SLAC-14
UC-28, Particle Accelerators
and High-Voltage Machines
UC-34, Physics
TID-4500

OPTICAL PROPERTIES OF QUADRUPOLE MULTIPLETS FOR SECTOR FOCUSING IN THE TWO-MILE ACCELERATOR

February 1963

by

R. H. Helm
Stanford Linear Accelerator Center

Technical Report
Prepared Under
Contract AT(04-3)-400
for the USAEC
San Francisco Operations Office

Printed in USA. Price $ .75. Available from the Office of Technical Services, Department of Commerce, Washington 25, D.C.

TABLE OF CONTENTS

I. INTRODUCTION

A. GENERAL CONSIDERATIONS

Two previous reports discussed the optical properties[1] and the effects of misalignments and quadrupole errors[2] for focusing systems for the two-mile accelerator. Numerical examples were based on a system consisting of quadrupoles of alternating sign, equal strength, and equal 40-foot spacing (hereafter referred to as "the 40-foot system").

Because of the exceptionally tight alignment tolerances imposed by the 40-foot system, it seems desirable to develop the properties of systems of quadrupole multiplets, consisting of closely grouped doublets or triplets having essentially the properties of circularly symmetric thin converging lenses, and spaced at intervals of (typically) one sector or ≈ 330 ft.

This sector-focusing system has several probable advantages over the 40-foot system:

(1) The quadrupoles of a given multiplet can be prealigned with respect to one another (probably within 0.001 to 0.002 inch) on a rigid, common support system. Any motion of the support system then would move the multiplet as a whole, a situation which tends to reduce the sensitivity of the system to misalignments.

(2) Because there would be a much smaller number of independent supports, the realignment procedure would be greatly facilitated.

(3) Since there are to be beam-position monitors at sector intervals, there would be unambiguous information as to the misalignment of a given lens, which again would facilitate realignment.

(4) Because of the one-to-one correspondence between lenses and beam position information, pulsed steering (with multiple beams) could be used if necessary.

There are also some disadvantages:

(1) At a given phase space admittance, the sector system will tend to have a somewhat higher low-energy cutoff. (See Section II below.)

---

[1]All references will be found at the end of the paper.

(2) The sector system probably will use more power than the 40-foot system, since quadrupoles are used less efficiently if closely grouped.

B. RESTRICTIONS AND CRITERIA

In discussing the properties of multiplet systems suitable for sector focusing, it will be convenient to introduce several limitations and general criteria:

(1) Regular Section. The discussion will concern in detail only that part of the machine in which the multiplet lenses are spaced at regular (sector) intervals.

(2) Use of Drift Space. It will be assumed that the regular lenses are contained within the 9-foot drift space which has been provided at the end of each sector. This is important because it allows use of economical, small-aperture quadrupoles.

(3) Constant (or Adiabatically Varying) Parameters. It will be assumed that the properties of the system (such as beam energy and quadrupole strength) are essentially constant or vary only slightly over lengths on the order of a focusing period or one sector.

(4) Focusing Strength Criterion. The focusing strength of the system will be considered as equivalent to the admittance. As in Ref. 1, admittance will be defined in terms of an integral in the (x, p_x) plane:

$$A = \int dp_x dx$$

where the integral is over all orbits which can be transmitted. In the present report it will be more convenient to define the admittance as Υ, given by

$$\Upsilon = \frac{1}{\pi} A$$

The maximum practical value of Υ as set by probable beam switchyard admittance, is roughly*

$$\Upsilon_{max} \approx 0.06\ V_f \qquad (1\text{-}1)$$

where V_f is final beam energy in Bev and Υ is in units of mc-cm, (i.e., momenta are measured in units of (rest-mass) × (velocity of light) or 0.511 Mev/c.

C. NOTATION AND UNITS

The following notation will be used:

x, y = transverse coordinates;

p_x, p_y = transverse momenta;

z = longitudinal coordinate (independent variable);

γ = total energy of electrons;

scalar momentum = $\sqrt{\gamma^2 - 1} \approx \gamma$ (the relativistic approximation will be assumed throughout);

$\mathbf{x}$, $\mathbf{y}$ will denote the vectors $\begin{pmatrix} x \\ p_x \end{pmatrix}$ and $\begin{pmatrix} y \\ p_y \end{pmatrix}$, respectively and it will be assumed that the transverse axes are chosen so that x and y are not coupled in first-order theory;

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ will denote unit transformations (i.e., over one section of the periodic focusing) in x and y, respectively;

$|Q| \equiv \int_{quad} \left| \frac{\partial B_y}{\partial x} \right| dz$ will define the "strength" of a quadrupole, where $\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$ is the magnetic field gradient and the integral is over the effective length of the particular quadrupole lens;

---

*Based on an angular acceptence of $\pm 10^{-4}$ radian and radial acceptance of ± 0.3 cm for the switchyard transport system.[3]

$$|g| = \left|\frac{\partial B_y}{\partial_x}\right| = \left|\frac{\partial B_x}{\partial_y}\right| \text{; and}$$

E = accelerating field seen by the electrons.

Unless otherwise noted, all lengths will be in units of cm; energy in units of mc^2 (0.511 Mev); momenta in units of mc (0.511 Mev/c); and Q, B, and E in units of $mc^2/e/cm$, which is equivalent to 1703 gauss or 0.511 Mv/cm.

Although the emphasis here is on the regular focusing sections, the transformation matrices will be given in a form general enough to be applied to problems involving special lenses with tapered spacings (for matching low energies into the regular spacing), and problems of misalignments and other perturbations.

In Section II the properties of an idealized system, consisting of equally spaced converging lenses, are described. This "singlet" system provides a good first approximation of the behavior of multiplets. In Section III detailed properties of several alternative multiplet combinations are described. Section IV gives a rough preliminary estimate of quadrupole space and power requirements.

II. SYSTEM OF THIN CONVERGING LENSES (SINGLET)

We consider here a periodic system of regularly-spaced thin lenses as illustrated by Fig. 2-1. The focal lengths are F and the spacing is Λ.

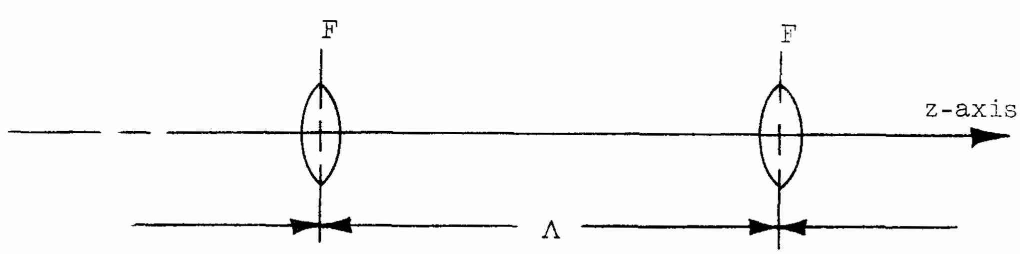

FIG. 2-1--System of thin converging lenses.

