Qt)_n}{24}\, e'_{2,n} \begin{pmatrix} -2 \\ Q_n \end{pmatrix}$$

$$\mathbf{m}_{1,n} \cong \frac{(Qt)_n}{24}\, e'_{1,n} \begin{pmatrix} 2 \\ Q_n \end{pmatrix}$$

Expansion to first order in T gives

$$\mathbf{d}_n \cong \frac{(Qt)_n}{24} \left(e'_{2,n} - e'_{1,n}\right) \begin{pmatrix} -2 \\ Q_n \end{pmatrix} \tag{2-18a}$$

Assuming the $\epsilon'_{1,n}$ to be random and independent, using the appropriate CD matrix elements from Ref. (2), and keeping only dominant terms, we find by Eq. (2-9)

$$\delta \overline{u_n} \approx \left[\frac{2\ Q^2 \lambda}{\sin\ \theta} \left(\frac{Qt}{24} \right)^2 \right]_n \overline{(e')^2} \qquad (2\text{-}19a)$$

Application of Eq. (2-8) now gives for the orbit perturbation (in the constant-parameter, constant-energy case)

$$\overline{\xi_n^2} \approx \frac{n}{16} \frac{\gamma_c^2}{\gamma^2 - \gamma_c^2} \overline{(e')^2} \qquad (2\text{-}20a)$$

Taking $\gamma = \gamma_c \sqrt{2}$, as in Section II.B.2, as the basis for the misalignment tolerance, we find

$$< e' >_{rms} \lesssim \frac{4}{\sqrt{n}} |\xi|_{max} \qquad (2\text{-}21a)$$

b) <u>Close Triplet (CT)</u> (See Fig. 1.1d)

In this case the definitions are

$$\Delta Z_1 = \Delta Z_3 = \tfrac{1}{4} T$$

$$\Delta Z_2 = \tfrac{1}{2} T$$

$$Q_1 = Q_3 = \tfrac{1}{2} Q$$

$$Q_2 = -Q$$

By a treatment analogous to the preceding example, one finds the following results:

Perturbation vector (to first order in T);

$$\mathbf{a}_n \approx \frac{(Qt)_n}{48} \left[-e'_{1,n} \begin{pmatrix} 1 \\ \frac{1}{4} Q_n \end{pmatrix} + e'_{2,n} \begin{pmatrix} 4 \\ 0 \end{pmatrix} + e'_{3,n} \begin{pmatrix} -1 \\ \frac{1}{4} Q_n \end{pmatrix} \right] \qquad (2\text{-}18b)$$

Phase-space increment;

$$\delta \overline{u_n} \approx \left[\frac{\frac{1}{8} Q^2 \lambda}{\sin \theta} \left(\frac{Qt}{48}\right)^2 \overline{(e')^2}\right]_n \tag{2-19b}$$

where $\overline{(\epsilon'_{1,n})^2} = \overline{(e')^2}$ = constant.

Orbit perturbation (for constant parameters and constant energy);

$$\overline{\xi_n^2} \approx \frac{n}{64} \frac{\gamma_c^2}{\gamma^2 - \gamma_c^2} \overline{(e')^2} \tag{2-20b}$$

Tolerance on independent skew misalignment of the quadrupoles (again taking $\gamma = \gamma_c \sqrt{2}$ at the low end of the energy band);

$$\langle e' \rangle_{rms} \lessapprox \frac{8}{\sqrt{n}} |\xi|_{max} \tag{2-21b}$$

4. Collinearity of the Multiplet Elements

In addition to the three components of transverse misalignments discussed so far, triplets are subject also to non-collinearity of the centers of the three quadrupoles. Figure 2.5 illustrates a suitable representation of this component of the misalignment, in the ST case; the situation is exactly analogous in the CT case.

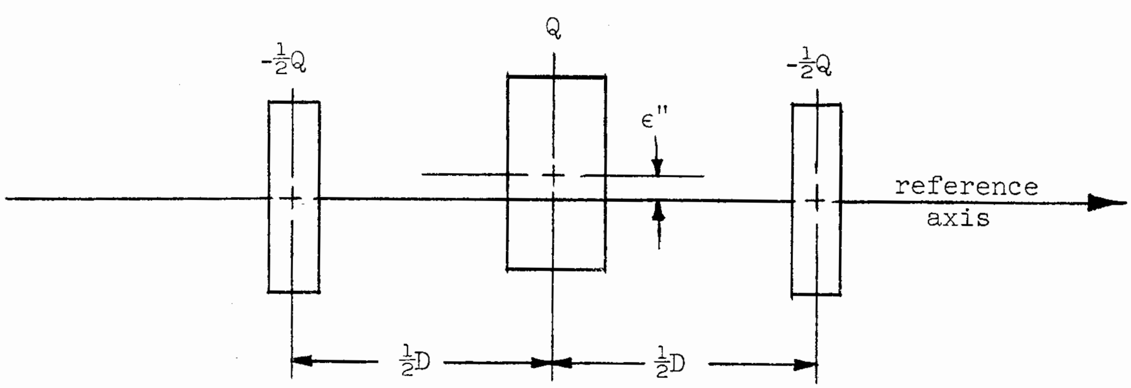

FIG. 2.5--Representation of non-collinearity.

a) Spaced Triplet

The perturbation vector for the misplaced (center) quadrupole is given in the thin-lens approximation by

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ -Q \end{pmatrix} \epsilon''$$

Hence the perturbation vector for the n-th section, referred as usual to the end of the multiplet, is

$$\mathbf{a}_n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2}d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -Q \end{pmatrix} \epsilon'' = - \begin{pmatrix} \frac{1}{2}Qd \\ Q(1 - \frac{1}{4}Qd) \end{pmatrix} \epsilon'' \qquad (2\text{-}22a)$$

