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SPIN AND PARITY ANALYSIS IN TWO-STEP DECAY PROCESSES

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ABSTRACT

Results are given for applying the helicity formalism to spin and parity analysis of a certain class of resonances decaying into another resonance and a non-resonant particle with the subsequent decay of the intermediate resonance into non-resonant particles. Integer and half integer cases are treated separately. The method outlined herein is hoped to be useful to those experimenters interested in performing a spin and parity analysis on resonance decay data.

I. INTRODUCTION

During the past few years theoretical physicists have shown an ever increasing interest in assisting their experimental colleagues in the problem of spin and parity analysis of resonances. It is our feeling that many of the treatments are somewhat too elaborate and formal to be of maximum value to experimentalists. We therefore would like to enter another work into the already large amount of literature in this field with hope that the treatment herein will show how these problems may be studied in what we believe to be a very simple manner. Rather than examine problems to which solutions have already been given, we illustrate the method in connection with a new avenue — the two-stage decay process where a parent resonance decays into a pseudoscalar meson and intermediate resonance which then subsequently decays into two or three final non-resonant particles.[1] An example for boson type resonances is $B \rightarrow \omega\pi$; $\omega \rightarrow 3\pi$, and an example for fermion type resonances is Ξ^* (1820) $\rightarrow \Xi^*$ (1530) + π; Ξ^* (1530) $\rightarrow \Xi + \pi$. In the actual detailed exercises which are treated here we have in mind the determination of the spin and parity of the parent resonance which decays via the two stage processes into particles and resonances whose quantum numbers are known. In particular we consider only those cases where the intermediate fermion resonance has spin 3/2, and the intermediate boson resonance has spin 1. These causes appear to be of great practical interest. Boson decays are treated in Section II and fermion decays in Section III. The method of calculation leading to the proposed weighted averages is outlined in the Appendix.

In all the analyses proposed here we have ignored the problem of separating the various resonances of fixed angular momentum from the background and have assumed that in the actual applications, the data will be sufficiently amenable to make this separation possible.

II. TWO-STEP BOSON DECAYS

We consider here a method for determining the spin and parity of a boson which decays into a spin-one and a spin-zero particle with the subsequent decay of the spin-one particle into two or three spin-zero particles. Some examples are (i) $A \to \rho\pi;\ \rho \to \pi\pi$, (ii) $B \to \omega\pi;\ \omega \to 3\pi$, (iii) $K^{**} \to K^* + \pi$. The analysis makes use of ratios of weighted averages over the angular distribution in both stages of decay and does not assume any particular dynamical mechanism for the production process. Furthermore the analysis can be carried out even if the produced boson is unpolarized.

A. Coordinate Systems

It is convenient for the description of the analysis presented here to introduce the following coordinate systems. The resonance whose spin and parity is to be analyzed is considered to be produced by the reaction

$$K + p_1 \to Q + p_2$$

where K and p_1 are the incident boson and target particle momentum respectively; Q is the momentum of the produced boson resonance and p_2 is the sum of the momenta of all other particles produced in the reaction. The resonance then decays in two stages where the intermediate state is a resonance state (ρ, ω or K^*) of momentum q and a $\pi(K)$ of momentum t. The vector meson (ρ, ω, K^*) eventually decays into 2π, 3π or $K\pi$ according to its known major decay channel. Thus we have

$$Q \to q + t;\ q \to \begin{cases} q_1 + q_2 + q_3 & (\omega \text{ decay}) \\ s + r & (\rho,\ K^*,\ \varphi \text{ decay}) \end{cases}$$

where s and r are the two momenta corresponding to the two bodies in the decays of ρ and K^* and q_1, q_2, q_3 are the momenta of the decay pions in the decay $\omega \to 3\pi$. A unit reactor $\hat{n}_\omega$ normal to the plane of the three pions in the ω decay is useful to consider and is defined in the ω rest frame as $\hat{n}_\omega = (\vec{q}_1 \times \vec{q}_2)/\left|\vec{q}_1 \times \vec{q}_2\right|$. It is also convenient to express $\hat{n}_\omega$ in a covariant normalized form as

$$- \left(\hat{n}_\omega\right)_\nu = \epsilon_{\mu\nu\sigma\tau}\, q_\mu\, q_{1\sigma}\, q_{2\tau}/n_o$$

where $\epsilon_{\mu\nu\sigma\tau}$ is the completely antisymmetric tensor of the fourth rank with $\epsilon_{0123} = +1$ and where n_o is simply $m_\omega\left|\vec{q}_1 \times \vec{q}_2\right|$ in the ω rest frame and may be expressed invariantly as[2]

$$n_o^2 = m_\omega^2 \left[m_\pi^4 - (q_1 \cdot q_2)^2\right] + (q \cdot q_1)\left[(q \cdot q_2)(q_1 \cdot q_2) - (q \cdot q_1)\, m_\pi^2\right] + (q \cdot q_2)\left[(q_1 \cdot q_2)(q \cdot q_1) - (q \cdot q_2)\, m_\pi^2\right]$$

A scalar of the form, for example, $t \cdot n_\omega = \epsilon_{\mu\nu\sigma\tau}\, q_\mu\, t_\nu\, q_{1\sigma}\, q_{2\tau}$ may be evaluated in any coordinate system by the 4×4 determinant

$$(t \cdot \hat{n}_\omega) = \begin{vmatrix} q_x & q_y & q_z & q_o \\ t_x & t_y & t_z & t_o \\ q_{1x} & q_{1y} & q_{1z} & q_{1o} \\ q_{2x} & q_{2y} & q_{2z} & q_{2o} \end{vmatrix}$$

Note that the matrix $a \cdot b = a_o\, b_o - \vec{a} \cdot \vec{b}$ is used throughout this presentation for invariant scalar products $a \cdot b$.

The two coordinate systems of interest are referred to as the Q frame and the q frame which are respectively the rest frame of the produced resonance, i.e., the system where Q has only a fourth component and the rest system of the vector meson, i.e., where q has only a fourth component. In the

Q frame the z axis is chosen as the beam direction K and the x axis along N the normal to the production plane; i.e., along the direction $\vec{K} \times \vec{p}_1$. The vector meson of momentum $\vec{q}$ in the Q frame makes angles θ and φ with respect to these axes as shown in Fig. 1. The direction of the vector $\vec{q}$ then becomes the z' axis in the q rest frame which is simply the opposite direction to the vector $\vec{t}$ while the y axis is chosen to be normal to the plane containing z and $\vec{q}$ and remains invariant in direction going from the Q to the q system, i.e., y is along $\vec{K} \times \vec{q}$ in the Q system or equivalently along $\vec{Q} \times \vec{K}$ in the q system. A unit vector $\hat{\eta}_\nu$ along the y axis may be expressed covariantly as

$$\hat{\eta}_\nu = - \frac{\epsilon_{\mu\nu\sigma\tau} Q_\mu K_\sigma q_\tau}{\eta_o}$$

where η_o is a normalizing factor which can be expressed invariantly as[2]

$$\eta_o^2 = m_q^2 \left[m_Q^2 m_b^2 - (Q.K)^2 \right] + (q.K) \left[(Q.K)(Q.q) - (q.K) m_Q^2 \right] + (Q.q) \left[(Q.K)(q.K) - m_b^2 (q.Q) \right]$$

where m_q is the mass associated with momentum q which in this case refers to the vector meson, ρ, ω, K^*; m_Q is the mass of the parent resonance boson associated with momentum Q, and m_b is the mass of the incident beam particle. In terms of the vectors thus defined, the relevant angles may be expressed in invariant form as

$$\cos\theta = \frac{\vec{q}.\vec{K}}{|\vec{q}|\,|\vec{K}|} = \frac{(Q.q)(K.Q) - (q.K) m_Q^2}{\sqrt{(Q.q)^2 - m_Q^2 m_q^2}\sqrt{(Q.K)^2 - m_Q^2 m_b^2}} \qquad (2.1)$$

$$\sin\theta\cos\varphi = \frac{\vec{q}\cdot\vec{N}}{|\vec{q}||\vec{N}|} = -\frac{\epsilon_{\mu\nu\sigma\tau}\, Q_\mu\, q_\nu\, K_\sigma\, p_{1\tau}}{N_o\sqrt{\dfrac{(Q\cdot q)^2}{m_Q^2} - m_q^2}} \tag{2.2}$$