It will be convenient to use the notation

$$\lambda = \int_{z}^{z+\Lambda} \frac{dz}{\gamma} \qquad (2\text{-}1)$$

which gives for constant energy

$$\lambda = \frac{\Lambda}{\gamma} \qquad (2\text{-}1a)$$

or for constant acceleration

$$\lambda = \frac{1}{E} \log \left[1 + \frac{\Lambda E}{\gamma(z)}\right] \qquad (2\text{-}1b)$$

where E is the accelerating field in units of $mc^2/e/cm$ (0.511 Mv/cm) and γ is the relativistic energy in mc^2 units ($\approx$ scalar momentum in mc units).

The matrix which transforms the vector $\begin{pmatrix} x \\ p_x \end{pmatrix}$ over one focusing period, if reference planes are chosen <u>just beyond the lenses</u>, is

$$A = \begin{pmatrix} 1 & 0 \\ -\frac{\gamma}{F} & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ -\frac{\gamma}{F} & 1 - \frac{\gamma\lambda}{F} \end{pmatrix} \qquad (2\text{-}2)$$

It is readily shown that the present value of a_{12} (namely, λ) is maximized with respect to choice of reference planes. Hence using Eq. (24) of Ref. 1, the admittance of the system is*

$$\Upsilon = \frac{a^2 \sin\theta}{a_{12}(\max)} = a^2\sqrt{\frac{\gamma}{F\lambda}\left(1 - \frac{1}{4}\frac{\gamma\lambda}{F}\right)} \qquad (2\text{-}3)$$

---

*The quantity Υ, designated as <u>admittance</u> in the present report, is related to the quantity A in Ref. 1 by $\Upsilon = (1/\pi)A$.

where

$$\cos\,\theta \equiv \tfrac{1}{2}\left(a_{11} + a_{22}\right) = 1 - \tfrac{1}{2}\frac{\gamma\lambda}{F}$$

For the simple multiplets discussed in Section III below, it will be found that to a fair approximation the multiplet focal length is proportional to γ^2:

$$F \propto \gamma^2$$

If we state this proportionality in terms of a parameter γ_c, such that

$$F \approx \frac{1}{4}\,\frac{\Lambda\gamma^2}{\gamma_c^2} \tag{2-4}$$

then Eq. (2-3) (for a beam with negligible acceleration) becomes

$$\Upsilon \approx \Upsilon_\infty\sqrt{1 - \frac{\gamma_c^2}{\gamma^2}} \tag{2-5}$$

where

$$\Upsilon_\infty \equiv \frac{2a^2\gamma_c}{\Lambda} \tag{2-6}$$

= asymptotic value of admittance for $\gamma >> \gamma_c$

It is clear from Eq. (2-5) that γ_c plays the part of a cutoff energy, below which the system does not transmit real phase space. Note that to the same approximation,

$$\cos\,\theta \approx 1 - 2\,\frac{\gamma_c^2}{\gamma^2} \tag{2-7}$$

so that γ_c also represents the energy below which

$$\cos \theta < -1$$

It will be convenient to define a practical low-energy band limit γ_o, such that

$$\gamma_o = \sqrt{2}\,\gamma_c = \frac{\Lambda \Upsilon_\infty}{\sqrt{2}a^2} \qquad (2\text{-}8)$$

Then

$$\Upsilon_o \equiv \Upsilon(\gamma_o) = \sqrt{\tfrac{1}{2}}\,\Upsilon_\infty \qquad (2\text{-}9)$$

i.e., at γ_o, the transmission (proportional to Υ^2) is down by a factor of 2 from its asymptotic value.

It may be shown that Υ is maximized for a particular γ and Λ if $\gamma_c = \gamma/\sqrt{2}$. Note that $\cos\theta(\gamma_o) = 0$; i.e., at $\gamma = \gamma_o$ the system is in the "$\pi/2$ mode" or four focusing sections per orbit wavelength.

A. NUMERICAL EXAMPLES

As an example, suppose we wanted the system to transmit energies down to 1 Bev (i.e., $\gamma_o = 2 \times 10^3$). Then using $a = 0.85$ cm and $\Lambda = 10^4$ cm (330 ft), we would have

$$\Upsilon_o \approx 0.14 \text{ (mc - cm)}$$

or

$$\Upsilon_\infty \approx 0.20 \text{ (mc - cm)}$$

On the other hand, if the system were to have an asymptotic admittance corresponding to the maximum useful value at 10 Bev, then Eq. (1-1) gives

$$\Upsilon_\infty \approx 0.6$$

from which Eq. (2-9) gives

$$\gamma_o \approx 5.9 \times 10^3 \text{ (3 Bev)}$$

$$\gamma_c \approx 4.2 \times 10^3 \text{ (2.1 Bev)}$$

Use of the maximum focusing strength would thus somewhat limit the low energy end of the band.

B. COMPARISON WITH 40-FOOT SYSTEM

Comparative numbers for the 40-foot system might be of some interest. Equation (37) of Ref. 1 gives

$$\Upsilon = \Upsilon_\infty \sqrt{\frac{1 - \frac{1}{2} Q\ell}{1 + \frac{1}{2} Q\ell}}$$

where Q is the quadrupole strength,

$$\ell \approx L/\gamma$$

L = quadrupole spacing, and

$$\Upsilon_\infty = \frac{1}{2} Qa^2$$

The criteria for practical low energy band limit and low energy cutoff are given by Ref. 1 as

$$\gamma_1 \approx \frac{QL}{1.24} \approx \frac{\Upsilon_\infty L}{0.62\ a^2}$$

and

$$\gamma_c \approx \frac{QL}{2} \approx \frac{\Upsilon_\infty L}{a^2}$$

from which, taking $\Upsilon_\infty = 0.6$ as in the above example, and taking $L = 40$ ft $= 1.2 \times 10^3$ cm, we find for the 40-foot system:

$$\gamma_1 \approx 1.6 \times 10^3 \quad (\approx 0.8 \text{ Bev})$$

$$\gamma_c \approx 1.0 \times 10^3 \quad (\approx 0.5 \text{ Bev})$$

III. OPTICAL PROPERTIES OF PERIODIC MULTIPLETS

A. SPACED-DOUBLET (SD) SYSTEM

In this system the basic section consists of a pair of quadrupoles of nominally equal strength and opposite sign, spaced by an amount fairly large compared to the thickness of the quadrupoles. Consider the basic focusing period defined by Fig. 3-1.