The phase-space increment, if only the dominant terms are kept, is by Eq. (2-9)

$$\overline{\delta u_n} \approx \left(\frac{Q^2\lambda}{\sin\theta}\right)_n \overline{(\epsilon'')^2} \qquad (2\text{-}23a)$$

The orbit perturbation in the constant-parameter, constant-energy case then is found to be

$$\overline{\xi_n^2} \approx \frac{2n\Lambda}{D} \frac{\overline{(\epsilon'')^2}}{1 - (\gamma_c/\gamma)^2} \qquad (2\text{-}24a)$$

Finally, the collinearity tolerance, based as usual on the low energy case where $\gamma = \gamma_c\sqrt{2}$, is

$$< \epsilon'' >_{rms} \lesssim \frac{1}{2} \left|\xi\right|_{max} \sqrt{\frac{D}{n\Lambda}} \qquad (2\text{-}25a)$$

b) Close Triplet

In this case the perturbation vector, referred to the end of the triplet, is

$$\mathbf{a}_n = - \begin{pmatrix} c' & \frac{1}{k\gamma} s' \\ -k\gamma s' & c' \end{pmatrix}_n \begin{pmatrix} C - 1 \\ k\gamma S \end{pmatrix}_n \epsilon''_n$$

where $c' \equiv \cos \frac{1}{2}\varphi$, $s' \equiv \sin \frac{1}{2}\varphi$, and the rest of the quantities (k, φ, C, and S) are as defined in Sect. II.B.2(b), above. Expansion in series to first order in T gives

$$\mathbf{a}_n \cong - \begin{pmatrix} \frac{1}{2}Qt \\ Q\left(1 - \frac{5}{48}Qt\right) \end{pmatrix}_n \epsilon''_n \tag{2-22b}$$

Use of the CT matrix elements[2] gives for the dominant part of the phase-space increment

$$\delta \overline{u_n} \approx \left(\frac{\lambda Q^2}{\sin\theta}\right)_n \overline{(\epsilon'')^2} \tag{2-23b}$$

and the orbit perturbation in the constant-parameter, constant-energy case is

$$\overline{\xi_n^2} \approx \frac{6n\Lambda}{T} \frac{\overline{(\epsilon'')^2}}{1 - (\gamma_c/\gamma)^2} \tag{2-24b}$$

The collinearity tolerance, based on $\gamma = \gamma_c\sqrt{2}$, then is

$$\langle \epsilon'' \rangle_{rms} \lesssim \frac{1}{2}|\xi|_{max}\sqrt{\frac{T}{3n\Lambda}} \tag{2-25b}$$

III. AXIAL ROTATIONS

A. DEFINITION AND FORMULATION

Axial rotation refers to misalignments in which optic elements (e.g., quadrupoles or multiplets) are rotated about the longitudinal reference axis. Figure 3.1 illustrates the situation in which the principal axes (X,Y) of the element are rotated by an angle ψ with respect to the fixed reference axes (x,y).

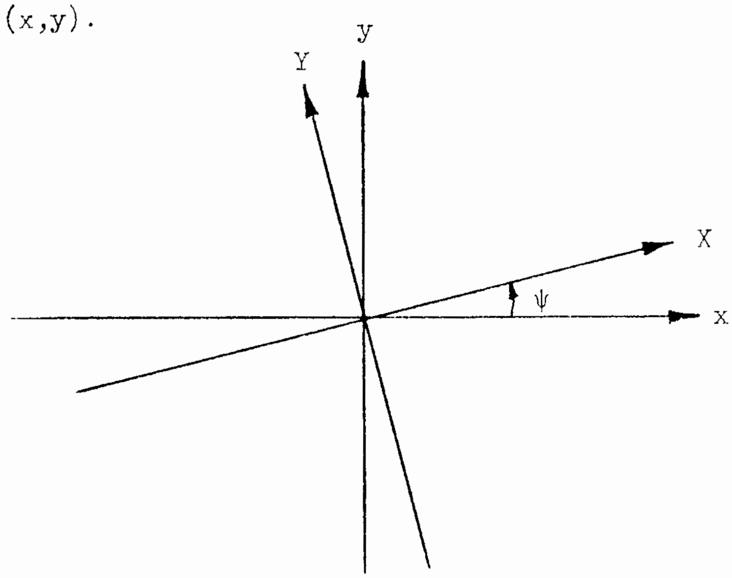

FIG. 3.1--Rotation of principal axes (X,Y) relative to transverse reference axes (x,y).

In this case the $(\mathbf{x},\mathbf{y})$ system is transformed into the $\mathbf{X},\mathbf{Y}$ system by a 4×4 rotation matrix, i.e.,

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = \begin{pmatrix} \mathbf{I}\cos\psi & \mathbf{I}\sin\psi \\ -\mathbf{I}\sin\psi & \mathbf{I}\cos\psi \end{pmatrix} \begin{matrix} \mathbf{x} \\ \mathbf{y} \end{matrix} \equiv \mathbf{R} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \qquad (3\text{-}1)$$

where

$$\mathbf{X} \equiv \begin{pmatrix} X \\ P_x \end{pmatrix} \qquad \mathbf{x} \equiv \begin{pmatrix} x \\ p_x \end{pmatrix}$$

$$\mathbf{Y} \equiv \begin{pmatrix} Y \\ P_y \end{pmatrix} \qquad \mathbf{y} \equiv \begin{pmatrix} y \\ p_y \end{pmatrix}$$

and

$$\mathbf{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

To first order in the rotational error ψ,

$$\mathbf{R} \approx \begin{pmatrix} \mathbf{I} & \mathbf{I}\psi \\ -\mathbf{I}\psi & \mathbf{I} \end{pmatrix} \tag{3-2}$$