where

$$N_o^2 = m_Q^2\left[m_b^2\, m_1^2 - (K\cdot p_1)^2\right] + (p_1\cdot Q)\left[(K\cdot p_1)(K\cdot Q) - (p_1\cdot Q)\, m_b^2\right] + (K\cdot Q)\left[(p_1\cdot Q)(p_1\cdot K) - (Q\cdot K)\, m_1^2\right] \tag{2.2a}$$

and m_1 is the target mass, i.e., the mass associated with the momentum p_1. For ρ and K^* decay, θ' the angle between the outgoing decay pion or kaon and the z' axis, can be expressed as

$$\cos\theta' = \frac{-\vec{s}\cdot\vec{t}}{|\vec{s}||\vec{t}|} = \frac{m_q^2\,(s\cdot t) - (q\cdot s)(q\cdot t)}{\sqrt{(s\cdot q)^2 - m_q^2\, m_s^2}\sqrt{(t\cdot q)^2 - m_q^2\, m_t^2}} \tag{2.3}$$

where m_s is the mass associated with the momentum s and m_t the mass associated with t. The azimuthal angle φ' can be determined from the angle between the outgoing pion or kaon and the y' axis as

$$\sin\theta'\sin\varphi' = \vec{s}\cdot\vec{\eta}/|\vec{s}|\,\eta_o = \frac{\epsilon_{\mu\nu\sigma\tau}\, s_\mu\, q_\nu\, Q_\sigma\, K_\tau}{\eta_o\sqrt{\dfrac{(s\cdot q)^2}{m_q^2} - m_s^2}} \tag{2.4}$$

where η_o is as above.

The angle θ' is defined between 0 and π and therefore determined by (2.3). With (2.4) however, we cannot distinguish between φ' and $\pi - \varphi'$. This ambiguity is harmless in the boson case as shown by (2.13) and (2.14). In the fermion case φ' (which is defined between 0 and 2π) must be determined precisely.

In order to eliminate this ambiguity, the frame of reference defined by the vector $\vec{q}$ (z' axis) and $\vec{K} \times \vec{q}$ (y' axis) has to be completed by a vector $(\vec{K} \times \vec{q}) \times \vec{q}$ (x' axis). A unit vector $\hat{\zeta}_\nu$ along the x' axis may be expressed covariantly as

$$\hat{\zeta}_\nu = \frac{\epsilon_{\mu\nu\sigma\tau}\, \epsilon_{\alpha\sigma\beta\gamma}\, Q_\mu\, Q_\alpha\, K_\beta\, q_\gamma\, q_\tau}{\zeta_o}$$

where

$$\zeta_o = \eta_o \sqrt{(Q\cdot q)^2 - m_Q^2\, m_q^2}$$

In that case one has

$$\sin\theta' \cos\varphi' = -\, m_q \frac{s\cdot\zeta}{\zeta_o\left((s\cdot q)^2 - m_q^2\, m_s^2\right)^{\frac{1}{2}}} \tag{2.5}$$

where

$$s\cdot\zeta = m_Q^2\Big((K\cdot s)\, m_q^2 - (s\cdot q)(K\cdot q)\Big) - (K\cdot Q)\Big((s\cdot Q)\, m_q^2 - (s\cdot q)(Q\cdot q)\Big) + (Q\cdot q)\Big((s\cdot Q)(K\cdot q) - (s\cdot K)(q\cdot Q)\Big) \tag{2.6}$$

For ω decay, the normal vector $\hat{n}_\omega$ has a zero time component in the ω rest system, hence

$$\cos\theta' = \frac{(\hat{n}_\omega\cdot t)}{\sqrt{\frac{(t\cdot q)^2}{m_q^2} - m_t^2}} \tag{2.7}$$

and

$$\sin\theta' \sin\varphi' = - n_\omega \cdot \eta$$

$$= \frac{-1}{(\eta_o)(n_o)} \Big\{ (Q\cdot q) \left[(q\cdot q_1)(K\cdot q_2) - (q\cdot q_2)(K\cdot q_1) \right] - (Q\cdot q_1)\left[q^2(K\cdot q_2) - (q\cdot q_2)(q\cdot K) \right] + (Q\cdot q_2)\left[q^2 (K\cdot q_1) - (q\cdot q_1)(q\cdot K) \right] \Big\} \qquad (2.8)$$

Expressing the relevant angles in invariant form allows the evaluation of these angles directly in terms of the measured laboratory values in the most direct and unambiguous manner. Thus one may use (2.1) to (2.8) to express the angles in the Q and q frames in terms of laboratory system or c.m. system values.

B. Types of Weighted Averages

The angular distribution of the decay products of the vector meson may be thought of as a function of four variables $\theta, \varphi, \theta', \varphi'$. Thus we can ask for the number of particles decaying into angles θ' and φ' for a fixed value of θ and φ. This quantity is labeled $I(\theta,\varphi;\theta',\varphi')$. It can be expressed in terms of the vector meson density matrix $\rho'_{ik}(\theta,\varphi)$ where i and k take on the values 1, 0, -1, referred to the vector meson direction as the axis of quantization (the z' axis). As shown in the appendix, I may be expressed as

$$I(\theta,\varphi;\theta',\varphi') = \sum_{ik} |G_o|^2 \rho'_{ik}(\theta,\varphi)\, D^1_{io}(\varphi',\theta',o)\, D^{1*}_{ko}(\varphi',\theta',o) \qquad (2.9)$$

where the D functions are the representation of the rotation group and are defined as in Rose[3] as

$$D^j_{m'm}(\alpha,\beta,\gamma) = e^{-im'\alpha}\, d^j_{m'm}(\beta)\, e^{-im\gamma}$$

and where $|G_o|^2$ is a coupling constant for the vector meson decay. The density matrix $\rho'_{ik}(\theta,\varphi)$ can in turn be expressed in terms of the density matrix ρ_{st} of the parent decaying boson resonance in the form

$$\rho'_{ik} = \sum_{st} \rho_{st} D^{j}_{si}(\varphi,\theta,0) D^{j*}_{tk}(\varphi,\theta,0) F^{*}_{i} F_{k} \qquad (2.10)$$

where j is the spin of the parent decaying boson and the F_i are the strengths of the decay amplitudes into a given vector meson helicity state. The values of i and k refer to the z' axis while s and t refer to the z axis. It can be shown that parity conservation in the decay of the parent boson yields the relation[4]

$$F_i = (-1)^j \epsilon F_{-i} \qquad (i = 1,0,-1) \qquad (2.11)$$

where ϵ is the relative parent boson vector-meson parity taking into account the negative parity of the pseudo scalar meson in the decay $Q \to q + t$. From equation (2.10) which relates ρ' to ρ one can derive the following useful result

$$\frac{\int 2\rho'_{1-1}(\varphi,\theta)\sin^2\theta\, d\Omega}{\int\left[\rho'_{11}(\varphi,\theta) + \rho'_{-1-1}(\varphi,\theta)\right](3\cos^2\theta - 1)\, d\Omega} = \epsilon\,(-1)^j \frac{j(j+1)}{3 - j(j+1)} \qquad (2.12)$$

The quantities ρ'_{11}, ρ'_{-1-1} and ρ'_{1-1} may be determined from $I(\theta,\varphi,\theta',\varphi')$ by the relations

$$\rho'_{11}(\varphi,\theta) + \rho'_{-1-1}(\varphi,\theta) = \frac{3}{8\pi|G_o|^2}\int I(\theta,\varphi;\theta',\varphi')\left[3-5\cos^2\theta'\right] d\Omega' \qquad (2.13)$$

and

$$\rho'_{1-1}(\varphi,\theta) = \frac{-3}{4\pi|G_o|^2}\int I(\theta,\varphi;\theta',\varphi')\cos 2\varphi'\, d\Omega' \qquad (2.14)$$

Equations (2.12), (2.13) and (2.14) can be used as a direct means for determining the spin and parity of the decaying boson. Also from equation (2.11) we see that the helicity state i = 0 is allowed only when $\epsilon = (-1)^j$ or that the helicity zero state is non vanishing for the choices $1^+, 2^-, 3^+$ etc., for the spin and parity of the decaying parent boson. Presence of the helicity state zero allows for a term $\cos^2\theta$ which would be absent if the helicity state zero was not allowed. For example, if the process under study was the decay $A \rightarrow \rho\pi$; $\rho \rightarrow \pi\pi$ then a $\cos^2\theta$ allows for all three final pions to be collinear in the A rest frame. The absence of the helicity zero state implies that the following weighted average should be found equal to zero

$$\int I(\theta',\varphi')\ (5\cos^2\theta' - 1)\ d\Omega' = 0 = \frac{8\pi|G|^2}{3}\ \rho'_{00}\ (\theta,\varphi).$$

Since no symmetrization has been performed in the analysis presented here, we note that the method does not apply to regions of the Dalitz plot where there are overlapping vector meson bands.[5]

III. FERMION DECAYS

We now apply the procedures outlined in Section II to the case of fermion two stage decays where the intermediate two particle state is assumed to be a spin 3/2 and a spin 0 particle. The intermediate spin 3/2 particle subsequently has a parity conserving decay into a spin 1/2 and a spin 0 particle. A complete determination of both spin and parity requires knowledge of either the longitudinal or transverse polarization state of the final spin 1/2 particle. The case when the intermediate particle has spin 1/2 rather than 3/2 has already been treated by Byers and Fenster.[6]

Unlike the boson case a complete determination of both the spin and parity of the parent fermion resonance requires it to have some net polarization resulting from the production process.