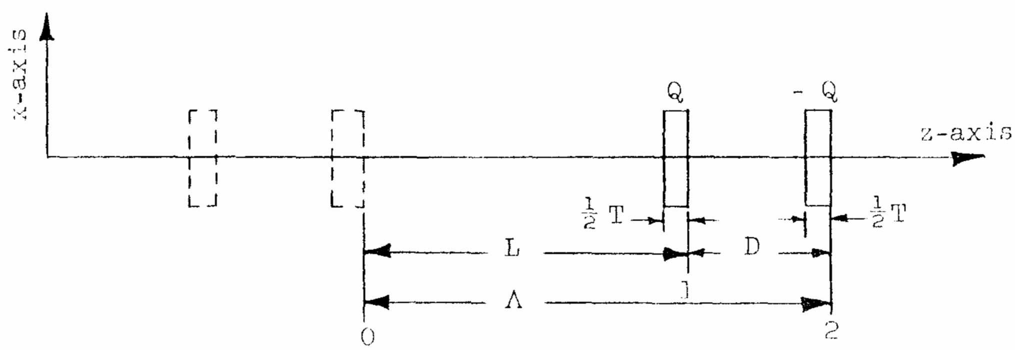

FIG. 3-1--"Spaced-doublet" system in **x**-plane.

On the assumption that thin-lens formulas apply for the individual quadrupoles, the transformation from plane 0 to plane 2 is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + Qd & \ell + d + Q\ell d \\ -Q^2 d & 1 - Qd - Q^2 \ell d \end{pmatrix} \tag{3-1}$$

where (see Section I-C, "Notation and Units");

$$\ell = \int_0^1 \frac{dz}{\gamma} \approx \frac{L}{\gamma}$$

$$d = \int_1^2 \frac{dz}{\gamma} \approx \frac{D}{\gamma}$$

Note that

$$\cos\theta \equiv \tfrac{1}{2}(a_{11} + a_{22}) = 1 - \tfrac{1}{2}Q^2\ell d \qquad (3\text{-}2)^*$$

If we call the situation in Fig. 3-1 the $\mathbf{x}$ plane, then the transformation, $\mathbf{B}$, in the y plane, is the same except that the sign of Q is reversed.

It will be stated without proof that the value of a_{12} given by Eq. (3-1) is maximized with respect to choice of reference planes; and that b_{12}(max) also has the same value [but b_{12}(max) occurs at the first quadrupole of the doublet, rather than the second]. Hence the admittance by Eq. (24) of Ref. 1, is

$$\Upsilon_x = \Upsilon_y = \Upsilon = \frac{Qa^2\sqrt{\ell d(1 - \frac{1}{4}Q^2\ell d)}}{\ell + d + Q\ell d} \qquad (3\text{-}3)$$

In order to use the thin lens "singlet" results of Section II, it will be sufficient to require that D be small in the sense that

$$Qd \approx \frac{QD}{\gamma} \ll 1 \qquad (3\text{-}4)$$

and

$$D \ll \Lambda \equiv L + D \qquad (3\text{-}5)$$

*It should be recalled that $\cos\theta$ is invariant under a longitudinal translation of reference planes.[4]

Equation (3-3) then takes the form of Eq. (2-5):

$$\Upsilon \approx \Upsilon_\infty \sqrt{1 - \frac{\gamma_c^2}{\gamma^2}}$$

where

$$\Upsilon_\infty = Qa^2\sqrt{\frac{D}{\Lambda}} \tag{3-6}$$

and

$$\gamma_c = \tfrac{1}{2}Q\sqrt{\Lambda D} = \frac{\Upsilon_\infty \Lambda}{2a^2} \tag{3-7}$$

The focal length of the multiplet is readily identified as

$$F = \frac{\gamma^2}{Q^2 D} \tag{3-8}$$

which justifies Eq. (2-4) for the present system.

The total length, T, required by the two quadrupoles is

$$T = \frac{2Q}{g} = \frac{2\Upsilon_\infty}{ga^2}\sqrt{\frac{\Lambda}{D}} \tag{3-9}$$

where g is the quadrupole gradient $\partial B_y/\partial x$ in units of $mc^2/e/(cm)^2$ or 1703 gauss/cm.

Numerical Example. As before, assume that $\Upsilon_\infty = 0.6$ mc-cm, $a = 0.85$ cm, and $\Lambda = 10^4$ cm. Take $D = 250$ cm ≈ 8 ft which would be about the maximum doublet spacing within the 9-ft drift space. Then Eqs. (3-6) and (3-7) give

$$Q = \frac{\Upsilon_\infty}{a^2}\sqrt{\frac{\Lambda}{D}} = 5.3 \ (\approx 9000 \text{ gauss})$$

and

$$\gamma_o = 5.9 \times 10^3 \quad (\approx 3 \text{ Bev})$$

The cutoff energy, by Eq. (2-8) is

$$\gamma_c = 4.2 \times 10^3 \quad (\approx 2.1 \text{ Bev})$$

On the basis of criteria assumed by Brown,[5] a reasonable estimate of the gradient is ≈ 1000 gauss/cm, or $g = 0.6$ in the present units. Then Eq. (3-9) gives for the total quadrupole length per sector,

$$T = 17.5 \text{ cm} \approx 7 \text{ in.}$$

or 3.5 in. for each quadrupole.

To see whether the treatment of the doublet as a thin lens is valid, note that in the present example

$$\frac{D}{\Lambda} \approx \frac{1}{40}$$

so that Eq. (3-5) is satisfied.

On the other hand, if we calculate Qd at $\gamma = \gamma_o$,

$$Qd_o = \frac{QD}{\gamma_o} \approx 0.225$$

so that Eq. (3-4) is not too well satisfied in the low-energy end of the transmission band. Calculation of admittance from Eq. (3-3) gives

$$\Upsilon_o \equiv \Upsilon(\gamma_o) \approx 0.58\, \Upsilon_\infty$$

where the approximate expression would give

$$\Upsilon_o = 0.707\, \Upsilon_\infty$$

Thus it can be seen that the approximate expressions are valuable in obtaining a quick first approximation, but that the exact expressions should be used if one needs precise answers.

It may also be pointed out that $T \ll D$ (i.e., $T/D \approx 0.03$ in the present example) so that the use of thin-lens formulas for the individual quadrupoles in Eq. (3-1) is justified.