The transformation through the optic element, in the reference system, now is given by

$$\mathbf{T}' = \mathbf{R}^{-1}\mathbf{T}\mathbf{R} \approx \begin{pmatrix} \mathbf{I} & -\mathbf{I}\psi \\ \mathbf{I}\psi & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{M} & \mathbf{O} \\ \mathbf{O} & \mathbf{N} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{I}\psi \\ -\mathbf{I}\psi & \mathbf{I} \end{pmatrix}$$

$$\approx \begin{pmatrix} \mathbf{M} & (\mathbf{M} - \mathbf{N})\psi \\ (\mathbf{M} - \mathbf{N})\psi & \mathbf{N} \end{pmatrix} \tag{3-3}$$

(to first order in ψ), where $\mathbf{M}$ and $\mathbf{N}$ are respectively the (2×2) transformations in the principal, or $\mathbf{X}$ and $\mathbf{Y}$, planes.

In a first-order perturbation treatment one makes the substitution

$$\left.\begin{aligned} \mathbf{x} &= \mathbf{X} + \xi \\ \mathbf{y} &= \mathbf{Y} + \eta \end{aligned}\right\} \tag{3-3a}$$

where ξ and η are presumed small. Equation (3-3), combined with Eq. (3-2), gives for the tranformation through the optic element

$$\begin{pmatrix} \mathbf{x}_2 \\ \mathbf{y}_2 \end{pmatrix} \approx \begin{pmatrix} \mathbf{M}\mathbf{x}_1 + (\mathbf{M} - \mathbf{N})\mathbf{Y}_1\psi \\ \mathbf{N}\mathbf{y}_1 + (\mathbf{M} - \mathbf{N})\mathbf{X}_1\psi \end{pmatrix} \tag{3-4}$$

Terms containing $\xi\psi$ and $\psi\eta$ are considered second-order, and are dropped.

Then the perturbed solutions are given in terms of the unperturbed solution by

$$\left.\begin{aligned} \mathbf{X}_2 &= \mathbf{M}\mathbf{X}_1 \\ \xi_2 &\approx \mathbf{M}\xi_1 + \mathbf{m} \end{aligned}\right\} \tag{3-4a}$$

where

$$\mathbf{m} = (\mathbf{M} - \mathbf{N})\mathbf{Y}_1\psi \tag{3-5}$$

with an analogous transformation for the $\mathbf{y}$ vector. The discussion in Section II.A.2 and the latter part of II.A.1, and Eqs. (2-3) through (2-9), thus apply in the present case.

B. APPLICATION TO PERIODIC MULTIPLETS

1. Axial Rotation of the Multiplet as a Whole

Consider first the component of axial rotation associated with accidental tilting of the common support of the multiplet quadrupoles. It will suffice to calculate this effect only for the Spaced-Doublet (SD), which is the most asymmetric combination. (Note that in a circularly symmetric lens there would be no effect because $\mathbf{M} - \mathbf{N} = 0$.)

From Ref. (2) one finds for the SD lens

$$\mathbf{M} - \mathbf{N} = \begin{pmatrix} 1 + Qd & d \\ -Q^2d & 1 - Qd \end{pmatrix} - \begin{pmatrix} 1 - Qd & d \\ -Q^2d & 1 + Qd \end{pmatrix}$$

$$= 2Qd \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

whence by Eq. (3-5) the perturbation vector for the n-th focusing section, referred to the end of the doublet, is

$$\boldsymbol{\alpha}_n = 2(Qd)_n \begin{pmatrix} -\hat{Y} \\ \hat{P}_y \end{pmatrix}_n \psi_n \qquad (3\text{-}6)$$

where

$$\hat{\mathbf{Y}}_n = \begin{pmatrix} \hat{Y} \\ \hat{P}_y \end{pmatrix}_n$$

is the (unperturbed) solution in the $\mathbf{y}$ plane, referred to the <u>beginning</u> of the doublet. We need the matrices

$$\mathbf{A} = \mathbf{ML} = \mathbf{M} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

[given in Ref. (2)], and

$$\hat{\mathbf{B}} = \mathbf{LN}$$

(the transformation for the $\hat{\mathbf{Y}}$ vector) which may be shown to be given by

$$\hat{\mathbf{B}} = \begin{pmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{pmatrix} \qquad (3\text{-}7)$$

where a_{ij} are the components of $\mathbf{A}$.

Use of Eqs. (3-7), (3-6), (2-9), and (2-7) gives, for the phase-space increment,

$$\delta\overline{u}_{x,n} = 4\left(\frac{Q^2d^2}{\sin\theta}\right)_n \overline{\psi^2}\left\{-\hat{b}_{21}\hat{Y}^2 + \left(\hat{b}_{11} - \hat{b}_{22}\right)\hat{Y}\hat{P}_y + \hat{b}_{12}\hat{P}_y^2\right\}_n$$

$$= 4\left(Q^2d^2\right)_n \overline{\psi^2}\, u_y \qquad (3\text{-}8)$$

where u_y is the (unperturbed) invariant function in the **y** plane.