The coordinate systems describing the two-stage fermion decay are similar to the systems used in the boson case except that the normal to the production plane N replaces the beam direction K as the z axis (axis of quantization) in the Q rest system. The reason for this difference is that most of the tests for spin and parity require the parent resonance to be polarized and thus one chooses the axis of quantization as that axis along which the polarization is maximum. If parity is conserved in the process which produces the parent resonance then the axis of maximum polarization is the normal to the production plane.

With this choice for the z axis we proceed as in the boson case and find for the Q system angles

$$\cos\theta = \frac{\vec{q}\cdot\vec{N}}{|\vec{q}||\vec{N}|} = -\frac{\epsilon_{\mu\nu\sigma\tau}\, Q_\mu\, q_\nu\, K_\sigma\, p_{1\tau}}{N_o\sqrt{\left(\frac{Q\cdot q}{m_Q}\right)^2 - m_q^2}} \qquad (3.1)$$

where N_o is given as before by Eq. (2.2a)

$$\sin\theta\cos\varphi = \frac{\vec{q}\cdot\vec{K}}{|\vec{q}|\,|\vec{K}|} = \text{same as Eq. (2.1)}$$

For the q system angles, $\cos\theta'$ is given as before by Eq. (2.3). The y' and x' axes are determined just as in the boson case except that the y' axis is along $(\vec{N}\times\vec{q})$ and the x' axis along $(\vec{N}\times\vec{q})\times\vec{q}$. Thus we find

$$\sin\theta'\sin\varphi' = \frac{1}{R_o}\Big\{(s\cdot K)\left[m_Q^2\,(q\cdot p_1) - (q\cdot Q)\,(p_1\cdot Q)\right] + (q\cdot K)\left[(s\cdot Q)\,(p_1\cdot Q) - m_Q^2\,(s\cdot p_1)\right] + (q\cdot Q)\,(K\cdot Q)\,(s\cdot p_1) - (s\cdot Q)\,(K\cdot Q)\,(q\cdot p_1)\Big\} \qquad (3.2)$$

$$\sin\theta'\cos\varphi' = \left\{(s\cdot N) - \frac{(q\cdot N)\quad (q\cdot Q)\,(s\cdot Q) - m_Q^2\,(s\cdot Q)}{\sqrt{(q\cdot Q)^2 - m_q^2\,m_Q^2}\,\sqrt{(s\cdot Q)^2 - m_s^2\,m_Q^2}}\right\}\frac{N_o}{N_o'} \qquad (3.3)$$

where

$$R_o = -\frac{N_o'}{m_Q}\sqrt{(s\cdot Q)^2 - m_s^2\,m_Q^2}\;\sqrt{(q\cdot Q)^2 - m_q^2\,m_Q^2}$$

$$(s\cdot N) = \frac{\epsilon_{\mu\nu\sigma\tau}\,Q_\mu\,s_\nu\,K_\sigma\,p_{1\tau}}{N_o\sqrt{\left(\frac{s\cdot Q}{m_Q}\right)^2 - m_s^2}}$$

$$(q\cdot N) = \frac{\epsilon_{\mu\nu\sigma\tau}\,Q_\mu\,q_\nu\,K_\sigma\,p_{1\tau}}{N_o\sqrt{\left(\frac{q\cdot Q}{m_Q}\right)^2 - m_q^2}}$$

and N_o is given by Eq. (2.2a) and $N_o' = N_o\sqrt{1 - (q\cdot N)^2}$.

Just as for the boson case the various angles are expressed invariantly which permits their evaluation in terms of coordinates measured in any frame. In particular the laboratory system values of the various momenta can be used in the above expressions to give directly the angles in the Q and q system.

Types of Weighted Averages

We proceed here in a manner similar to the boson case of Section II and describe the parent resonance by a density matrix ρ_{st} referred to fixed axis defined by the production mechanism. The axis of quantization can be chosen as any fixed axis and, as mentioned above, in order to obtain the maximum polarization we choose the normal to the production plane as the axis of quantization in the Q system.

The density matrix of the intermediate resonance ρ'_{ik}, where i and k refer to an axis along the momentum of the resonance as quantization axis, can be expressed in terms of ρ_{st} precisely in the same manner as (2.10),

$$\rho'_{ik} (\theta,\varphi) = \sum_{st} \rho_{st} D^{j}_{si} (\varphi,\theta,0) D^{j*}_{tk} (\varphi,\theta,0) F_i F^*_k \qquad (3.4)$$

where θ and φ are the polar and azimuthal angles of the intermediate resonance in the parent resonance rest system and j is the spin of the parent resonance. Parity conservation in the decay relates the strength of the helicity amplitudes by the relation [4]

$$F_{-i} = \epsilon(-1)^{j-1/2} F_i$$

where ϵ is the relative parity between the parent resonance and the intermediate spin 3/2 particle. The intermediate resonance is then assumed to decay into a spin 1/2 and spin 0 particle. The quantities of interest for the decay spin 1/2 particle are its angular distribution $I(\theta,\varphi;\ \theta',\varphi')$, its longitudinal polarization $p_L(\theta,\varphi;\ \theta',\varphi')$ and $p_T(\varphi,\theta;\ \varphi',\theta')$ the component of transverse polarization in the (z',s) plane of Fig. 1. The component of the transverse polarization which is perpendicular to the (z',s) plane does not depend on the diagonal elements of ρ_{st} and is not treated here.

As shown in the Appendix, these quantities can be expressed in terms of ρ'_{ik} as

$$I(\theta,\varphi;\theta',\varphi') = |G|^2 \sum_{ik} \rho'_{ik}(\theta,\varphi) \left[D^{S*}_{i\frac{1}{2}}(\varphi',\theta',0) + D^{S}_{k\frac{1}{2}}(\varphi',\theta',0) + D^{S*}_{i-\frac{1}{2}}(\varphi',\theta'0)\ D^{S}_{k-\frac{1}{2}}(\varphi',\theta'0)\right] \tag{3.6}$$

$$Ip_L(\theta,\varphi;\theta',\varphi') = |G|^2 \sum_{ik} \rho'_{ik}(\theta,\varphi) \left[D^{S*}_{i\frac{1}{2}}(\varphi',\theta',0)\ D^{S}_{k\frac{1}{2}}(\varphi',\theta',0) - D^{S*}_{i-\frac{1}{2}}(\varphi',\theta',0)\ D^{S}_{k-\frac{1}{2}}(\varphi',\theta',0)\right] \tag{3.7}$$

$$Ip_T(\theta,\varphi;\theta',\varphi') = \epsilon_o(-1)^{S+\frac{1}{2}}|G|^2 \sum_{ik} \rho'_{ik}(\theta,\varphi) \left[D^{S*}_{i\frac{1}{2}}(\varphi',\theta',0)\ D^{S}_{k-\frac{1}{2}}(\varphi',\theta',0) + D^{S*}_{i-\frac{1}{2}}(\varphi',\theta',0)\ D^{S}_{k\frac{1}{2}}(\varphi',\theta',0)\right] \tag{3.8}$$

where S and ϵ_o are the spin and parity of the intermediate resonance and $|G|^2$ is the coupling constant for the decay of the intermediate resonance into a spin 1/2 and a spin zero particle. In the subsequent discussion we will take the values $S = 3/2$ and $\epsilon_o = +1$, i.e., the resonance is a member of the $3/2^+$ decimet.