B. CLOSE DOUBLET (CD) SYSTEM

The CD configuration is illustrated in Fig. 3-2. In this case it is not immediately obvious exactly where the periodic reference planes should be chosen in order to define the maximum value of the matrix element a_{12},

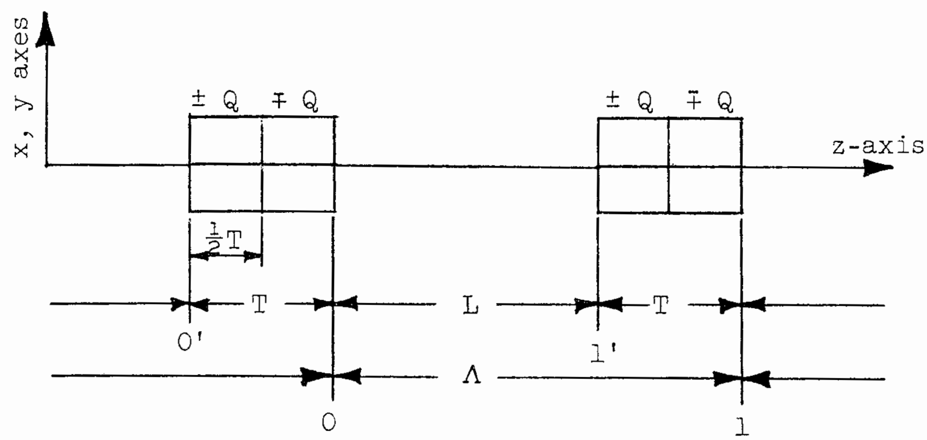

FIG. 3-2--"Close Doublet" System. The choice in the sign of Q refers to the x and y planes, respectively.

which determines the admittance of the system. It may be shown by qualitative arguments that in the (initially defocusing) configuration shown in Fig. 3-2, a_{12}(max) occurs very close to the end of the focusing section, so long as $F \gg T$. In this case (plane 0 to plane 1 in Fig. 3-2) the transformation is*

$$A = \begin{pmatrix} c & \frac{1}{k\gamma}s \\ -k\gamma s & c \end{pmatrix} \begin{pmatrix} C & \frac{1}{k\gamma}S \\ k\gamma S & C \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} cC + sS & \ell(cC + sS) + \frac{1}{k\gamma}(sC + cS) \\ -k\gamma(sC - cS) & cC - sS - k\gamma\ell(sC - cS) \end{pmatrix} \qquad (3\text{-}10)$$

where

$$k \equiv \sqrt{g/\gamma};$$

$$g = \left|\frac{dB_y}{dx}\right| \text{ is the magnetic gradient;}$$

$$\varphi \equiv \tfrac{1}{2}\, kT;$$

$$s = \sin\varphi; \quad S = \sinh\varphi;$$

$$c = \cos\varphi; \quad C = \cosh\varphi;$$

and

$$\ell = \int_{z(o)}^{z(z')} \frac{dz}{\gamma} \approx \frac{L}{\gamma}$$

If φ is small, as it will be in the present case, series expansions of the terms will be useful. To second order in T, the result is

$$A \approx \begin{bmatrix} 1 + \frac{1}{2}Qt - \frac{1}{24}\,Q^2t^2 & \lambda + \frac{1}{2}Q\lambda t - \frac{1}{2}Qt^2 - \frac{1}{24}\,Q^2\lambda t^2 \\ -\frac{1}{3}\,Q^2t & 1 - \frac{1}{2}Qt + \frac{7}{24}\,Q^2t^2 - \frac{1}{3}\,Q^2\lambda t \end{bmatrix} \qquad (3\text{-}10a)$$

---

*It is necessary to use the thick-lens formulas for the close-spaced quadrupoles. The appropriate matrices are given for example by Septier.[4]

where

$$t = \frac{T}{\gamma}, \qquad \lambda = \ell + t,$$

and

$$Q = \tfrac{1}{2}\, Tg = \text{strength of the individual quadrupoles.}$$

Note that

$$\cos\,\theta \equiv \tfrac{1}{2}\left(a_{11} + a_{22}\right) \approx 1 - \tfrac{1}{6}Q^2\lambda t + \tfrac{1}{8}Q^2t^2 \qquad (3\text{-}11)$$

The transformation $\mathbf{B}$ in the $\mathbf{y}$ plane (initially <u>focusing</u> doublet configuration), at the same reference planes, may be found from Eq. (3-10) or Eq. (3-10a) by changing the sign of g (or Q) which is the same as replacing k by ik. However, the maximum value of the matrix element b_{12} occurs near the <u>beginning</u> of the doublet rather than the end, i.e., in the system in which the basic transformation is from plane 0' to plane 1' in Fig. 3-2. If we call the transformation in this case $\hat{\mathbf{B}}$, then it is readily shown that

$$\hat{\mathbf{B}} = \begin{pmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{pmatrix} \qquad (3\text{-}12)$$

where the a_{ij} are given by Eq. (3-10) or Eq. (3-10a); thus $b_{12}(\max) = a_{12}(\max)$, as in the SD case.

It will be assumed that the quadrupole thickness T is small in the sense that

$$Qt = \frac{QT}{\gamma} \ll 1 \qquad (3\text{-}13)$$

and

$$T \ll \Lambda = L + T \tag{3-14}$$

[Condition (3-13) is equivalent to the assumption that φ is small in Eq. (3-10) since, as may be seen from the defining equations, $Qt = 2\varphi^2$.] Under these assumptions, keeping correction terms of only up to first order in Qt and T/Λ, the admittance is given by

$$\Upsilon_x = \Upsilon_y \equiv \Upsilon = \frac{a^2 \sin\theta}{a_{12}(\max)} \approx \frac{a^2 Q\sqrt{\frac{1}{3}\lambda t\left(1 - \frac{1}{12} Q^2\lambda t\right)}}{\lambda(1 + \frac{1}{2}Qt)} \tag{3-15}$$

In the approximation corresponding to the "singlet" formulation of Section II,

$$\Upsilon \approx \Upsilon_\infty \sqrt{1 - \frac{\gamma_c^2}{\gamma^2}}$$

where

$$\Upsilon_\infty \approx a^2 Q\sqrt{\frac{T}{3\Lambda}} \tag{3-16}$$

and

$$\gamma_c \approx \tfrac{1}{2}Q\sqrt{\frac{\Lambda T}{3}} \approx \frac{\Upsilon_\infty \Lambda}{2a^2} \tag{3-17}$$

