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FIRST AND SECOND ORDER BEAM OPTICS OF A CURVED, INCLINED MAGNETIC FIELD BOUNDARY IN THE IMPULSE APPROXIMATION

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by

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I. INTRODUCTION

The first-order optical effects of the entrance and exit field boundaries of a wedge-type magnetic spectrometer with external source and image have been described by Cross and others.[1,2,3] In particular, it has been shown that rotation of the boundary relative to a radius through the center of curvature is equivalent to a thin stigmatic lens located at the boundary, in the approximation of negligibly small azimuthal extent of the fringing field.

Numerous investigators[1,3,4,5] have extended these calculations to include second-order effects in the magnetic midplane of a spectrometer whose boundaries are curved as well as inclined, essentially by introducing geometric corrections to the first-order edge focusing. These results have been summarized in terms of second-order expansion coefficients by Brown,[6] who has developed also a second-order matrix formalism[6] in which the expansion coefficients are particularly useful.

In the present report the second-order calculation of the magnetic boundary is extended to include off-midplane rays. In this case, the appearance of second derivatives in the field expansion introduces new terms which would not be found in a purely geometric calculation.* The calculation of Ikegami[5] does not include these new terms and therefore may be significantly incorrect insofar as off-midplane rays are concerned.

The present calculation is essentially an impulse approximation and does not treat explicitly the finite extent of an actual fringing field. However, in the case of the first-order edge focusing, it is well known that the impulse approximation gives the dominant effect, and that semi-empirical corrections to first order in the gap height give a very good representation of the first-order coefficients. Thus it seems reasonable that the impulse calculation may give a useful first approximation for the second-order coefficients.

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*The writer is indebted to R. Belbeoch who, in a conversation in August 1961, suggested the implications of the second derivative terms in the off-midplane field expansion.

II. FORMULATION

Consider the system shown in Fig. 1, which represents a slight generalization of the problem of entrance to (exit from) a wedge magnet with curved field boundaries. The mean ray is supposed to have constant curvature $\frac{1}{r_1}$ in Region I, to the left of the boundary BB, and constant curvature $\frac{1}{r_2}$ in Region II, to the right of the boundary; the field is supposed to be azimuthally constant along the mean ray in both Regions I and II, but to vary discontinuously across a negligibly small region at the boundary.

It is desired that the net effect of the boundary on a ray near the mean ray be represented by a fictitious optical element of zero thickness, located at the reference plane $z = \zeta = 0$. This breaks down naturally into three steps:

1. A transformation $T(1|0)$ from the initial point (0) to point (1), just to the left of the boundary;

2. The transformation $T(2|1)$ across the boundary, from point (1) to point (2);

3. The transformation $T(f|2)$ from point (2) back to the final point (f), just to the right of the reference plane.

The net transformation across the fictitious element then is represented schematically by

$$T(f|0) = T(f|2)T(2|1)T(1|0)$$

A. TRANSFORMATION FROM REFERENCE PLANE TO BOUNDARY, IN REGION I

It is convenient to make the calculation in the rectangular (x, y, z) system. The coordinate transformation (in Region I) is

$$\left.\begin{aligned} \xi &= \sqrt{z^2 + (r_1 + x)^2} - r_1 \\ \zeta &= r_1 \arctan\left(\frac{z}{r_1 + x}\right) \end{aligned}\right\} \qquad (1.a)$$