Ratios of weighted averages of ρ'_{ik} can be related to the spin and parity of the parent resonance just as in the boson case. In particular we have

$$\frac{< \sin\theta \; 2\mathrm{Re}\; \rho'_{\frac{1}{2}-\frac{1}{2}}(\theta,\varphi) >}{< \cos\theta \left\{ \rho'_{\frac{1}{2}\frac{1}{2}}(\theta,\varphi) - \rho'_{-\frac{1}{2}-\frac{1}{2}}(\theta,\varphi) \right\} >} = \epsilon(-1)^{j-\frac{1}{2}}(2j+1) \qquad (3.9)$$

$$\frac{< \sin^3\theta \; 2\mathrm{Re}\; \rho'_{\frac{3}{2}-\frac{3}{2}}(\theta,\varphi) >}{< (5\cos^3\theta - 3\cos\theta) \left\{ \rho'_{\frac{3}{2}\frac{3}{2}}(\theta,\varphi) - \rho'_{-\frac{3}{2}-\frac{3}{2}}(\theta,\varphi) \right\} >} = \frac{(2j+3)(2j+1)(2j-1)}{3[49-12j(j+1)]}\,\epsilon(-1)^{j-\frac{1}{2}} \qquad (3.10)$$

where the symbol $< >$ stands for averaging over the angles θ and φ. Both the numerator and denominator of these weighted averages are proportional to the parent isobars net polarization.

The various density matrix elements of ρ'_{ik} appearing in (3.9) and (3.10) can be determined in two independent ways by suitable averages over the longitudinal and transverse polarizations Ip_L and Ip_T. In terms of averages over Ip_L we have

$$\rho'_{\frac{1}{2}\frac{1}{2}}(\theta,\varphi) - \rho'_{-\frac{1}{2}-\frac{1}{2}}(\theta,\varphi) = \frac{5}{8\pi|G|^2}\int \cos\theta' \,(7\cos^2\theta' - 3)\; Ip_L\,(\theta,\varphi;\theta',\varphi')\,d\Omega' \qquad (3.11)$$

$$\rho'_{\frac{3}{2}\frac{3}{2}}(\theta,\varphi) - \rho'_{-\frac{3}{2}-\frac{3}{2}}(\theta,\varphi) = \frac{5}{24\pi|G|^2}\int \cos\theta' \,(15 - 7\cos^2\theta')\; Ip_L\,(\theta,\varphi;\theta',\varphi')\,d\Omega' \qquad (3.12)$$

$$2\mathrm{Re}\;\rho'_{\frac{1}{2}-\frac{1}{2}}(\theta,\varphi) = \frac{5}{8\pi|G|^2}\int \cos\varphi' \sin\theta' \,(7\cos^2\theta' + 1)\; Ip_L\,(\theta,\varphi;\theta',\varphi')\,d\Omega' \qquad (3.13)$$

$$2\mathrm{Re}\;\rho'_{\frac{3}{2}-\frac{3}{2}}(\theta,\varphi) = \frac{-5}{4\pi|G|^2}\int \cos 3\varphi' \sin\theta' \; Ip_L\,(\theta,\varphi;\theta',\varphi')\,d\Omega' \qquad (3.14)$$

Independently in terms of p_T we have

$$\rho'_{\frac{1}{2}\frac{1}{2}}(\theta,\varphi) - \rho'_{-\frac{1}{2}-\frac{1}{2}}(\theta',\varphi') = \frac{-15\,\epsilon_o}{32\pi|G|^2}\int \sin\theta'(7\cos^2\theta' - 1)\ Ip_T(\theta,\varphi;\theta',\varphi')d\Omega' \qquad (3.15)$$

$$\rho'_{\frac{3}{2}\frac{3}{2}}(\theta,\varphi) - \rho'_{-\frac{3}{2}-\frac{3}{2}}(\theta,\varphi) = \frac{5\epsilon_o}{32\pi|G|^2}\int \sin\theta'(7\cos^2\theta' - 5)\ Ip_T(\theta,\varphi;\theta',\varphi')d\Omega' \qquad (3.16)$$

$$2\mathrm{Re}\left\{\rho'_{\frac{1}{2}-\frac{1}{2}}(\theta,\varphi)\right\} = \frac{5\epsilon_o}{4\pi|G|^2}\int \cos\varphi'\ \cos\theta'(7\cos^2\theta' - 3)\ Ip_T(\theta,\varphi;\theta',\varphi')d\Omega' \qquad (3.17)$$

$$2\mathrm{Re}\left\{\rho'_{\frac{3}{2}-\frac{3}{2}}(\theta,\varphi)\right\} = \frac{-5\,\epsilon_o}{\pi|G|^2}\int \cos 3\varphi'\ \cos\theta'\ Ip_T(\theta,\varphi;\theta',\varphi')d\Omega' \qquad (3.18)$$

The polarizations Ip_L and Ip_T can in turn be determined from the decay asymmetry of the spin 1/2 baryon. They are respectively obtained from the forward backward asymmetry and the up-down asymmetry as given below by Eq. (4.3) and Eq. (4.4).

Even though the determination of such polarizations is considered as a well known procedure, we give, for the sake of completeness, invariant expressions similar to the ones which we have introduced for the other decay angles in Section IV.

IV. GUIDE FOR THE DETERMINATION OF p_L AND p_T.

To complete the presentation given here we include the necessary angles for the determination of p_T and p_L.

The spin 1/2 baryon of momentum s which is present at the final stage of the two step process is assumed to decay into a spin 1/2 and a spin zero particle via a parity violating decay. Such a decay allows for the determination of p_L and p_T by suitable averages over the decay angular distribution.

For definiteness we assign momenta s_1 and s_2 to the decay products of s with s_1 being the momentum of the spin 1/2 particle. Thus the various steps of the decay may be characterized by the relations $Q \to q + t$; $q \to s + r$; $s \to s_1 + s_2$.

(We recall that in the s system the angular distribution is given by the familiar equation $I(\theta) = 1 - \alpha|\vec{p}| \cos \theta_0$ where $\vec{p}$ is the polarization of particle s and θ_0 is the angle between $\vec{p}$ and $\vec{s}_1$.)

Just as in the previous coordinate systems we take the z'' axis in the s system to be along the direction of the vector $\vec{s}$ which is along $-\vec{r}$ in the s system. The y'' axis will be along $\vec{s} \times \vec{t}$ and the x'' axis along $(\vec{s} \times \vec{t}) \times \vec{s}$. For the longitudinal polarization we need $\cos \theta''$ where θ'' is the angle between s_1 and z'' as shown in Fig. 1. While for the component of transverse polarization in the (z'',x'') plane we need $\sin \theta'' \cos \varphi''$ where φ'' is the azimuthal angle around the z'', with x'' as the line of $\varphi'' = 0$.

The angles may be expressed invariantly as

$$\cos \theta'' = \frac{m_s^2 \, (s_1 \cdot r) - (s_1 \cdot s)(r \cdot s)}{\sqrt{(s_1 \cdot s)^2 - m_s^2 \, m_{s_1}^2} \, \sqrt{(r \cdot s)^2 - m_s^2 \, m_r^2}} \tag{4.1}$$

$$\begin{aligned} \sin \theta'' \cos \varphi'' = \Big\{ & m_q^2 \left[m_s^2 \, (s_1 \cdot t) - (s_1 \cdot s)(s \cdot t) \right] \\ & - (q \cdot t) \left[(s_1 \cdot q) \, m_s^2 - (s_1 \cdot s)(q \cdot s) \right] \\ & + (q \cdot s) \left[(s_1 \cdot q)(s \cdot t) - (s_1 \cdot t)(q \cdot s) \right] \Big\} / A_0 \end{aligned}$$

$$A_0^2 = \left[\frac{(s_1 \cdot s)^2}{m_s^2} - m_{s_1}^2\right] \left[\; m_s^2\, m_q^2 - (s \cdot q)^2\right]$$

$$\times \left[m_s^2\, m_q^2\, m_t^2 - m_t^2\,(s\cdot q)^2 - m_s^2\,(q\cdot t)^2 - m_q^2 (s\cdot t)^2 + 2(s\cdot q)(t\cdot q)(s\cdot t)\right] \qquad (4.2)$$