The perturbation is again seen to be strongest at low energy; hence it suffices to calculate the orbit perturbation at constant energy, which by use of Eqs. (3-8), (2-8), and (2-7) turns out to be

$$\overline{\xi_n^2} \approx 8n \frac{D}{\Lambda} \frac{\gamma_c^2}{\gamma^2} (Y)^2_{max} \overline{\psi^2} \qquad (3\text{-}9)$$

where $\gamma_c \approx \frac{1}{2}Q\sqrt{\Lambda D}$ in the SD case.[2]

With $\gamma = \gamma_c\sqrt{2}$ as the practical minimum energy, the tolerance on axial rotations for the doublet is

$$< \psi >_{rms} \lesssim \frac{1}{2} \frac{|\xi|_{max}}{|Y|_{max}} \sqrt{\frac{\Lambda}{nD}} \qquad (3\text{-}10)$$

With typical numbers $|\xi|_{max} = 0.1$ cm, $|Y|_{max} = 1$ cm, $\Lambda = 4000$ inches, $D = 80$ inches, and $n = 15$, one gets

$$< \psi >_{rms} \lesssim .091 \text{ radian} = 5.2^\circ$$

Since the other combinations (CD, ST, and CT) are expected to have even larger tolerances, it appears that axial rotation of the multiplet support is no problem.

2. Relative Axial Rotation of the Quadrupole

Consider now the case where the individual quadrupoles of a multiplet have independent random axial rotations relative to one another. It is necessary to evaluate the orbit perturbation for each of the multiplets.

(a) <u>Spaced Doublet</u> (<u>SD</u>) (See Figure 1.1a.)

In this case the perturbation vector in the $\mathbf{x}$ plane, referred to the end of the multiplet, is

$$\mathbf{a}_n = \left[\mathbf{QD}(\hat{\mathbf{Q}} - \mathbf{Q})\hat{\mathbf{Y}}\psi_1 + (\mathbf{Q} - \hat{\mathbf{Q}})\mathbf{D}\mathbf{Q}\hat{\mathbf{Y}}\psi_2\right]_n \tag{3-11a}$$

where $\mathbf{Q}, \hat{\mathbf{Q}}$ are the transformations through a quadrupole in the focusing and defocusing planes, respectively;*

$$\mathbf{D} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

is the transformation through the space between the quadrupoles; $\hat{\mathbf{Y}}_n$ is the (unperturbed) solution in the $\mathbf{y}$ plane, measured at the beginning of the doublet; $\psi_{1,n}$ and $\psi_{2,n}$ are the rotations of the first and second quadrupoles, respectively.

Expanding Eq. (3-11a) and keeping only the most dominant terms, we find

$$\mathbf{a}_n \approx \begin{pmatrix} 0 \\ 2Q\hat{Y} \end{pmatrix}_n (\psi_{1,n} - \psi_{2,n}) \tag{3-12a}$$

We define the relative rotation, $\delta\psi_n$, by

$$\psi_{2,n} - \psi_{1,n} \equiv \delta\psi_n$$

and note that $\delta\psi_n$ is the quantity which is expected to be a random variable.

---

*In the thin-lens approximation,

$$\mathbf{Q} = \hat{\mathbf{Q}}^{-1} \quad \begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix}$$

Then evaluation of the phase-space increment [Eq. (2-9)], with appropriate SD matrix elements from Ref. (2), gives

$$\overline{\delta u_n} \approx 4 \left(\frac{Q^2 \lambda}{\sin \theta} \hat{Y}^2 \right)_n \overline{(\delta\psi)^2} \tag{3-13a}$$

The orbit perturbation in the constant-parameter, constant-energy case then is found [Eq. (2-8)] to be

$$\overline{\xi_n^2} \approx 2 \frac{\Lambda}{D} \frac{\overline{(\delta\psi)^2}}{1 - (\gamma_c/\gamma)^2} \sum_{m=1}^{n} \hat{Y}_m^2$$

$$\approx n \frac{\Lambda}{D} \frac{\overline{(\delta\psi)^2}\,(Y)^2_{max}}{1 - (\gamma_c/\gamma)^2} \tag{3-14a}$$

where $\gamma_c \approx \frac{1}{2} Q \sqrt{\Lambda D}$ in the SD case; and, because of the quasi-periodic form of the orbits in periodic focusing systems,[3,1] $\hat{Y}_m^2$ has been replaced by $\frac{1}{2}(Y)^2_{max}$ in the summation.

Basing the tolerance, as usual, on the condition $\gamma = \gamma_c \sqrt{2}$, we find

$$< \delta\psi >_{rms} \lesssim \frac{|\xi|_{max}}{|Y|_{max}} \sqrt{\frac{D}{2n\Lambda}} \tag{3-15a}$$

(b) Close Doublet (CD) (See Figure 1.1b)

In this case the perturbation vector in the $\mathbf{x}$ plane is

$$\mathbf{a}_n = \mathbf{Q}(\hat{\mathbf{Q}} - \mathbf{Q})\hat{\mathbf{Y}}_{\psi_1} + (\mathbf{Q} - \hat{\mathbf{Q}})\mathbf{Q}\hat{\mathbf{Y}}_{\psi_2}\Big]_n \tag{3-11b}$$

where in the CD case

$$\mathbf{Q} = \begin{pmatrix} c & \frac{1}{k\gamma} s \\ - k\gamma s & c \end{pmatrix}$$

$$\hat{\mathbf{Q}} = \begin{pmatrix} C & \frac{1}{k\gamma} S \\ k\gamma S & C \end{pmatrix}$$

c, s, C, S, and k are as defined in Sect. II.B.2(b); $\hat{Y}_n$, $\psi_{1,n}$ and $\psi_{2,n}$ are as defined in (a), above.

Expanding Eq. (3-11b) and keeping only the dominant terms, we again find

$$\mathbf{a}_n \approx - \begin{pmatrix} 0 \\ 2Q\hat{Y} \end{pmatrix}_n \delta\psi_n \qquad (3\text{-}12b)$$

where $\delta\psi_n \equiv \psi_{2,n} - \psi_{1,n}$.