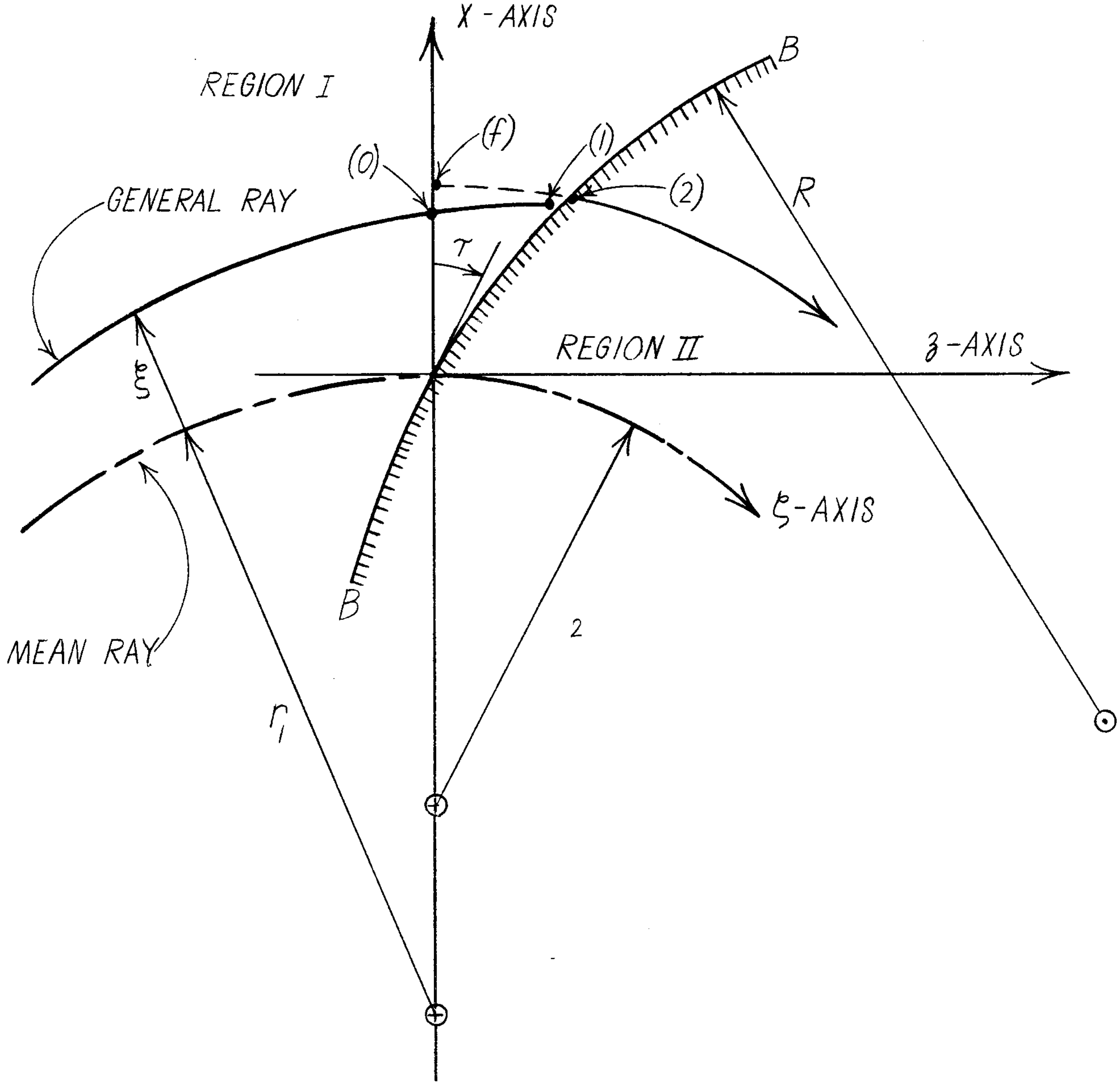

FIG. 1--Midplane geometry. The ζ and ξ coordinates are measured, respectively, along and perpendicular to the mean ray. The y axis, not shown, is normal to (ζ,ξ) and (z,x).

Expansion to second order in small quantities gives

$$\left.\begin{aligned} \xi &= x + \frac{1}{2} C_1 z^2 + \ldots \\ \zeta &= z - C_1 z x + \ldots \end{aligned}\right\} \qquad (1.b)$$

where $C_1 = \frac{1}{r_1}$ = curvature of mean ray in Region I.