In terms of these angles we find

$$< I\ (\theta,\varphi;\theta',\varphi';\theta'',\varphi'')\ \cos'' > = -\left(\frac{2\pi}{3}\right) \alpha\ I p_L\ (\theta,\varphi;\theta',\varphi') \qquad (4.3)$$

$$< I\ (\theta,\varphi;\theta',\varphi';\theta'',\varphi'')\ \sin\theta''\ \cos\varphi'' > = -\left(\frac{2\pi}{3}\right) \alpha\ I p_T\ (\theta,\varphi;\theta',\varphi') \qquad (4.4)$$

where $I\ (\theta,\varphi;\theta',\varphi';\theta'',\varphi'')$ is the angular distribution of the spin 1/2 particle of momentum s_1 for fixed θ,φ and θ',φ' and α is the usual asymmetry parameter[7]

$$\alpha = -\frac{2\mathrm{Re}\ S^*P}{|S|^2 + |P|^2}$$

V. TESTS FOR SPIN ONLY

In certain cases the polarization state of the final spin 1/2 baryon may not be readily determinable. For example, the two stage decay $N^*(1688) \rightarrow N^*(1230) + \pi, N^*(1230) \rightarrow N\pi$ has a nucleon as the final spin 1/2 particle whose polarization is usually more difficult to measure as compared to the Λ, Σ or Ξ. On the other hand the parent resonance may be produced in an unpolarized state as, for example, in the reaction $\pi p \rightarrow N^*(1688)$. In either of these cases the method outlined in Section III could not be applied. Nevertheless, it is still possible to get information on the spin state of the parent isobar by considering moments of the decay distribution of the intermediate spin 3/2 particle.

Since the tests described below for this case do not depend on the parent resonance being polarized, the z axis in the parent particles system may be chosen as either the incident beam direction or the normal to the production plane (or any other direction which is convenient). In some cases it may even be preferable to use the beam direction in which case the invariant expressions given in the beginning of Section II are applicable.

Using the same notation as in Section III we refer to the angular distribution of the spin 1/2 particle which results from the decay of the intermediate spin 3/2 particle as $I(\theta,\varphi;\theta',\varphi')$. The angles $\theta,\varphi,\theta',\varphi'$ are as in Section II or III depending on the choice of z axis. The alignment properties of the intermediate spin 3/2 particle are obtained from the angular distribution by the equations

$$a_1(\theta,\varphi) \equiv \left[\rho'_{\frac{1}{2}\frac{1}{2}}(\theta,\varphi) + \rho'_{-\frac{1}{2}-\frac{1}{2}}(\theta,\varphi)\right] = \frac{3}{4\pi|G|^2} \int I(\theta,\varphi;\theta',\varphi')(5\cos^2\theta' - 1)d\Omega' \quad (5.1)$$

and

$$a_3(\theta,\varphi) \equiv \left[\rho'_{\frac{3}{2}\frac{3}{2}}(\theta,\varphi) + \rho'_{-\frac{3}{2}-\frac{3}{2}}(\theta,\varphi)\right] = \frac{1}{4\pi|G|^2} \int I(\theta,\varphi;\theta',\varphi')(7-15\cos^2\theta')d\Omega' \quad (5.2)$$

where G is a coupling constant for the second stage decay $3/2 \rightarrow 1/2 + 0$ and ρ' refers to the density matrix elements of the intermediate spin 3/2 particle. Ratios of those combinations of density matrix elements, i.e., of $a_1(\theta,\varphi)$ and $a_3(\theta,\varphi)$ can be related to the spin j of the initial parent resonance by the equations

$$\frac{\int (3\cos^2\theta - 1)\, a_3(\theta,\varphi)d\Omega}{\int (3\cos^2\theta - 1)\, a_1(\theta,\varphi)d\Omega} = \frac{|F_{\frac{3}{2}}|^2}{|F_{\frac{1}{2}}|^2}\left[\frac{4j(j+1) - 27}{4j(j+1) - 3}\right] \quad (5.3)$$

and

$$\frac{\int (35\cos^4\theta - 30\cos^2\theta + 3)\ a_3(\theta,\varphi)}{\int (35\cos^4\theta - 30\cos^2\theta + 3)\ a_1(\theta,\varphi)} = \frac{|F_{\frac{3}{2}}|^2}{|F_{\frac{1}{2}}|^2}\left[\frac{2j(j+1)\left\{j(j+1) - \frac{49}{2}\right\} + \frac{(15)(83)}{8}}{2j(j+1)\left\{j(j+1) - \frac{9}{2}\right\} + \frac{45}{8}}\right] \quad (5.4$$

where $F_{\frac{3}{2}}$ and $F_{\frac{1}{2}}$ are the helicity 3/2 and 1/2 coupling constants for the decay of the parent resonance into the intermediate spin 3/2 particle. The ratio of these coupling constants is in general unknown. Equation (5.3) may be applied for $j \leq 3/2$ and Eq. (5.4) for $j \geq 5/2$. For $j = 1/2$ neither (5.3) nor (5.4) are applicable but $I\ (\theta,\varphi;\theta',\varphi')$ is uniform in θ and φ. Equations (5.3) and (5.4) may be used to determine the spin in the following manner.

(i) Apply (5.3). If $j = 3/2$ then (5.3) is negative.

(ii) If (5.3) is positive then j must be greater than 3/2. In that case apply the combination Eq. (5.3) + (3/5) Eq. (5.4). This combination is negative for $j = 5/2$ and positive for $j \geq 7/2$.

(iii) If (ii) yields a positive number then apply (5.4). For $j = 7/2$ (5.4) is negative and for $j \geq 9/2$ (5.4) is positive.

The positive negative tests (i), (ii) and (iii) can be used to determine the spin of the parent resonance if its spin is less than 11/2. For spins 11/2 and greater, similar tests can be devised using higher powers of $\cos\theta$ as test functions.

Although in general the ratio $|F_{\frac{3}{2}}|^2/|F_{\frac{1}{2}}|^2$ is arbitrary if the dynamical assumption is made that only the lowest orbital angular momentum state ℓ_o contributes in the decay of the parent resonance into the spin 3/2 particle, then this ratio is determinable. Making this assumption yields that

$$|F_{\frac{3}{2}}|^2/|F_{\frac{1}{2}}|^2 = C\ \frac{2j+3}{2j-1}$$

where C is equal to 3 (or 1/3) according to the ℓ_o values $j - 1/2$ (or $j - 3/2$). The lowest orbital value ℓ_o depends on the parity ϵ of the parent isobar and is related to the parent spin j by the relations

$$\ell_o = j - 1/2 \qquad \text{for} \qquad \epsilon = (-1)^{j+\frac{1}{2}}$$

$$\ell_o = j - 3/2 \qquad \text{for} \qquad \epsilon = (-1)^{j-\frac{1}{2}}$$

We conclude by remarking that a parity determination is possible from a study of the angular distributions alone, only with additional dynamical assumptions about the first stage decay which relate $F_{\frac{1}{2}}$ to $F_{\frac{3}{2}}$.

APPENDIX

This appendix is devoted to a derivation of the main results presented in Sections II, III and IV.[8]

For the decay of a parent particle of spin j into two particles of spin S_1 and S_2 we need the relationship between a state $|\theta,\varphi\ \lambda>$ describing the decay with angles θ and φ in the rest system of the parent particle and a proper angular momentum state $|\ j\ m\ \lambda>$. The quantity λ is the helicity of the two particle system and its values correspond to the eigenvalues of the component of the total angular momentum along the momentum direction of the decaying system. The direction of particle S_1 is taken positive. The magnetic quantum number m refers to the eigenvalues of j along a fixed axis independent of the decay. The relationship between these two states is obtained by the Wigner[9] method. We express this relationship as

$$|j\ m\ \lambda> = \int D^{j*}_{m\lambda}(\varphi,\ \theta,0)\ |\varphi,\theta,\lambda> \sin\theta\ d\theta\ d\varphi \qquad \text{(X-1)}$$

The decay amplitude $T_{m\lambda}$ from a pure state m to a pure state λ is then given by

$$T_{m\lambda} = D^{j*}_{m\lambda}(\varphi,\theta,0)\ F_\lambda \quad (\lambda \text{ not summed}) \qquad \text{(X-2)}$$

where F_λ is the coupling constant for the state λ. Although the F_λ are in general arbitrary, if the decay is parity conserving then F_λ and $F_{-\lambda}$ are related by the equation $F_{-\lambda} = \eta_1\eta_2\epsilon_p\ (-1)^{j-S_1-S_2}\ F_\lambda$ where η_1,η_2,ϵ_p are the parities of particles S_1,S_2 and the parent particle respectively.[3] If the parent particle is not in a pure state but rather in a mixture of states described by a density matrix ρ_{st}, then by the usual rule of quantum mechanics,

($\rho' = T\rho T^{+}$) we can express the density matrix ρ'_{ik} of the decay particles by

$$\rho'_{ik}(\varphi,\theta) = \sum_{s,t} \rho_{st}\, D^{j*}_{si}(\varphi,\theta,0)\, D^{j}_{tk}(\varphi,\theta,0)\, F_i F^*_k \qquad \text{(X-3)}$$

In the examples treated here S_2 is always zero, so that ρ'_{ik} refers to the other decay particle of spin S_1. From (X-1) we see that the indices s and t for ρ_{st} refer to a fixed axis while the indices i and k refer to the momentum direction of particle S_1.