The practical low energy limit is

$$\gamma_o = \gamma_c\sqrt{2} \approx \frac{\Upsilon_\infty \Lambda}{\sqrt{2a^2}} \tag{3-18}$$

and the focal length of the doublet is

$$F = \frac{3\gamma^2}{Q^2 T} \tag{3-19}$$

The total length of the doublet is found from Eq. (3-16), by using the definition $Q = \frac{1}{2} Tg$;

$$T \approx \left(\frac{2\Upsilon_\infty \sqrt{3\Lambda}}{ga^2} \right)^{2/3} \tag{3-20}$$

Numerical example: As before, take $\Upsilon_\infty = 0.6$, $a = 0.85$ cm, $\Lambda \approx L \approx 10^4$cm, and $g = 0.6$ ($\approx$ 1000 gauss/cm). Then in the "singlet" approximation,

$$\gamma_o \approx 5.9 \times 10^3 \quad (\approx 3 \text{ Bev})$$

and

$$\gamma_c \approx \gamma_o/\sqrt{2} \approx 4.2 \times 10^3 \quad (\approx 2.1 \text{ Bev})$$

which of course are the same as in the examples in Sections II and III-A above. From Eq. (3-20) we find

$$T \approx \left[\frac{(2)(0.6)\sqrt{3 \times 10^4}}{(0.6)(0.85)^2} \right]^{2/3} \approx 61 \text{ cm} \approx 24 \text{ inches}$$

The quadrupole strength from Eq. (3-16) is

$$Q \approx \frac{\Upsilon_\infty}{a^2} \sqrt{\frac{3\Lambda}{T}} \approx 18.3 \quad (\approx 31{,}000 \text{ gauss})$$

Condition (3-14) is satisfied since

$$T/\Lambda \approx \frac{61}{10^4} \approx \frac{1}{160}$$

Condition (3-13) is not quite so well satisfied, since (at $\gamma = \gamma_o$)

$$Qt_o = \frac{QT}{\gamma_o} \approx 0.185;$$

however, notice that the correction to the admittance is only half this or $\approx 9\%$, since the quantity enters Eq. (3-15) as $(1 + \frac{1}{2}Qt)^{-1}$.

C. SPACED TRIPLET (ST) SYSTEM

Figure 3-3 illustrates the ST system. In this case it may be shown that the maximum value of b_{12} (the matrix element in the y plane or initially <u>defocusing</u> configuration) occurs at the center quadrupole; and that the maximum value of a_{12} (the matrix element in the x plane or initially <u>focusing</u> configuration) occurs at the beginning of the triplet.

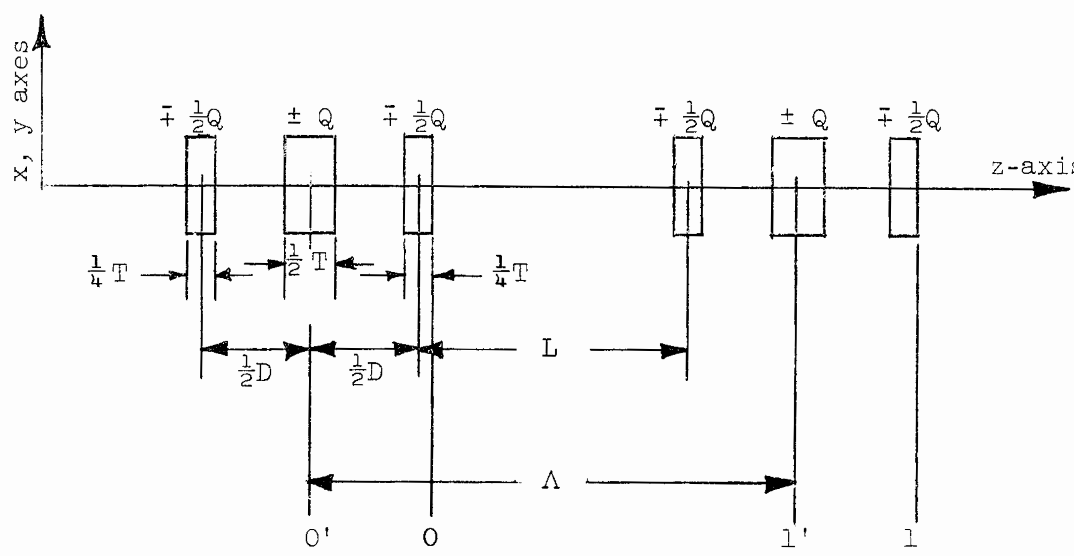

FIG. 3-3--Spaced triplet system in x plane.

It will be assumed as in Section III-A above that the individual quadrupoles may be treated as thin lenses.

In the **x** plane, it will be appropriate to calculate the transformation with reference planes at the end of the triplet (from plane 0 to plane 1 in Fig. 3-3):

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2}d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2}d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1 - \frac{1}{8} Q^2d^2 & \lambda - \frac{1}{8} Q^2\lambda d^2 + \frac{1}{4} Qd^2 + \frac{1}{8} Q^2d^3 \\ -\frac{1}{4}Q^2d(1 - \frac{1}{4}Qd) & 1 + \frac{1}{8} Q^2d^2 - \frac{1}{4}Q^2\lambda d\ (1 - \frac{1}{4}Qd) - \frac{1}{16} Q^3d^3 \end{bmatrix} \qquad (3\text{-}21a)$$

where

$$d \approx D/\gamma$$

$$\ell \approx L/\gamma$$

$$\lambda = \ell + d \approx \Lambda/\gamma$$

In the **y** plane, it will be appropriate to calculate the transformation from center-to-center of the middle quadrupoles (from 0' to 1' in Fig. 3-3);

$$\hat{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2}d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2}Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2}Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2}d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}Q & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1 - \frac{1}{8} Q^2\lambda d - \frac{1}{32} Q^3\ell d^2 & \lambda + \frac{3}{8} Q\lambda d - \frac{1}{4}Qd^2 + \frac{1}{32} Q^2\ell d^2 \\ -\frac{1}{4}Q^2d\ (1 - \frac{1}{4}Qd - \frac{1}{16}Q^2\ell d) & 1 - \frac{1}{8} Q^2\lambda d - \frac{1}{32} Q^3\ell d^2 \end{bmatrix} \qquad (3\text{-}22)$$

Another matrix which may be of some interest is **B** , the transformation from plane 0 to plane 1 in the **y** plane. This is derived from