The magnetic field in the midplane is assumed given in Region I by

$$B_y^I(x,0,z) \equiv B_1(x,z) = b_1(1 + \alpha_1 \xi + \beta_1 \xi^2 + \ldots) \qquad (2.a)$$

or with the substitution of Eq. (1.b),

$$B_1(x,z) = b_1(1 + \alpha_1 x + \beta_1 x^2 + \frac{1}{2} C_1 \alpha_1 z^2 + \ldots) \qquad (2.b)$$

With the help of Maxwell's Equations and the symmetry of the field about the midplane, one finds the general relationships

$$\left.\begin{aligned} B_y &= \left[1 - \frac{1}{2} y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) + \ldots\right] B_y(x,0,z) \\ B_z &= \left[y - \frac{1}{6} y^3 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) + \ldots\right] \frac{\partial}{\partial z} B_y(x,0,z) \\ B_x &= \left[y - \frac{1}{6} y^3 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) + \ldots\right] \frac{\partial}{\partial x} B_y(x,0,z) \end{aligned}\right\} \qquad (3)$$

which in the present case gives

$$\left.\begin{aligned} B_y &= b_1\left[1 + \alpha_1 x + \beta_1 x^2 - (\beta_1 + \frac{1}{2} C_1 \alpha_1) y^2 + \frac{1}{2} C_1 \alpha_1 z^2 + \ldots\right] \\ B_z &= b_1\left[C_1 \alpha_1 y z + \ldots\right] \\ B_x &= b_1\left[\alpha_1 y + 2\beta_1 x y + \ldots\right] \end{aligned}\right\} \qquad (4)$$

With z as the independent variable, the equations of motion may be written

$$\left.\begin{aligned} P\left(\frac{x'}{u}\right)' &= y'B_z - B_y \\ P\left(\frac{y'}{u}\right)' &= B_x - x'B_z \end{aligned}\right\} \qquad (5)$$

where P = scalar momentum expressed as magnetic rigidity ($B\rho$); the prime (') indicates differentiation by z (e.g., $x' = \frac{dx}{dz}$, etc.); and

$$u = \sqrt{1 + x'^2 + y'^2}$$

Expansion of Eq. (5) to second-order (considering x, x', y, y' to be small) and substitution of Eq. (4) for the fields gives

$$\left.\begin{aligned} x'' &= -\frac{b_1}{P}\left[1 + \alpha_1 x + \beta_1 x^2 - (\beta_1 + \tfrac{1}{2} C_1\alpha_1)y^2 + \tfrac{1}{2} C_1\alpha_1 z^2 + \ldots\right] \\ y'' &= \frac{b_1}{P}\left[\alpha_1 y + 2\beta_1 xy + \ldots\right] \end{aligned}\right\} \qquad (6)$$

The condition that the mean ray must have curvature C_1 is expressed by

$$b_1 = C_1 P_o \qquad (7)$$

where P_o is the magnetic rigidity of the mean ray and is related to P by

$$P = (1 + \epsilon)P_o \qquad (8)$$

where ϵ also is assumed small. (It will, however, be convenient to keep P in parametric form and not expand in powers of ϵ until the final step of the transformation.)

The solution of Eq. (6) is readily found as a Taylor expansion in z, and is given to second order by

$$\left.\begin{aligned} x &= x_o + x_o'z - \tfrac{1}{2} C_1 z^2 + \ldots . \\ x' &= x_o' - \frac{P_o}{P} C_1 z - C_1\alpha_1 x_o z + \ldots . \\ y &= y_o + y_o'z + \ldots . \\ y' &= y_o' + C_1\alpha_1 y_o z + \ldots . \end{aligned}\right\} \tag{9}$$

The equation of the boundary BB (Fig. 1) is given by

$$\begin{aligned} z &= R \cos \tau - \sqrt{R^2 - (x + R \sin \tau)^2} \\ &= x \tan \tau + \tfrac{1}{2} Kx^2 \sec^3 \tau + \ldots . \end{aligned} \tag{10}$$

where

$$K \equiv \frac{1}{R} = \text{curvature of boundary.}$$

Simultaneous solution of Eqs. (9) and (10) gives z_1 , the abscissa at the point of intersection of the ray and the boundary:

$$z_1 = x_o \tan \tau + \tfrac{1}{2}\left(K \sec^3 \tau - C_1 \tan^3 \tau\right)x_o^2 + x_o x_o' \tan^2 \tau + \ldots \tag{11}$$