If we use the Clebsch-Gordan series for the product of the two D-functions then (X-3) may be expressed as

$$\rho'_{ik}(\varphi,\theta) = \sum_{s,t,\ell} \rho_{st}(-1)^{s-i}C(jj\ell;\ -st)C(jj\ell;-ik)D^{\ell}_{t-s,k-i}(\varphi,\theta,0)F_i F^*_k \qquad \text{(X-4)}$$

where the Clebsch-Gordan coefficients $C(jj\ell;\ -st)$ and $C(jj\ell;\ -ik)$ are in the notation of Rose.[3]

If we think of (X-4) as describing the parity conserving decay of a boson resonance into a vector meson and a pion then the indices i and k run over the values 1, 0 and -1. To derive Eq. (2.12) we consider the density matrix elements ρ'_{11}, ρ'_{-1-1} and ρ'_{1-1}. If these elements are expressed by (X-4) and weighted by D^2_{00} and D^2_{02}, and then averaged over the angles θ and φ we have by use of the orthogonality properties of the D-functions that the ratio

$$\frac{\int 2\rho'_{1-1}(\theta,\varphi)\, D^2_{02}(\varphi,\theta,0)\, d\Omega}{\int (\rho'_{11} + \rho'_{-1-1})D^2_{00}(\varphi,\theta,0)d\Omega} = \frac{\epsilon(-1)^{j-1}C(jj2;\ 11)}{C(jj2;\ 1-1)} \qquad \text{(X-5)}$$

The numerator and denominator are in effect both proportional to the same function of ρ which cancels out in the ratio. Substituting the explicit form for the D-functions given at the end of this Appendix and specifying the Clebsch-Gordan coefficients yields immediately Eq. (2.12).

In order to complete the discussion, the relations between the density matrix elements and some directly measurable quantities must be given. If the vector meson decays into two spinless particles, both of the same parity, then we can determine these elements from the angular distribution of these final decay particles. Since the final particles have spin zero in the vector meson rest system there is only one density matrix element for this decay system which is simply the angular distribution. Thus applying the same rule which led to (X-3) we find for the angular distribution of the final spinless particles the expression

$$I(\theta,\varphi;\theta',\varphi') = \left|G_o\right|^2 \sum_{ik} \rho'_{ik}(\theta,\varphi)\ D^{1*}_{io}\ (\varphi',\theta',0)\ D^1_{ko}\ (\varphi',\theta',0) \qquad \text{(X-6)}$$

The angles θ' and φ' are measured with respect to the z' axis as shown on Fig. 1. Furthermore since there is only one helicity state for the final spinless particle system there is only one final decay coupling constant which is labeled by G_o in (X-6). If we expand (X-6) by the Clebsch-Gordan series, then applying the orthogonality relations for the D-functions yields that

$$\int \left[1 - \frac{5}{2}\ D^2_{oo}\ (\varphi',\theta',0)\right]\ I(\varphi,\theta;\ \varphi',\theta')\ d\Omega = 2\pi\left[\rho'_{11}\ (\varphi,\theta) + \rho'_{-1-1}(\varphi,\theta)\right]\left|G_o\right|^2$$

and substituting the values of the D-functions yields (2.13). Equation (2.14) may be derived directly by multiplying (X-6) by cos $2\varphi'$ and averaging over Ω'.

For the case of Section III where the parent fermion decays into an intermediate spin 3/2 particle with subsequent decay of the spin 3/2 particle into a spin 1/2 and spin zero particle we proceed just as in the boson case. However, in the fermion case we have in addition to the angular distribution of the final spin 1/2 particle its longitudinal and transverse polarization distributions.

The density matrix for the intermediate spin 3/2 particle may be expressed in terms of the parent fermion resonance density matrix just as in the boson case by (X-3) or (3.4) with the parity conservation condition (3.5).

For the fermion case we think of (X-4) as determining the spin 3/2 density matrix in which case the indices i and k run from - 3/2 to + 3/2 in integer steps.

Proceeding as in the boson case, we consider various values of i and k and calculate with the use of the orthogonality of the D-functions the following averages

$$\int \rho'_{\frac{1}{2}\frac{1}{2}}(\theta,\varphi)\, D^1_{oo}(\varphi,\theta,0)\, d\Omega = -\int \rho'_{-\frac{1}{2}-\frac{1}{2}}(\theta,\varphi)\, D^1_{oo}(\varphi,\theta,0)\, d\Omega$$

$$= \left|F_{\frac{1}{2}}\right|^2 (4\pi/3)\, C(jj1;\, -\tfrac{1}{2}\,\tfrac{1}{2}) \times \sum_s \rho_{ss}\, (-1)^{s-\frac{1}{2}}\, C(jj1;\, -ss) \qquad \text{(X-7)}$$

$$\int \rho'_{\frac{1}{2}-\frac{1}{2}} (\varphi,\theta)\ D^1_{01} (\varphi,\theta,0)\ d\Omega = -\ (4\pi/3)\ \epsilon_0(-1)^{j+\frac{1}{2}} \left|F_{\frac{1}{2}}\right|^2 C(jj1;\ \tfrac{1}{2}\ \tfrac{1}{2})$$

$$\sum_s (-1)^{s-\frac{1}{2}} \rho_{ss}\ C(jj1;\ -ss) \qquad \text{(X-8)}$$

$$\int \rho'_{\frac{3}{2}\frac{3}{2}} (\varphi,\theta)\ D^3_{00} (\varphi,\theta,0)\ d\Omega = -\int \rho'_{-\frac{3}{2}-\frac{3}{2}} (\varphi,\theta)\ D^3_{00} (\varphi,\theta,0)\ d\Omega$$

$$= \left|F_{\frac{3}{2}}\right|^2 (4\pi/7)\ C(jj3;\ -\tfrac{3}{2}\ \tfrac{3}{2}) \sum_s \rho_{ss}\ (-1)^{s-\frac{1}{2}}\ C(jj3;\ -ss) \qquad \text{(X-9)}$$

$$\int \rho'_{\frac{3}{2}} (\varphi,\theta)\ D^3_{03} (\varphi,\theta,0)\ d\Omega = (4\pi/7)\ \epsilon(-1)^{j-\frac{1}{2}} \left|F_{\frac{3}{2}}\right|^2 C(jj3;\ 3/2\ 3/2)$$

$$\sum_s \rho_{ss}(-1)^{s-\frac{3}{2}}\ C(jj3;\ -ss) \qquad \text{(X-10)}$$

Taking the ratios (X-7) to (X-8) and (X-10) and expressing the D-functions as well as the Clebsch-Gordan coefficients explicitly yields immediately Eqs. (3.9) and (3.10). This generalizes the relation of Byers and Fenster.[6]

Before giving the relevant equations from which ρ'_{ik} can be determined we remark that there is an important distinction between the boson and fermion cases. This difference is in the quantity

$$\sum_s \rho_{ss}\ (-)^{s-\frac{1}{2}}\ C(jj3;\ -ss) \qquad \text{for fermions}$$

to which the numerator and denominator of (3.9) and (3.10) are proportional and

$$\sum_s \rho_{ss}\ (-)^{s-1}\ C(jj2;\ -ss) \qquad \text{for bosons}$$

to which the numerator and denominator of (2.12) are proportional. Comparing negative and positive values of the index s we see that

$$C(jj3;\ s-s)\ (-1)^{-s-\frac{1}{2}} = -\ C(jj3;\ -ss)\ (-1)^{s-\frac{1}{2}} \qquad j \text{ is half integer}$$

and

$$C(jj2;\ s-s)\ (-1)^{-s-1} = C(jj2;\ -ss)\ (-1)^{s-1} \qquad j \text{ is integer}$$

Thus for the fermion case the numerator and denominator of (3.9) and (3.10) vanish if the parent resonance is unpolarized which is the reason for choosing the normal to the production plane as the z axis. On the other hand, the numerator and denominator of (2.12) do not vanish if the boson resonance has no net polarization and therefore any choice of direction for the z axis is permissible.