The remainder of the derivation is analogous to (a), above. Using the CD matrix element from Ref. (2), we find:

Phase-space increment;

$$\overline{\delta u_n} \approx 4 \left(\frac{Q^2\lambda}{\sin\,\theta}\, Y^2\right)_n \overline{(\delta\psi)^2} \qquad (3\text{-}13b)$$

Orbit perturbation (constant parameters, constant energy);

$$\overline{\xi_n^2} \approx 3n \,\frac{\Lambda}{T}\, \frac{\overline{(\delta\psi)^2}\ (Y)^2_{max}}{1 - (\gamma_c/\gamma)^2} \qquad (3\text{-}14b)$$

where

$$\gamma_c = \tfrac{1}{2}Q\sqrt{\tfrac{1}{3}\,\Lambda T}$$

in the CD case.[2]

Tolerance on relative axial rotation;

$$< \delta\psi >_{rms} \lesssim \frac{|\xi|_{max}}{|Y|_{max}} \sqrt{\frac{T}{6n\Lambda}}$$

(c) Spaced Triplet (ST) (See Figure 1.1c)

The treatment is analogous to the preceding cases. The results are:
Perturbation vector (dominant terms);

$$a_n \approx - \begin{pmatrix} 0 \\ \hat{QY} \end{pmatrix}_n \left(\psi_{1,n} - 2\psi_{2,n} + \psi_{3,n}\right)$$

$$\approx \begin{pmatrix} 0 \\ \hat{QY} \end{pmatrix}_n \left(\delta\psi_{1,n} - \delta\psi_{3,n}\right) \qquad (3\text{-}12c)$$

where $\psi_{1,n}$, $\psi_{2,n}$, and $\psi_{3,n}$ are respectively the axial rotations of the three quadrupoles;

$$\delta\psi_{1,n} \equiv \psi_{2,n} - \psi_{1,n}$$

$$\delta\psi_{3,n} \equiv \psi_{3,n} - \psi_{2,n}$$

[$\delta\psi_{1,n}$ and $\delta\psi_{3,n}$, the rotations of the outer quadrupoles relative to the center quadrupole, are assumed to be independent random variables.]

Phase-space increment;

$$\overline{\delta u_n} \approx 2 \left(\frac{Q^2\lambda}{\sin\,\theta} \hat{Y}^2 \right)_n \overline{(\delta\psi)^2} \qquad (3\text{-}13c)$$

Orbit perturbation (constant parameters, constant energy);

$$\overline{\xi_n^2} \approx 2n \frac{\Lambda}{D} \frac{\overline{(\delta\psi)^2}\ (Y)^2_{max}}{1 - (\gamma_c/\gamma)^2} \qquad (3\text{-}14c)$$

where $\gamma_c = \frac{1}{4}Q\sqrt{\Lambda D}$ in the ST case.[2]

Tolerance on relative axial rotation;

$$< \delta\psi >_{rms} \lesssim \frac{1}{2} \frac{|\xi|_{max}}{|Y|_{max}} \sqrt{\frac{D}{n\Lambda}} \qquad (3\text{-}15c)$$

(d) Close Triplet (CT)

The results in this case are:

Perturbation vector (dominant terms);

$$\mathbf{a}_n \approx \begin{pmatrix} 0 \\ \\ \hat{QY} \end{pmatrix}_n (\delta\psi_{1,n} - \delta\psi_{2,n}) \qquad (3\text{-}12d)$$

where $\delta\psi_{1,n}$ and $\delta\psi_{2,n}$, as in the ST case, are the axial rotations of the outer quadrupoles relative to the center quadrupole.

Phase-space increment;

$$\overline{\delta u_n} \approx 2 \left(\frac{Q^2\lambda}{\sin\,\theta} \hat{Y}^2 \right)_n \overline{(\delta\psi)^2} \qquad (3\text{-}13d)$$

Orbit perturbation (constant parameters, constant energy);

$$\overline{\xi_n^2} \approx 6n \, \frac{\Lambda}{T} \, \frac{\overline{(\delta\psi)^2} \; (Y)^2_{max}}{1 - (\gamma_c/\gamma)^2} \tag{3-14d}$$

where $\gamma_c = \frac{1}{4}Q\sqrt{\frac{1}{3}\,\Lambda T}$ in the CT case.[2]

Tolerance on relative axial rotation;

$$< \delta\psi >_{rms} \lesssim \frac{1}{2} \, \frac{|\xi|_{max}}{|Y|_{max}} \sqrt{\frac{T}{3n\Lambda}}$$

IV. LONGITUDINAL MISALIGNMENTS

A. FORMULATION

An optic element which has a small accidental longitudinal displacement from its correct position may be described by the matrix

$$\mathbf{M}' = \begin{pmatrix} 1 & -\delta\ell \\ 0 & 1 \end{pmatrix} \mathbf{M} \begin{pmatrix} 1 & \delta\ell \\ 0 & 1 \end{pmatrix}$$

where $\delta\ell = \frac{\delta L}{\gamma}$ and δL is the longitudinal displacement. To first order in δL,

$$\delta\mathbf{M} \equiv \mathbf{M}' - \mathbf{M} \approx \begin{pmatrix} -m_{21} & m_{11} - m_{22} \\ 0 & m_{21} \end{pmatrix} \delta\ell \tag{4-1}$$

If we make the substitution

$$\mathbf{x} = \mathbf{X} + \xi$$

where $\mathbf{X}$ is the solution of the unperturbed system and ξ is assumed small, then the transformation through the optic element is

$$\left.\begin{aligned} \mathbf{X}_2 &= \mathbf{M}\mathbf{X}_1 \\ \xi_2 &\approx \mathbf{M}\xi_1 + \mathbf{m} \end{aligned}\right\} \tag{4-2}$$

where

$$\mathbf{m} = (\delta\mathbf{M})\mathbf{X}_1 = (\delta\mathbf{M})\mathbf{M}^{-1}\mathbf{X}_2 \tag{4-3}$$

The solution for the orbit perturbation thus is analogous to the cases in Sections II and III.