Substitution of Eq. (11) in Eq. (9) now yields $T(1|0)$, the first part of the desired transformation:

$$\left.\begin{aligned} x_1 &= x_o - \left[\tfrac{1}{2} C_1 \tan^2 \tau\right]x_o^2 + \left[\tan \tau\right]x_o x_o' + \ldots . \\ x_1' &= - \left[\frac{P_o}{P} C_1 \tan \tau\right]x_o + x_o' \\ &\quad - \left[\tfrac{1}{2} C_1(K \sec^3 \tau - C_1 \tan^3 \tau + 2\alpha_1 \tan \tau)\right]x_o^2 - \left[C_1 \tan^2 \tau\right]x_o x_o' + \ldots \\ y_1 &= y_o + \left[\tan \tau\right]x_o y_o' + \ldots . \\ y_1' &= y_o' + \left[C_1\alpha_1 \tan \tau\right]x_o y_o + \ldots . \end{aligned}\right\} \tag{12}$$

Another quantity which will be required is the slope of the boundary at the intersection, defined by

$$\tan\tau_1 \equiv \left(\frac{dz}{dx}\right)_{BB} = \tan\tau + Kx_1 \sec^3\tau + \ldots \tag{13}$$

(It turns out that $\tan\tau_1$ is needed only to first order.)

It is also of some interest to calculate the <u>path-length</u> <u>difference</u> between the general ray and the mean ray. This is given in the present case (note that the mean ray has zero path length to the boundary) by

$$\delta\ell_1 = \delta\ell_0 + \int_0^{z_1} u\,dz$$

where $\delta\ell_0$ is the initial value. Calculation to second order gives

$$\delta\ell_1 = \delta\ell_0 + z_1 + \ldots \tag{14}$$

with z_1 given by Eq. (11).

B. TRANSFORMATION ACROSS THE BOUNDARY

This part of the transformation is most readily carried out in a rotated coordinate system (t, y, w) as illustrated in Fig. 2.

The usefulness of this system arises from the field derivatives which appear in the expansion [Eq. (3)]; the first derivatives in the t-direction are essentially finite while the variation in the w-direction is discontinuous.

The coordinate transformation (with w treated as the independent variable) is

$$\left.\begin{aligned} t &= (x - x_1)\cos\tau_1 + (z - z_1)\sin\tau_1 \\ w &= -(x - x_1)\sin\tau_1 + (z - z_1)\cos\tau_1 \\ \dot{t} &\equiv \frac{dt}{dw} = \frac{\tan\tau_1 + x'}{1 - x'\tan\tau_1} \\ \dot{y} &\equiv \frac{dy}{dw} = \frac{y'\sec\tau_1}{1 - x'\tan\tau_1} \end{aligned}\right\} \tag{15}$$