Returning to the determination of ρ'_{ik} for the fermion case we note that the density matrix ρ''_{ab} of the final spin $\frac{1}{2}$ particle can be expressed in terms of ρ'_{ik} the same manner as (X-3). Thus we have

$$\rho''_{ab}\ (\theta,\varphi,\theta',\varphi') = \sum_{ik} \rho'_{ik}\ (\theta,\varphi)\ D^{\frac{3}{2}*}_{ia}\ (\varphi',\theta',0)\ D^{\frac{3}{2}}_{kb}\ (\varphi',\theta',0)\ F'_a\ F'^*_b \qquad \text{(X-11)}$$

The indices a and b take on the values $\frac{1}{2}$, $-\frac{1}{2}$ and refer to the helicity of the final spin $\frac{1}{2}$ particle (z'' of Fig. 1). In terms of ρ''_{ab} the various properties of the final spin $\frac{1}{2}$ particle can be determined in the usual manner.

The angular distribution

$$I(\theta,\varphi,\theta',\varphi') = \rho''_{\frac{1}{2}\frac{1}{2}} + \rho''_{-\frac{1}{2}-\frac{1}{2}} \qquad \text{(X-12)}$$

the longitudinal polarization distribution

$$Ip_L(\theta, \varphi; \theta', \varphi') = \rho''_{\frac{1}{2}\,\frac{1}{2}} - \rho''_{-\frac{1}{2}\,-\frac{1}{2}} \tag{X-13}$$

the component of transverse polarization in the (z', s) plane

$$Ip_T(\theta, \varphi, \theta', \varphi') = \rho''_{\frac{1}{2}\,-\frac{1}{2}} + \rho''_{-\frac{1}{2}\,\frac{1}{2}} \tag{X-14}$$

(We know that (X-14) is in the (z', s) plane because it is given by $\mathrm{Tr} < \sigma_{x'} \rho'' >$ where the axis x' is shown on Fig. 1. It is by definition in the (z', s) plane.)

Since we have assumed that the spin 3/2 particle decay is parity conserving there is only one F'_a and we have that

$$F'_{-\frac{1}{2}} = \epsilon_o F'_{\frac{1}{2}} = \epsilon_o G$$

where ϵ_o is the parity of the intermediate spin 3/2 particle and $F_{\frac{1}{2}}$ has been labeled as G in the notation of Section III. Expanding Ip_L by the Clebsch-Gordan series and applying the orthogonality relation yields

$$\int Ip_L(\theta, \varphi, \theta', \varphi') D^1_{oo}(\varphi', \theta', 0) d\Omega' = (4\pi/60) |G|^2 \left[3(\rho'_{\frac{3}{2}\,\frac{3}{2}} - \rho'_{-\frac{3}{2}-\frac{3}{2}}) + (\rho'_{\frac{1}{2}\,\frac{1}{2}} - \rho'_{-\frac{1}{2}\,-\frac{1}{2}}) \right] \tag{X-15}$$

and

$$\int Ip_L(\theta,\varphi;\theta',\varphi')\, D^3_{00}(\varphi',\theta',0)\, d\Omega' = (3\pi/35)\,|G|^2$$

$$\left[3(\rho'_{\frac{1}{2}\frac{1}{2}} - \rho'_{-\frac{1}{2}-\frac{1}{2}}) - (\rho'_{\frac{3}{2}\frac{3}{2}} - \rho'_{-\frac{3}{2}-\frac{3}{2}})\right] \qquad \text{(X-16)}$$

These equations may be directly solved for $(\rho'_{\frac{1}{2}\frac{1}{2}} - \rho'_{-\frac{1}{2}-\frac{1}{2}})$ and $(\rho'_{\frac{3}{2}\frac{3}{2}} - \rho'_{-\frac{3}{2}-\frac{3}{2}})$ and the result is given by (3.11) and (3.12). For the determination of $\rho'_{\frac{1}{2}-\frac{1}{2}}$ we proceed as for (X-15) and (X-16) and find

$$\int Ip_L(\theta,\varphi;\theta',\varphi')\, D^{3*}_{10}(\varphi',\theta',0)\, d\Omega' = (3\pi/35)\,|G|^2$$

$$\left[2\rho'_{\frac{1}{2}\frac{3}{2}} - \sqrt{12}\,\rho'_{-\frac{1}{2}\frac{1}{2}} + 2\rho'_{-\frac{3}{2}-\frac{1}{2}}\right] \qquad \text{(X-17)}$$

$$\int Ip_L(\theta,\varphi;\theta',\varphi')\, D^1_{10}(\varphi',\theta',0)\, d\Omega' = -(1/15)\,|G|^2$$

$$\left[\sqrt{6}\,\rho'_{\frac{1}{2}\frac{3}{2}} + \sqrt{8}\,\rho'_{-\frac{1}{2}\frac{1}{2}} + \sqrt{6}\,\rho'_{-\frac{3}{2}-\frac{1}{2}}\right] \qquad \text{(X-18)}$$

These equations can then be solved for $\rho'_{-\frac{1}{2}\frac{1}{2}}$ yielding (3.13). Finally, (3.14) may be derived directly from the definition (X-13).

The determination of the relevant density matrix elements ρ'_{ik} by means of the transverse polarization as given by (3.15) to (3.18) proceeds in a similar manner to the case of the longitudinal polarization. The method is essentially the same as for Ip_L except that for the off-diagonal terms the definition (X-11) is more straightforward than using (X-11) expanded with the Clebsch-Gordan series.

Rather than use other more oscillatory test functions to get new ratios of the type given by (2.12), (3.9), and (3.10) as checks on the analysis, we suggest that the analysis be repeated changing the direction of the z axis with the reminder that for the fermion case the beam direction as z axis will yield vanishing numerator and denominator in (3.9) and (3.10).

Finally, we list the d functions which are useful for the analysis of the decay of particles of spin less than or equal to 3. Not all the d functions are given. The missing ones are easily obtained using the simple symmetry relations

$$d^{j}_{m'm}(\beta) = (-1)^{m-m'}\, d^{j}_{-m'-m}(\beta)$$

$$d^{j}_{m'm}(\beta) = (-1)^{m-m'}\, d^{j}_{m\,m'}(\beta)$$

The relevant d are now listed below.

Spin $\frac{1}{2}$

$$d_{\frac{1}{2}\frac{1}{2}}(\beta) = \cos\frac{\beta}{2} \qquad d_{\frac{1}{2}\frac{1}{2}}(\beta) = \sin\frac{\beta}{2}$$

Spin 1

$$d_{11}(\beta) = \frac{1+\cos\beta}{2} \qquad d_{01}(\beta) = \frac{\sin\beta}{\sqrt{2}}$$

$$d_{1-1}(\beta) = \frac{1-\cos\beta}{2} \qquad d_{00}(\beta) = \cos\beta$$

Spin $\frac{3}{2}$

$$d_{\frac{3}{2}\frac{3}{2}}(\beta) = \frac{1 + \cos\beta}{2} \quad \cos\frac{\beta}{2}$$

$$d_{\frac{3}{2}\frac{1}{2}}(\beta) = -\sqrt{3}\,\frac{1 + \cos\beta}{2}\sin\frac{\beta}{2}$$

$$d_{\frac{3}{2}-\frac{1}{2}}(\beta) = \sqrt{3}\,\frac{1 - \cos\beta}{2}\cos\frac{\beta}{2}$$

$$d_{\frac{3}{2}-\frac{3}{2}}(\beta) = -\frac{1 - \cos\beta}{2}\sin\frac{\beta}{2}$$

$$d_{\frac{1}{2}\frac{1}{2}}(\beta) = \frac{3\cos\beta - 1}{2}\cos\frac{\beta}{2}$$

$$d_{\frac{1}{2}-\frac{1}{2}}(\beta) = -\frac{1 + 3\cos\beta}{2}\sin\frac{\beta}{2}$$