B. MULTIPLET SYSTEMS

1. Longitudinal Displacement of the Multiplet

Suppose that the individual quadrupoles of a multiplet all have an equal longitudinal displacement – e.g., displacement of the common support system. Then by Eq. (4-1) (using the singlet approximation[2] for the multiplet), we find

$$\delta \mathbf{M}_n \approx \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{\gamma}{F_n}\right) \delta\ell_n$$

whence by Eq. (4-3)

$$\mathbf{a}_n \approx \begin{pmatrix} \hat{X} \\ -\hat{P}_x \end{pmatrix} \left(\frac{\gamma}{F}\right)_n \delta\ell_n \tag{4-4}$$

where $\hat{X}$, $\hat{P}_x$ are the coordinates measured just ahead of the lens, and $\mathbf{a}$ is the perturbation measured just beyond the lens.

Use of the results of Section II.A and the singlet matrix elements[2] then gives, for the phase-space increment

$$\delta\overline{u_n} \approx \left(\frac{1}{F}\right)_n^2 \overline{\delta L^2} u \approx 16 \frac{\gamma_c{}^4}{\gamma^4} \frac{\overline{\delta L^2}}{\Lambda^2} u \tag{4-5}$$

where u is the adiabatic invariant in the unperturbed system. It then follows that the orbit perturbation in the constant-parameter, constant-energy case is

$$\overline{\xi_n^2} \approx 8n \frac{\gamma_c{}^4}{\gamma^4} \frac{\overline{\delta L^2}}{\Lambda^2} (X)^2_{max} \tag{4-6}$$

Then the tolerance on longitudinal misalignment, based on $\gamma = \sqrt{2}\,\gamma_c$, is

$$< \delta L >_{rms} \lesssim \frac{|\xi|_{max}}{|X|_{max}} \frac{\Lambda}{\sqrt{2n}} \qquad (4\text{-}7)$$

2. Independent Relative Longitudinal Displacements of the Quadrupoles

(a) Spaced Doublet (SD)

Suppose that the first and second quadrupoles of the pair are displaced by $-\frac{1}{2}\delta D$ and $\frac{1}{2}\delta D$, respectively, from their correct positions. Then the perturbation vector is given, with the help of Eqs. (4-1) - (4-3), by

$$\mathbf{a}_n = \left[\mathbf{Q}\,\mathbf{D}\,(\delta\mathbf{Q}_1)\,\hat{\mathbf{Q}}^{-1}\mathbf{D}^{-1}\mathbf{Q}^{-1} + (\delta\mathbf{Q}_2)\mathbf{Q}^{-1}\right]_n \mathbf{X}_n$$

where $\mathbf{Q}$, $\hat{\mathbf{Q}}$, and $\mathbf{D}$ are as defined in Sect. III.B.2(a);

$$\left(\delta\mathbf{Q}_1\right) = \left(\delta\mathbf{Q}_2\right) = \tfrac{1}{2}Q\delta d \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ;$$

and

$$\delta d = \frac{\delta D}{\gamma}$$

Expanding and keeping only the dominant terms, we find

$$\mathbf{a}_n \approx (Q\delta d)_n \begin{pmatrix} 1 & 0 \\ -Q_n & -1 \end{pmatrix} \mathbf{X}_n \qquad (4\text{-}8a)$$

The dominant part of the phase-space increment then is found to be

$$\overline{\delta u_n} \approx \left(\frac{Q^4 \lambda \, \overline{\delta d^2} \, X^2}{\sin \theta} \right)_n \qquad (4\text{-}9a)$$

The orbit perturbation in the constant-parameter, constant-energy case then is found to be*

$$\overline{\xi_n^2} \approx n \, \frac{\overline{\delta D^2}}{D^2} \, \frac{(\gamma_c/\gamma)^2}{1 - (\gamma_c/\gamma)^2} \, (X)^2_{max} \qquad (4\text{-}10a)$$

The tolerance on relative longitudinal misalignment, for $\gamma = \sqrt{2}\,\gamma_c$, is

$$< \delta D >_{rms} \stackrel{\sim}{<} \frac{|\xi|_{max}}{|X|_{max}} \, \frac{D}{\sqrt{n}} \qquad (4\text{-}11a)$$

(b) Close Doublet (CD)

In this case it is helpful to imagine that there is a small gap of length D, between the two quadrupoles, such that

$$< \delta D > \ll D \ll T$$

— i.e., the gap has negligibly small effect on the gross properties of the doublet, but allows the errors to have a symmetrical distribution.

The calculation is then analogous to the preceding (SD) case. The results for the perturbation vector and phase-space increment are formally the same as Eqs. (4-8a) and (4-9a).

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* X_n^2 is replaced by $\frac{1}{2}(X)^2_{max}$ in the summation.