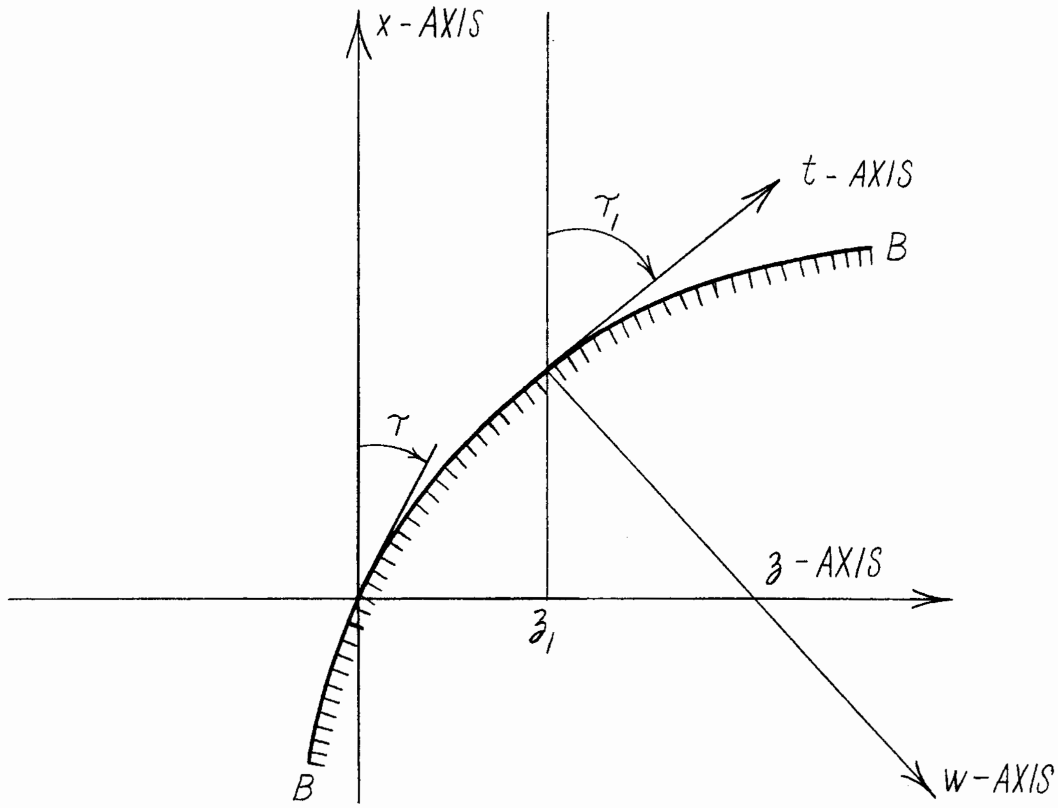

FIG. 2--Rotated coordinate system for boundary transformation calculation. The t and w axes are respectively tangent and normal to the boundary at the point of intersection; y is unchanged.

Note that $\dot{t}$, the slope of the t-coordinate, may not be assumed small in this system because of the zero order term $\tan \tau_1$. It will be convenient to introduce the notation

$$\dot{t} \equiv \dot{t}_1 + \dot{\eta} \tag{16}$$

where $\dot{t}_1$ is the initial value; $\dot{\eta}_1 = \eta_1 = 0$; and $\dot{\eta}$ may hopefully be treated as small.

Use of the analogues of Eqs. (3) and (5) (which, of course, are invariant in form) gives

$$P \frac{d}{dw}\left(\frac{\dot{t}}{U}\right) = - B + \tfrac{1}{2} y^2 \left(\frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial t^2}\right) B + y\dot{y} \frac{\partial B}{\partial w} + \ldots .$$

$$P \frac{d}{dw}\left(\frac{\dot{y}}{U}\right) = y \frac{\partial B}{\partial t} - y\dot{t} \frac{\partial B}{\partial w} + \ldots . \tag{17}$$

$$U = \sqrt{1 + \dot{t}^2 + \dot{y}^2}$$

where the midplane field, B, is [see Eq. (2.b)]

$$B = \begin{cases} B_1 = b_1(1 + \alpha_1 x + \beta_1 x^2 + \tfrac{1}{2} C_1 \alpha_1 z^2 + \ldots) \text{ (in Region I)} \\ B_2 = b_2(1 + \alpha_2 x + \beta_2 x^2 + \tfrac{1}{2} C_2 \alpha_2 z^2 + \ldots) \text{ (in Region II)} \end{cases} \tag{18}$$

Second-order expansion and use of the identities

$$\left.\begin{aligned} \frac{\partial B}{\partial w} &= \frac{dB}{dw} - \dot{t} \frac{\partial B}{\partial t} \\ \frac{\partial^2 B}{\partial w^2} &= \frac{d}{dw}\frac{\partial B}{\partial w} - \dot{t} \frac{d}{dw}\frac{\partial B}{\partial t} + \dot{t}^2 \frac{\partial^2 B}{\partial t^2} \end{aligned}\right\} \tag{19}$$

in Eq. (17) gives

$$\left.\begin{aligned} \frac{d}{dw}\left(\dot{\eta} - \frac{1}{2}\,\dot{t}_1\dot{y}^2 - \frac{3}{2}\,\frac{\dot{t}_1\dot{\eta}^2}{1 + \dot{t}_1^2}\right) &= \frac{\left(1 + \dot{t}_1^2\right)^{3/2}}{P}\Bigg[- B + \frac{1}{2}\,\frac{d}{dw}\left(y^2\,\frac{\partial B}{\partial w}\right) \\ &\quad - \frac{1}{2}\,\dot{t}_1 y^2\,\frac{d}{dw}\,\frac{\partial B}{\partial t} + \frac{1}{2}\left(1 + \dot{t}_1^2\right)y^2\,\frac{\partial^2 B}{\partial t^2}\Bigg] + \ldots. \\ \frac{d}{dw}\left(\dot{y} - \frac{\dot{t}_1\dot{\eta}\dot{y}}{1 + \dot{t}_1^2}\right) &= \frac{\left(1 + \dot{t}_1^2\right)^{1/2}}{P}\left[- \dot{t}y\,\frac{dB}{dw} + \left(1 + \dot{t}^2\right)y\,\frac{\partial B}{\partial t}\right] + \ldots. \end{aligned}\right\} \quad (20)$$