Spin 2

$$d_{22}(\beta) = \left(\frac{1 + \cos\beta}{2}\right)^2$$

$$d_{21}(\beta) = -\frac{1 + \cos\beta}{2}\sin\beta$$

$$d_{20}(\beta) = \frac{\sqrt{6}}{4}\sin^2\beta$$

$$d_{2-1}(\beta) = -\frac{1 - \cos\beta}{2}\sin\beta$$

$$d_{2-2}(\beta) = \left(\frac{1 - \cos\beta}{2}\right)^2$$

$$d_{11}(\beta) = \frac{1 + \cos\beta}{2}(2\cos\beta - 1)$$

$$d_{10}(\beta) = -\sqrt{\frac{3}{2}}\sin\beta\cos\beta$$

$$d_{1-1}(\beta) = \frac{1 - \cos\beta}{2}(2\cos\beta + 1)$$

$$d_{00}(\beta) = \frac{3\cos^2\beta - 1}{2}$$

Spin $\frac{5}{2}$

$$d_{\frac{5}{2}\frac{5}{2}}(\beta) = \left(\frac{1+\cos\beta}{2}\right)^2 \cos\frac{\beta}{2}$$

$$d_{\frac{5}{2}\frac{3}{2}}(\beta) = -\sqrt{5}\left(\frac{1+\cos\beta}{2}\right)^2 \sin\frac{\beta}{2}$$

$$d_{\frac{5}{2}\frac{1}{2}}(\beta) = \frac{\sqrt{10}}{4}\sin^2\beta\cos\frac{\beta}{2}$$

$$d_{\frac{5}{2}-\frac{1}{2}}(\beta) = -\frac{\sqrt{10}}{4}\sin^2\beta\sin\frac{\beta}{2}$$

$$d_{\frac{5}{2}-\frac{3}{2}}(\beta) = \sqrt{5}\left(\frac{1-\cos\beta}{2}\right)^2\cos\frac{\beta}{2}$$

$$d_{\frac{5}{2}\frac{5}{2}}(\beta) = -\left(\frac{1+\cos\beta}{2}\right)^2\sin\frac{\beta}{2}$$

$$d_{\frac{3}{2}\frac{3}{2}}(\beta) = \frac{5\cos\beta - 3}{2}\cos^3\frac{\beta}{2}$$

$$d_{\frac{3}{2}\frac{1}{2}}(\beta) = \frac{-(5\cos\beta-1)}{\sqrt{2}}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}$$

$$d_{\frac{3}{2}-\frac{1}{2}}(\beta) = \frac{1+5\cos\beta}{\sqrt{2}}\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}$$

$$d_{\frac{3}{2}-\frac{3}{2}}(\beta) = -\frac{5\cos\beta+3}{2}\sin^3\frac{\beta}{2}$$

$$d_{\frac{1}{2}\frac{1}{2}}(\beta) = \frac{5\cos^2\beta-2\cos\beta-1}{2}\cos\frac{\beta}{2}$$

$$d_{\frac{1}{2}-\frac{1}{2}}(\beta) = -\frac{5\cos^2\beta+2\cos\beta-1}{2}\sin\frac{\beta}{2}$$

Spin 3

$$d_{33}(\beta) = \left(\frac{1+\cos\beta}{2}\right)^3$$

$$d_{32}(\beta) = -\frac{\sqrt{6}}{8}\sin\beta\,(1+\cos\beta)^2$$

$$d_{31}(\beta) = \frac{\sqrt{15}}{8}\sin^2\beta\,(1+\cos\beta)$$

$$d_{30}(\beta) = -\frac{\sqrt{5}}{4}\sin^3\beta$$

$$d_{3-1}(\beta) = \frac{\sqrt{15}}{8}\sin^2\beta\,(1-\cos\beta)$$

$$d_{3-2}(\beta) = -\frac{\sqrt{6}}{8}\sin\beta\,(1-\cos\beta)^2$$

$$d_{3-3}(\beta) = \left(\frac{1-\cos\beta}{2}\right)^3$$

$$d_{22}(\beta) = \left(\frac{1+\cos\beta}{2}\right)^2(3\cos\beta-2)$$

$$d_{21}(\beta) = -\frac{\sqrt{5}}{4\sqrt{2}} \sin\beta\,(3\cos^2\beta + 2\cos\beta - 1) \qquad d_{20}(\beta) = \frac{\sqrt{15}}{2\sqrt{2}} \cos\beta \sin^2\beta$$

$$d_{2-1}(\beta) = \frac{\sqrt{5}}{4\sqrt{2}} \sin\beta\,(3\cos^2\beta - 2\cos\beta - 1) \qquad d_{2-2}(\beta) = \left(\frac{1-\cos\beta}{2}\right)^2 (3\cos\beta + 2)$$

$$d_{11}(\beta) = \frac{1+\cos\beta}{8} (15\cos^2\beta - 10\cos\beta - 1) \qquad d_{10}(\beta) = -\frac{\sqrt{3}}{4} \sin\beta\,(5\cos^2\beta - 1)$$

$$d_{1-1}(\beta) = \frac{1-\cos\beta}{8} (15\cos^2\beta + 10\cos\beta - 1) \qquad d_{00}(\beta) = \frac{5\cos^3\beta - 3\cos\beta}{2}$$

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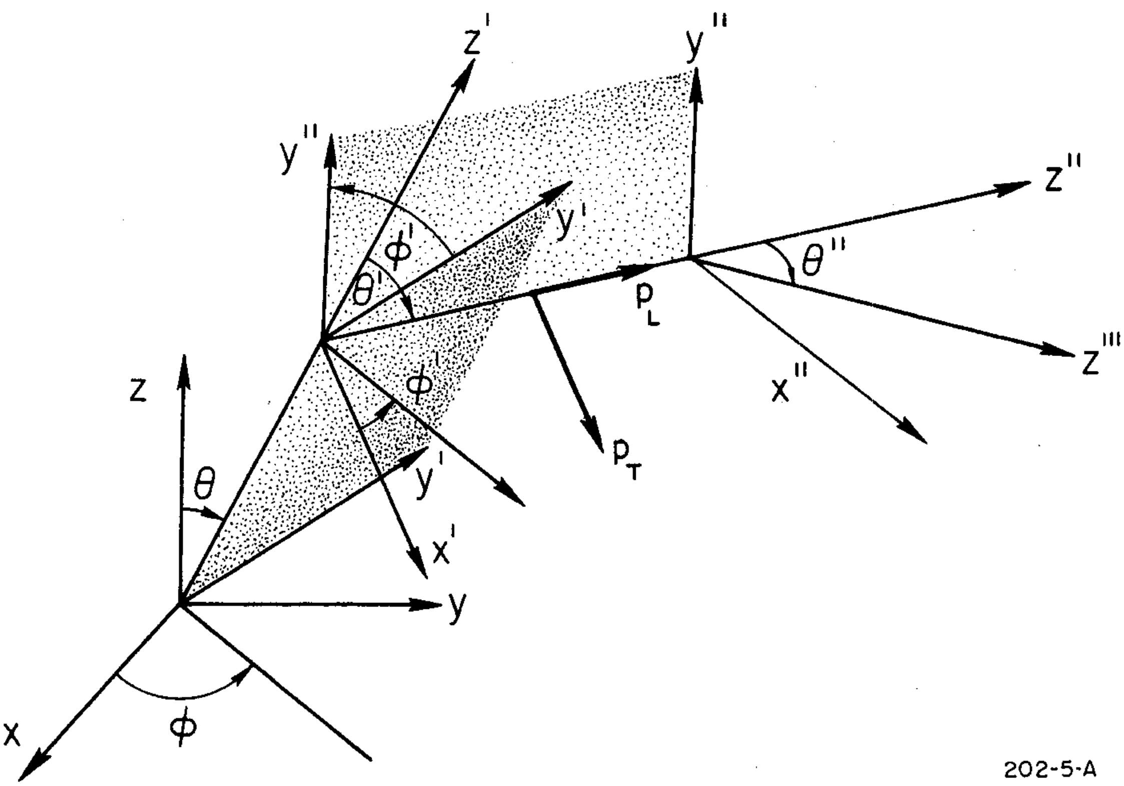

FIGURE I --TWO-STEP DECAY

(xyz) REFER TO THE PARENT RESONANCE REST SYSTEM.
(x'y'z') REFER TO THE INTERMEDIATE RESONANCE REST SYSTEM.
(x''y''z'') IS APPLICABLE TO THE CASE WHERE THE FINAL SPIN 1/2 PARTICLE DECAYS IN WHICH CASE IT REFERS TO THE FINAL SPIN 1/2 PARTICLES REST SYSTEM.