The orbit perturbation (for constant parameters and energy) is

$$\overline{\xi_n^2} \approx 9n \, \frac{\overline{\delta D^2}}{T^2} \, \frac{(\gamma_c/\gamma)^2}{1 - (\gamma_c/\gamma)^2} \, (X)^2_{max} \tag{4-10b}$$

and the tolerance on the doublet spacing error is

$$< \delta D >_{rms} \lesssim \frac{1}{3} \, \frac{|\xi|_{max}}{|X|_{max}} \, \frac{T}{\sqrt{n}} \tag{4-11b}$$

(c) Spaced Triplet (ST)

In this case it is appropriate to consider the displacements of the outer quadrupoles, relative to the center quadrupoles, to be independent random variables. Then in the same approximation as Eq. (4-8a), the perturbation vector is found to be

$$\mathbf{a}_n \approx \tfrac{1}{2}Q_n \begin{pmatrix} 1 & 0 \\ -\tfrac{1}{2}Q_n & -1 \end{pmatrix} \mathbf{X}_n \left(\delta d_{1,n} + \delta d_{3,n}\right) \tag{4-8c}$$

where $\delta d_{1,n} = \dfrac{\delta D_{1,n}}{\gamma_n}$, etc.; $\delta D_{1,n}$ and $\delta D_{3,n}$ are the errors in position of the first and third quadrupoles, relative to the second.

The results of the remainder of the calculation are:

Phase-space increment;

$$\overline{\delta u_n} \approx \frac{1}{8} \left(\frac{Q^2 \, \overline{\delta d^2} \, \lambda X^2}{\sin \theta} \right)_n \tag{4-9c}$$

Orbit perturbation (constant parameters and energy);

$$\overline{\xi_n^2} \approx 2n \frac{\overline{\delta D^2}}{D^2} \frac{(\gamma_c/\gamma)^2}{1 - (\gamma_c/\gamma)^2} (X)^2_{max} \qquad (4\text{-}10c)$$

Tolerance on relative longitudinal errors;

$$< \delta D >_{rms} \lesssim \frac{|\xi|_{max}}{|X|_{max}} \frac{D}{\sqrt{2n}} \qquad (4\text{-}11c)$$

(d) <u>Close Triplet</u> (CT)

The calculation is analogous to the preceding cases. The perturbation vector and phase-space increment, considering only the dominant terms, are formally the same as Eqs. (4-8c) and (4-9c), above. The remainder of the results are:

Orbit perturbation (constant parameters and energy);

$$\overline{\xi_n^2} \approx 18n \frac{\overline{\delta D^2}}{T^2} \frac{(\gamma_c/\gamma)^2}{1 - (\gamma_c/\gamma)^2} (X)^2_{max} \qquad (4\text{-}10d)$$

Tolerance on relative longitudinal errors;

$$< \delta D >_{rms} \lesssim \frac{1}{3} \frac{|\xi|_{max}}{|X|_{max}} \frac{T}{\sqrt{2n}} \qquad (4\text{-}11d)$$

V. ERRORS IN QUADRUPOLE STRENGTH

A. FORMULATION

Suppose that the j-th quadrupole in a multiplet has an error, δQ_j in its strength, Q_j. Then the perturbed matrix for the multiplet will be

$$\mathbf{M}' = \mathbf{M} + (\delta\mathbf{M}) \tag{5-1}$$

where $\mathbf{M}$ is the unperturbed transformation and

$$(\delta\mathbf{M}) \approx \sum \frac{\partial}{\partial Q_j}(\mathbf{M})\ \delta Q_j \tag{5-2}$$

(to first order in the error).

The situation is then analogous to the case of a longitudinally displaced quadrupole, so that Eqs. (4-2) and (4-3) of the preceding Section apply, with $\delta\mathbf{M}$ defined by Eq. (5-2).

B. APPLICATION

1. Variation in the multiplet as a whole

Consider an error in which all the quadrupoles of a multiplet vary proportionately. In this case we can use the singlet **approximation,**[2] which gives

$$(\delta\mathbf{M}) = \frac{\partial}{\partial F}\begin{pmatrix} 1 & 0 \\ -\frac{\gamma}{F} & 1 \end{pmatrix}\delta F$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\frac{\gamma\delta F}{F^2}$$

Proceeding as in the previous sections, one finds the following results: Perturbation vector;

$$\mathbf{a}_n = \begin{pmatrix} 0 \\ X_n \end{pmatrix}\frac{\gamma\delta F}{F^2} \tag{5-3}$$

Phase-space increment (assuming all multiplets vary independently);

$$\overline{\delta u}_n = \left[\left(\frac{\gamma}{F}\right)^2 \frac{\lambda}{\sin\theta} \frac{\overline{\delta F^2}}{F^2} X^2\right]_n \tag{5-4}$$

Orbit perturbation (for constant parameters and energy);

$$\overline{\xi^2_n} = n \frac{(\gamma_c/\gamma)^2}{1 - (\gamma_c/\gamma)^2} \frac{\overline{\delta F^2}}{F^2} (X)^2_{max}$$

Tolerance on random variations in multiplet focal length (for $\gamma = \sqrt{2}\,\gamma_c$);

$$\frac{\langle F \rangle_{rms}}{F} \lesssim \frac{|\xi|_{max}}{\sqrt{n}\,|X|_{max}} \tag{5-5}$$

This may be related to current-regulation tolerance, on the assumption that all quadrupoles of a given multiplet have a common power supply. For all the multiplets considered here,

$$F \propto Q^{-2} \propto I^{-2}$$

where I is the quadrupole exciting current.
Hence

$$\frac{\langle \delta I \rangle_{rms}}{|I|} \lesssim \frac{1}{2} \frac{|\xi|_{max}}{\sqrt{n}\,|X|_{max}} \tag{5-6}$$

2. Independent Variations of the Quadrupoles

The effect of a variation in quadrupole strength on the matrix for the quadrupole is given (in the focusing plane) by

$$(\delta \mathbf{Q}) = \frac{\partial}{\partial Q} \begin{pmatrix} c & \frac{1}{k\gamma} s \\ -k\gamma s & c \end{pmatrix} \delta Q$$

where the notation is the same as in Section II.B.3. Expansion of the terms gives (to first order in the thickness, ΔZ)

$$(\delta \mathbf{Q}) \approx - \begin{pmatrix} \frac{1}{2}\frac{\Delta Z}{\gamma} & 0 \\ 1 - \frac{1}{3}\frac{Q\Delta Z}{\gamma} & \frac{1}{2}\frac{\Delta Z}{\gamma} \end{pmatrix} \delta Q \qquad (5\text{-}7)$$

In the defocusing plane the result is the same except that the signs of Q and δQ are changed.