A, Eq. (3-21a), by changing the sign of Q;

$$\mathbf{B} = \begin{bmatrix} 1 - \frac{1}{8} Q^2 d^2 & \lambda - \frac{1}{8} Q^2 \lambda d - \frac{1}{4} Q d^2 + \frac{1}{8} Q^2 d^3 \\ - \frac{1}{4} Q^2 d (1 + \frac{1}{4} Q d) & 1 + \frac{1}{8} Q^2 d^2 - \frac{1}{4} Q^2 \lambda d (1 + \frac{1}{4} Q d) + \frac{1}{16} Q^3 d^3 \end{bmatrix} \tag{3-21b}$$

This transformation is important because, with sector focusing the quadrupole aperture can be made somewhat larger than the accelerator aperture. In this case the admittance may be determined by b_{12} rather than $\hat{b}_{12}$, even though $\hat{b}_{12} > b_{12}$, because it is actually $\left[a^2/b_{12}\right]_{min}$ which limits the admittance. Thus Eq. (3-22) should be used to describe the **y** plane focusing if the limiting aperture is within the triplet; and (3-21b) should be used if the beam aperture through the lens is greater than the accelerator aperture.

For illustrative purposes it will be assumed that the former situation, Eq. (3-22), applies.

Using the assumption as in Section III-A that

$$Qd = \frac{QD}{\gamma} << 1 \tag{3-23}$$

and

$$D << \Lambda = L + D \tag{3-24}$$

and keeping only the correction terms of first order in Qd and D/Λ, we find

$$a_{12}(\max) = a_{12} \approx \lambda \tag{3-25}$$

$$b_{12}(\max) = \hat{b}_{12} \approx \lambda \left(1 + \frac{3}{8} Qd\right) \tag{3-26}$$

and

$$\cos \theta_{x,y} \approx 1 - \frac{1}{8} Q^2 \lambda d \left(1 \mp \frac{1}{4} Qd\right) \tag{3-27}$$

In the "singlet" approximation corresponding to Section II,

$$a_{12}(\max) \approx b_{12}(\max) \approx \lambda \tag{3-25a}$$

$$\cos \theta_x \approx \cos \theta_y \approx 1 - \frac{1}{8} Q^2 \lambda d \tag{3-27a}$$

and the equivalent singlet focal length is

$$F \approx \frac{4\gamma^2}{Q^2 D} \tag{3-28}$$

Equations (2-5) through (2-9) follow, with the definition

$$\Upsilon_\infty = \frac{1}{2} Qa^2 \sqrt{\frac{D}{\Lambda}} \tag{3-29}$$

The total length of quadrupole per focusing section (see Fig. 3-3) is

$$T = \frac{2Q}{g} = \frac{4\Upsilon_\infty}{ga^2}\sqrt{\frac{\Lambda}{D}} \tag{3-30}$$

where as before g is the magnetic gradient in the quadrupoles.

<u>Numerical Example.</u> As in previous examples, take $\Upsilon_\infty = 0.6$, $a = 0.85$ cm, $\Lambda = 10^4$ cm, $D = 250$ cm, and $g = 0.6$. As before,

$$\gamma_o \approx 5.9 \times 10^3 \approx 3 \text{ Bev}$$

$$\gamma_c \approx 4.2 \times 10^3 \approx 2.1 \text{ Bev}$$

The total length of quadrupole per sector is by Eq. (3-30)

$$T \approx 35 \text{ cm} \approx 14 \text{ in.}$$

and the strength of the quadrupoles is

$$Q \approx \tfrac{1}{2}\, Tg \approx 10.5 \approx 18{,}000 \text{ gauss}$$

Corresponding to conditions (3-24) and (3-23) we have

$$D/\Lambda = \frac{1}{40} \ll 1;$$

and

$$Qd_o = \frac{QD}{\gamma_o} \approx 0.44$$

so that Eq. (3-24) is satisfied but Eq. (3-23) is only qualitatively true. If the admittances at γ_o are calculated using the first-order corrections of Eq. (3-25), (3-26) and (3-27), then

$$\Upsilon_x(\gamma_o) = \left(\frac{a^2 \sin \theta_y}{a_{12}(\max)}\right)_{\gamma_o} \approx \frac{\Upsilon_\infty}{\sqrt{2}} \quad \text{(no first-order correction in this case)}$$

$$\Upsilon_y(\gamma_o) = \left(\frac{a^2 \sin \theta_x}{b_{12}(\max)}\right)_{\gamma_o} \approx \frac{0.86\, \Upsilon_\infty}{\sqrt{2}} \quad \text{(14\% correction)}$$

The singlet approximation thus is fairly good for energies $\gamma \gtrsim \gamma_o$.

In order to make the admittance in the y plane essentially equal to that in the x plane, the beam aperture through the lens would need a radius b such that

$$\frac{b}{a} \geq \sqrt{\frac{\hat{b}_{12}}{b_{12}}} \approx \sqrt{1 + \tfrac{3}{8}\, Qd}$$

With $Qd = 0.44$ as above, this implies

$$b \geq 1.08\ a$$

D. CLOSE TRIPLET (CT) SYSTEM

The CT system is shown in Fig. 3-4. In this case the maximum value of b_{12} (matrix element of **B**, the transformation in the **y**-plane or initially <u>defocusing</u> configuration) occurs at the center of the middle quadrupole. The maximum value of a_{12} (matrix element of **A**, the transformation in the **x**-plane or initially <u>focusing</u> configuration) occurs very close to the beginning and the end of the triplet, provided that $F \gg T$.

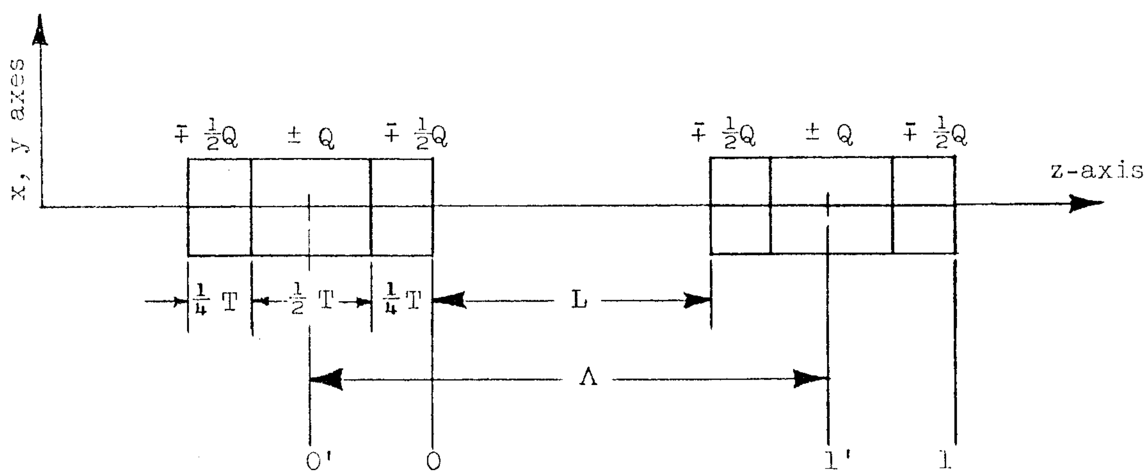

FIG. 3-4--Close triplet system.