In order to integrate Eq. (20), it is assumed that the midplane field may be represented by

$$B = B_1 + (B_2 - B_1)\; S(w - \tfrac{1}{2}\,Kt^2) \quad (21)$$

where B_1 and B_2 are defined by Eq. (18), and $S(w)$ is essentially the unit step function. Note that Eq. (21) expresses the fact that the midplane field is B_1 to the left of the boundary and B_2 to the right of the boundary, since the equation of the boundary in the (t, y, w) system is

$$w = \tfrac{1}{2}\,Kt^2 + \ldots.$$

Equation (20) may now be integrated by successive approximations, with the help of Eq. (21). After the second approximation one obtains

$$
\left.
\begin{aligned}
t_2 &= \eta_2 = \tfrac{1}{2}\,\frac{\left(1 + \dot{t}_1^2\right)^{3/2}}{P}\,(B_2 - B_1)y_1^2 + \ldots \\
\dot{t}_2 &= \dot{t}_1 + \dot{\eta}_2 \\
&= \dot{t}_1 - \frac{\dot{t}_1^2\left(1 + \dot{t}_1^2\right)^{1/2}}{P}\,(B_2 - B_1)y_1\dot{y}_1 \\
&\quad + \tfrac{1}{2}\,y_1^2\left\{\frac{\left(1 + \dot{t}_1^2\right)\dot{t}_1^3}{P^2}\,(B_2 - B_1)^2 + \frac{\left(1 + \dot{t}_1^2\right)^{3/2}}{P}\left[\left(\frac{\partial B}{\partial w} - \dot{t}_1\,\frac{\partial B}{\partial t}\right)_2 - \left(\frac{\partial B}{\partial w} - \dot{t}_1\,\frac{\partial B}{\partial t}\right)_1\right]\right. \\
&\qquad \left. - \frac{\left(1 + \dot{t}_1^2\right)^{5/2}}{P}\,K(B_2 - B_1)\right\} + \ldots \\
y_2 &= y_1 + \ldots \\
\dot{y}_2 &= -\,\frac{\dot{t}_1\left(1 + \dot{t}_1^2\right)^{1/2}}{P}\,(B_2 - B_1)y_1 + \dot{y}_1 + \ldots
\end{aligned}
\right\}
\quad (22)
$$

Terms which contain first and second integrals of B, $\frac{\partial B}{\partial t}$, etc., have been dropped after the integration, because it is assumed that B and its tangential first derivatives are everywhere finite and that the thickness of the boundary is negligibly small. However, one notices that there is a term containing $\frac{\partial B}{\partial w}$ (in the expression for $\dot{\eta}$); this means that the assumption of negligible boundary thickness must be used with caution, since $\frac{\partial B}{\partial w}$ (and consequently $\dot{\eta}$) would become infinite at the boundary if B were really discontinuous, and the series expansion of U in powers of $\dot{\eta}$ would not be valid. This difficulty is avoided by assuming that the boundary actually has a "finite but small" thickness, e.g., of the same order of smallness as x and y, so that $\frac{\partial B}{\partial w}$ and $\dot{\eta}$ remain finite everywhere (the troublesome term is in any event only of the order y^2). Final dropping of terms which depend on details of boundary structure, as is usually done in the ordinary calculation of first-order edge focusing, may still give a useful first approximation for the effect of real fringe fields.

It is necessary to evaluate the partial derivative terms which occur in Eq. (22). With the help of Eq. (15), it is readily shown that

$$\frac{\partial B}{\partial w} - \dot{t}_1 \frac{\partial B}{\partial t} = - 2b\alpha \sin \tau_1 + \ldots .$$

(where $b\alpha = b_1\alpha_1$ in Region I, etc.).