(a) Spaced Doublet (SD). The perturbation vector in this case is

$$\mathbf{a}_n = \left[\mathbf{Q}\mathbf{D}(\delta\mathbf{Q}_1)\,\hat{\mathbf{Q}}^{-1}\mathbf{D}^{-1}\mathbf{Q}^{-1} + (\delta\mathbf{Q}_2)\mathbf{Q}^{-1} \right]_n \mathbf{X}_n$$

where [for the thin-lens limit of Eq. (5-7)]

$$(\delta\mathbf{Q}_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \delta Q_1; \qquad (\delta\mathbf{Q}_2) = - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \delta Q_2;$$

and the other quantities are as defined in Section III.B.2 (a).

Carrying out the products and keeping only the dominant terms, we find

$$\mathbf{a}_n \approx \begin{pmatrix} 0 \\ X_n \end{pmatrix} \left(\delta Q_1 - \delta Q_2\right) \qquad (5\text{-}8a)$$

The errors δQ_1 and δQ_2 both are considered random and independent - e.g., errors in the construction or in the magnetic material of the quadrupoles. The phase-space increment (dominant part) is found to be

$$\overline{\delta u}_n \approx 2\overline{\delta Q^2} \left(\frac{\lambda X^2}{\sin\theta}\right)_n \qquad (5\text{-}9a)$$

The orbit perturbation (for constant parameters and energy) is

$$\overline{\xi_n^2} \approx \frac{n}{2}\frac{\Lambda}{D}\,\frac{\overline{\delta Q^2}}{Q^2}\,\frac{(X)^2_{max}}{1 - \left(\gamma_c/\gamma\right)^2} \qquad (5\text{-}10a)$$

The quadrupole uniformity tolerance, calculated for $\gamma = \sqrt{2}\,\gamma_c$, is

$$\frac{<\delta Q>_{rms}}{Q} \lesssim \frac{|\xi|_{max}}{|X|_{max}} \sqrt{\frac{D}{n\Lambda}} \tag{5-11a}$$

(b) Close Doublet (CD). In this case a calculation analogous to the preceding shows that the perturbation vector and phase-space increment (if only dominant terms are kept) are formally the same as given in Eqs. (5-8a) and (5-9a). The remainder of the calculation gives:
Orbit perturbation (for constant parameters and energy);

$$\overline{\xi_n^2} \approx \tfrac{3}{2}n \, \frac{\Lambda}{T} \, \frac{\overline{\delta Q^2}}{Q^2} \, \frac{(X)^2_{max}}{1 - (\gamma_c/\gamma)^2} \tag{5-10b}$$

Quadrupole uniformity tolerance (for $\gamma = \sqrt{2}\,\gamma_c$);

$$\frac{<\delta Q>_{rms}}{Q} \lesssim \frac{|\xi|_{max}}{|X|_{max}} \sqrt{\frac{T}{3n\Lambda}} \tag{5-11b}$$

(c) Spaced Triplet (ST). In this case the dominant part of the perturbation vector is found to be

$$\mathbf{a}_n \approx \begin{pmatrix} 0 \\ X_n \end{pmatrix} \left(-\tfrac{1}{2}\delta Q_{1,n} + \delta Q_{2,n} - \tfrac{1}{2}\delta Q_{3,n} \right) \tag{5-8c}$$

The phase-space increment to the same approximation is

$$\delta\bar{u}_n \approx \tfrac{3}{2}\,\overline{\delta Q^2} \left(\frac{\lambda X^2}{\sin\theta} \right)_n \tag{5-9c}$$

The orbit perturbation in the constant-parameter, constant energy case is

$$\overline{\xi_n^2} \approx \tfrac{3}{2}n \, \frac{\Lambda}{D} \, \frac{\overline{\delta Q^2}}{Q^2} \, \frac{(X)^2_{max}}{1 - (\gamma_c/\gamma)^2} \tag{5-10c}$$

The quadrupole uniformity tolerance (for $\gamma = \sqrt{2}\,\gamma_c$) is

$$\frac{\langle \delta Q \rangle_{rms}}{Q} \lesssim \frac{\xi_{max}}{X_{max}} \frac{D}{3n\Lambda} \qquad (5\text{-}11c)$$

(d) Close Triplet (CT). In this case the dominant part of the perturbation vector and phase-space increment are found to be formally the same as Eqs. (5-8c) and (5-9c). The remainder of the calculation gives:

Orbit perturbation (constant parameters and energy);

$$\overline{\xi_n^2} \approx \frac{9}{2} n \frac{\Lambda}{T} \frac{\delta Q^2}{Q^2} \frac{(X)^2_{max}}{1 - (\gamma_c/\gamma)^2} \qquad (5\text{-}10d)$$

quadrupole uniformity tolerance;

$$\frac{\langle \delta Q \rangle_{rms}}{Q} \lesssim \frac{1}{3} \frac{\xi_{max}}{X_{max}} \sqrt{\frac{T}{n\Lambda}} \qquad (5\text{-}11d)$$

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