In the **x** plane it will be sufficient to calculate the matrix for the system in which the triplet is at the end of the section (from plane 0 to plane 1 in Fig. 3-4):

$$\mathbf{A} = \begin{pmatrix} c' & \frac{1}{k\gamma}s' \\ -k\gamma s' & c' \end{pmatrix} \begin{pmatrix} C & \frac{1}{k\gamma}S \\ k\gamma S & C \end{pmatrix} \begin{pmatrix} c' & \frac{1}{k\gamma}s' \\ -k\gamma s' & c' \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} cC & \ell cC + \frac{1}{k\gamma}(sC + S) \\ -k\gamma(sC - S) & cC - k\gamma\ell(sC - S) \end{bmatrix} \qquad (3\text{-}31)$$

where the notation is the same as defined under Eq. (3-10), Section III-B, with the additional definitions

$$c' = \cos\tfrac{1}{2}\varphi = \cos\tfrac{1}{4}kT; \qquad C' = \cosh\tfrac{1}{2}\varphi$$

$$s' = \sin\tfrac{1}{2}\varphi \qquad S' = \sinh\tfrac{1}{2}\varphi$$

In the **y** plane we need the transformation from center to center of the middle quadrupoles (from plane 0' to plane 1' in Fig. 3-4);

$$\hat{\mathbf{B}} = \begin{pmatrix} c' & \frac{1}{k\gamma}s' \\ -k\gamma s' & c' \end{pmatrix} \begin{pmatrix} C' & \frac{1}{k\gamma}S' \\ k\gamma S' & C' \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C' & \frac{1}{k\gamma}S' \\ k\gamma S' & C' \end{pmatrix} \begin{pmatrix} c' & \frac{1}{k\gamma}s' \\ -k\gamma s' & c' \end{pmatrix}$$

$$= \begin{bmatrix} cC - \frac{1}{2}k\gamma\ell(s - cS) & \frac{1}{2}\ell(C + c + sS) + \frac{1}{k\gamma}(sC + S) \\ -k\gamma(s - cS) + \frac{1}{2}k^2\gamma^2\ell(C - c - sS) & cC - \frac{1}{2}k\gamma\ell(s - cS) \end{bmatrix} \qquad (3\text{-}32)$$

Alternatively, if the beam aperture through the lens can be made greater than the accelerator aperture, the appropriate reference in the **y** plane is at the end of the triplet (from plane 0 to plane 1 in Fig. 3-4), for which the matrix is found by replacing k by ik in Eq. (3-31);

$$\mathbf{B} = \begin{bmatrix} cC & \ell cC + \frac{1}{k\gamma}(s + cS) \\ -k\gamma(s - cS) & cC - k\gamma\ell(s - cS) \end{bmatrix} \qquad (3\text{-}33)$$

In the small angle approximation, with the assumptions

$$Qt = Q\frac{T}{\gamma} = 2\varphi^2 \ll 1 \qquad (3\text{-}34)$$

and

$$T \ll \Lambda = L + T, \qquad (3\text{-}35)$$

expansion of the matrix elements gives (to second order in T):

$$\mathbf{A} \approx \begin{bmatrix} 1 - \frac{1}{24} Q^2t^2 & \lambda + \frac{1}{8} Qt^2 - \frac{1}{24} Q^2\lambda t^2 \\ -\frac{1}{12} Q^2t\left(1 - \frac{1}{8} Qt\right) & 1 + \frac{1}{24} Q^2t^2 - \frac{1}{12} Q^2\lambda t\left(1 - \frac{1}{8} Qt\right) \end{bmatrix} \qquad (3\text{-}31a);$$

$$\hat{\mathbf{B}} \approx \begin{bmatrix} 1 - \frac{1}{24} Q^2\lambda t\left(1 + \frac{1}{8} Qt\right) & \lambda + \frac{1}{4} Q\lambda t - \frac{1}{8} Qt^2 + \frac{1}{96} Q^2\lambda t^2 \\ -\frac{1}{12} Q^2t\left(1 - \frac{1}{8} Qt - \frac{1}{48} Q^2\lambda t\right) & 1 - \frac{1}{24} Q^2\lambda t\left(1 + \frac{1}{8}Qt\right) \end{bmatrix} \qquad (3\text{-}32a);$$

and

$$\mathbf{B} \approx \begin{bmatrix} 1 - \frac{1}{24} Q^2t^2 & \lambda - \frac{1}{8} Qt^2 - \frac{1}{24} Q^2\lambda t^2 \\ -\frac{1}{12} Q^2t\left(1 + \frac{1}{8} Qt\right) & 1 + \frac{1}{24} Q^2t^2 - \frac{1}{12} Q^2\lambda t\left(1 + \frac{1}{8} Qt\right) \end{bmatrix} \qquad (3\text{-}33a)$$

The characteristic phase angle is given by

$$\cos\theta_{x,y} \approx 1 - \frac{1}{24} Q^2\lambda t\left(1 \mp \frac{1}{8} Qt\right) \qquad (3\text{-}36)$$

Keeping only first order correction terms, we have

$$a_{12}(\max) \approx a_{12} \approx \lambda \tag{3-37}$$

$$b_{12}(\max) = \hat{b}_{12} \approx \lambda\,(1 + \tfrac{1}{4}Qt) \tag{3-38}$$

$$b_{12} \approx \lambda \tag{3-39}$$

In the "singlet" approximation we make the identification

$$F = \frac{12\gamma^2}{Q^2 T} \tag{3-40}$$

and Eqs. (2-5) through (2-9) apply, with the asymptotic admittance defined by

$$\Upsilon_\infty = \tfrac{1}{2}\, Qa^2 \sqrt{\frac{T}{3\Lambda}} \tag{3-41}$$

The total length of the triplet thus is

$$T = \frac{2Q}{g} \cong \left(\frac{4\Upsilon_\infty\sqrt{3\Lambda}}{ga^2}\right)^{\frac{2}{3}} \tag{3-42}$$

Numerical Example. Use of the same parameters as in the previous examples ($\Upsilon_\infty = 0.6$, $\Lambda = 10^4$ cm, $g = 0.6$, and $a = 0.85$ cm) gives

$$T \approx 100 \text{ cm} \approx 40 \text{ inches;}$$

$$Q = \tfrac{1}{2}\, Tg \approx 30 \approx 51{,}000 \text{ gauss;}$$

and as before (in the singlet approximation)

$$\gamma_o \approx 5.9 \times 10^3, \quad \gamma_c \approx 4.2 \times 10^3$$

The validity of the singlet approximation is established by

$$T/\Lambda \approx \frac{100}{10^4} = 10^{-2} \ll 1$$

and

$$Qt_o = \frac{QT}{\gamma_o} \approx 0.5$$

The latter result is seen from Eq. (3-38) to imply a 12% correction in $\Upsilon_y(\gamma_o)$ $\left[\text{which is proportional to } (1 + \tfrac{1}{4}Qt_o)^{-1}\right]$.