In order to transform back to the original (x, y, z) coordinate system, one uses Eq. (15) to transform the initial conditions (x_1, x_1', y_1, y_1') to $(t_1, \dot{t}_1, y_1, \dot{y}_1)$; the inverse of Eq. (15) to transform the final values $(t_2, \dot{t}_2, y_2, \dot{y}_2)$ to (x_2, x_2', y_2, y_2'); and Eq. (13) to express $\tan \tau_1$ in terms of $\tan \tau$. The result of this straightforward but tedious calculation gives $T(2|1)$, the second part of the desired transformation:

$$\left.\begin{aligned}
x_2 &= x_1 + \left[\tfrac{1}{2}(C_2 - C_1) \sec^2 \tau\right] y_1^2 + \ldots . \\
x_2' &= x_1' + \left[\tfrac{1}{2}(C_2 - C_1)^2 \tan^3 \tau - (C_2\alpha_2 - C_1\alpha_1) \tan \tau\right] y_1^2 \\
&\quad - \left[(C_2 - C_1) \tan^2 \tau\right] y_1 y_1' + \ldots . \\
y_2 &= y_1 + \ldots . \\
y_2' &= - \left[\frac{P_0}{P} (C_2 - C_1) \tan \tau\right] y_1 + y_1' \\
&\quad - \left[K(C_2 - C_1) \sec^3 \tau + (C_2\alpha_2 - C_1\alpha_1) \tan \tau\right] x_1 y_1 \\
&\quad - \left[(C_2 - C_1) \sec^2 \tau\right] x_1' y_1 + \ldots .
\end{aligned}\right\} \quad (23)$$

The substitutions [Eq. (7)]

$$b_1 = C_1 P_o$$

and

$$b_2 = C_2 P_o$$

have been made.

The path-length difference is found readily; it is

$$\delta \ell_2 = \delta \ell_1 + \int_0^{w_2} dw \sqrt{1 + \dot{t}^2 + \dot{y}^2}$$

To the same approximation as Eqs. (22), this gives

$$\begin{aligned} \delta \ell_2 &= \delta \ell_1 + \left(1 + \dot{t}_1^2\right)^{-1/2} \dot{t}_1 \eta_2 + \ldots . \\ &= \delta \ell_1 + \left[\tfrac{1}{2}(C_2 - C_1) \tan \tau \sec^2 \tau\right] y_1^2 + \ldots . \end{aligned} \qquad (24)$$

C. TRANSFORMATION FROM BOUNDARY BACK TO REFERENCE PLANE, IN REGION II

The trajectory of the ray is again found by a Taylor's series solution of the differential equations. By analogy with Eq. (9),

$$\left.\begin{aligned} x &= x_2 + x_2'(z - z_2) - \tfrac{1}{2} C_2 (z - z_2)^2 + \ldots . \\ x' &= x_2' - \frac{P_o}{P} C_2 (z - z_2) - C_2 \alpha_2 x_2 (z - z_2) + \ldots . \\ y &= y_2 + y_2'(z - z_2) + \ldots . \\ y' &= y_2' + C_2 \alpha_2 y_2 (z - z_2) + \ldots . \end{aligned}\right\} \qquad (25)$$

The final conditions (x_f, x_f', y_f, y_f') are given by Eq. (25) by setting $z = 0$. Equation (10) gives z_2 in terms of x_2;

$$z_2 = x_2 \tan \tau + \frac{1}{2} K x_2^2 \sec^3 \tau + \ldots . \tag{26}$$

With these substitutions Eq. (25) yields $T(f|2)$, the third part of the desired transformation:

$$\left.\begin{aligned}
x_f &= x_2 - \left[\frac{1}{2} C_2 \tan^2 \tau\right] x_2^2 - \left[\tan \tau\right] x_2 x_2' + \ldots . \\
x_f' &= \left[\frac{P}{P_o} C_2 \tan \tau\right] x_2 + x_2' + \left[\frac{1}{2} K C_2 \sec^3 \tau + C_2 \alpha_2 \tan \tau\right] x_2^2 + \ldots \\
y_f &= y_2 - \left[\tan \tau\right] x_2 y_2' + \ldots . \\
y_f' &= y_2' - \left[C_2 \alpha_2 \tan \tau\right] x_2 y_2 + \ldots .
\end{aligned}\right\} \tag{27}$$

The path-length difference in this case is

$$\delta\ell_f = \delta\ell_2 + \int_{z_2}^{0} u\,dz$$

which when evaluated to second order gives

$$\delta\ell_f = \delta\ell_2 - z_2 + \ldots . \tag{28}$$

where z_2 is given by Eq. (26).