In order to make the admittance essentially the same in the y plane as in the x plane, the beam aperture through the lens needs to have a radius, b, given by

$$\frac{b}{a} \geq \sqrt{\frac{b_{12}}{b_{12}}} \approx \sqrt{1 + \tfrac{1}{4}Qt}$$

from which (taking $Qt = 0.5$, $a = 0.85$ cm)

$$b \geq 1.06\ a = 0.90 \text{ cm}$$

IV. PRELIMINARY SPECIFICATION FOR THE QUADRUPOLES

In the interest of providing a feeling for quadrupole space requirements, some further numerical examples will be given.

A. STANDARD CONDITIONS

The radius of the accelerator hole is assumed to be given by 2a = 0.670 inches or a = 0.85 cm.

The length of the sector is taken as

$$\Lambda \approx 330 \text{ ft} \approx 10^4 \text{ cm}$$

The maximum spacing, D, is taken as $D \approx 250$ cm.

An estimate for the quadrupole gradient may be obtained from the specifications of a commercially available quadrupole,* which has a nominal field gradient of 4000 gauss/inch in a 1.5 inch-diameter gap. On the assumption of the same winding and yoke cross section but a slightly smaller gap, a conservative estimate would be 5 kilogauss/inch or $g \approx 1.15$. Other numbers applying to this particular quadrupole are:

Diameter of outside yoke = 12 inches, from which

Weight (pounds) $\approx$ 33 T (in.);

Maximum power per quadrupole of 4-inch length $\approx$ 128 watts, of which about 1/3 appear to be in the exterior ends of the windings; hence, very roughly,

$$\text{Power (watts)} \approx 80 + 20\ T\ (\text{in.}) \quad (\text{doublets})$$

and

$$\text{Power (watts)} \approx 120 + 20\ T\ (\text{in.}) \quad (\text{triplets})$$

where T (in.) is total quadrupole length in inches.

B. NUMERICAL RESULTS

Table I lists approximate values of switchyard transport admittance for several final beam energies. Table II lists some quadrupole specifications, and the cutoff-and-half-transmission energies, vs various choices of admittance.

---

*Model Q1540, Spectromagnetic Industries, Hayward, California.

TABLE I. Switchyard transport admittance vs final beam energy

V_f, Bev	Υ_{max}, by Eq. (1-1)
40	2.4
20	1.2
10	0.6

TABLE II. Cutoff energy, half-transmission energy, and quadrupole requirements vs design admittance. Air-cooled quadrupoles of 12 inch outside diameter and ≈ 1 inch aperture diameter are assumed; see Standard Conditions, above.

Υ_∞	Cutoff energy V_c (Bev)	Half-transmission energy V_o (Bev)	Multiplet type	Total quad. length per sector	Total quad. weight per sector	Max quad. power per sector
2.4	8.6	12.2	SD	14.3 inches	470 lb	370 watts
			CD	39.2	1300	860
			ST	28.6	980	700
			CT	62.2	2060	1360
1.2	4.3	6.1	SD	7.2	240	220
			CD	24.8	820	580
			ST	14.3	470	410
			CT	39.2	1300	900
0.6	2.1	3.0	SD	3.6	120	160
			CD	15.6	520	370
			ST	7.2	240	260
			CT	24.8	820	620

C. CONCLUSIONS

1) For the separated combinations, SD (Spaced Doublet) and ST (Spaced Triplet), the quadrupole requirements are rather modest and it would be practical to design for somewhere near the maximum usable transmission.

2) For the contiguous combinations CD (Close Doublet) and CT (Close Triplet) the quadrupoles become rather large. If further analysis tends to favor one of these types, and if maximum transmission is desired, it might be necessary to design with considerably higher gradients than the 5 kilogauss/inch assumed here. For example, reduction of the quadrupole length by a factor of 2 would require an increase in gradient by a factor of $(2)^{2/3} \approx 1.59$ (i.e., 8 kilogauss/inch). Water-cooled conductor probably would be required.

3) On the basis of the low energy cutoff, it is evident that simultaneous transmission of high and low energy beams is incompatible with maximum focusing strengths. One has several alternatives:

(a) Decrease the lens spacing [see Eq. (2-6)].

(b) Design for minimal focusing (e.g., admittance corresponding to final energy $\lesssim 10$ Bev).

(c) Program experiments so that multiple-beam operations are confined to a fairly narrow energy band.

Alternative (a) is undesirable on account of expense and lens-alignment problems. (Note that in any event regions of tapered spacing will be required to match low energies into the regular-sector spacing; but this should be confined to a range of not more than one or two sectors per injector.) Some combination of alternatives (b) and (c) seems to be indicated.

4) Finally, it should be clear that none of the above results is sufficient criteria for final choice of one of the possible alternative systems. Study of the lens alignment problems will be the next step in formulating definite recommendations.

LIST OF REFERENCES

1. R. H. Helm, "Discussion of focusing requirements for the Stanford two-mile accelerator," SLAC Report No. 2, Stanford Linear Accelerator Center, Stanford University, Stanford, California, August 1962.

2. R. H. Helm, "Misalignment and quadrupole error effects in a focusing system for the two-mile accelerator," SLAC Report No. 11, Stanford Linear Accelerator Center, Stanford University, Stanford, California (now being printed).

3. H. Butler, private communication.

4. For review article and bibliography on quadrupole systems, see for example Albert Septier, "Strong focusing lenses," Advances in Electronics and Electron Optics XIV, Academic Press, New York, 1961; pp. 85-205.

5. K. L. Brown, "Design of quadrupoles along the accelerator," TN-61-3, Stanford Linear Accelerator Center, Stanford University, Stanford, California.