D. EQUIVALENT TRANSFORMATION AT THE REFERENCE PLANE $z = 0$

By successive substitution of Eqs. (23) and (12) in (27), the final transformation

$$T(f|0) = T(f|2)T(2|1)T(1|0)$$

is formed. The result is

$$
\left.
\begin{aligned}
x_f &= x_o - \left[\tfrac{1}{2}(C_2 - C_1)\tan^2\tau\right]x_o^2 + \left[\tfrac{1}{2}(C_2 - C_1)\sec^2\tau\right]y_o^2 + \ldots \\
x_f' &= \left[(C_2 - C_1)\tan\tau\right]x_o + x_o' \\
&\quad + \left\{\tfrac{1}{2}K(C_2 - C_1)\sec^3\tau - \tfrac{1}{2}C_1(C_2 - C_1)\tan^3\tau + (C_2\alpha_2 - C_1\alpha_1)\tan\tau\right\}x_o^2 \\
&\quad + \left[(C_2 - C_1)\tan^2\tau\right]x_o x_o' - \left[(C_2 - C_1)\tan\tau\right]x_o\epsilon \\
&\quad + \left\{\tfrac{1}{2}(C_2 - C_1)\left[C_2\sec^2\tau + (C_2 - C_1)\tan^2\tau\right]\tan\tau - (C_2\alpha_2 - C_1\alpha_1)\tan\tau \right. \\
&\quad \left. - \tfrac{1}{2}K(C_2 - C_1)\sec^3\tau\right\}y_o^2 - \left[(C_2 - C_1)\tan^2\tau\right]y_o y_o' + \ldots \\
y_f &= y_o + \left[(C_2 - C_1)\tan^2\tau\right]x_o y_o + \ldots \\
y_f' &= -\left[(C_2 - C_1)\tan\tau\right]y_o + y_o' \\
&\quad - \left[K(C_2 - C_1)\sec^3\tau - C_1(C_2 - C_1)\tan\tau\sec^2\tau + 2(C_2\alpha_2 - C_1\alpha_1)\tan\tau\right]x_o y_o \\
&\quad - \left[(C_2 - C_1)\tan^2\tau\right]x_o y_o' + \left[(C_2 - C_1)\tan\tau\right]y_o\epsilon \\
&\quad - \left[(C_2 - C_1)\sec^2\tau\right]x_o' y_o + \ldots
\end{aligned}
\right\} \quad (29)
$$

where, by way of summary, the notation is

$$C_1 = \frac{1}{r_1} = \text{curvature of mean ray to the left of the boundary;}$$

$$C_2 = \frac{1}{r_2} = \text{curvature of mean ray to the right of the boundary;}$$

$$K = \frac{1}{R} = \text{curvature of the boundary;}$$

τ = angle of rotation of the boundary relative to a normal to the mean ray (See Fig. 1 for sign convention of τ, r_1, r_2, and K);

$$\epsilon = \frac{P - P_o}{P_o} = \text{momentum deviation relative to the mean ray.}$$

Similarly, substitution of Eqs. (26), (24), (14), and (11) in (28) yields

$$\delta\ell_f = \delta\ell_o + \ldots . \tag{30}$$

i.e., there is no net first- or second-order path-length difference in the present approximation.

III. APPLICATIONS

By way of illustration, the general expressions [Eq. (29)] will now be applied to the examples of entrance to and exit from a wedge magnet with curved boundaries.

Example 1: Magnet Entrance

In this case, the definitions are (see Fig. 3)

$$C_1 = 0 \qquad C_2 = 1$$

$$\tau = \tau_1 \qquad K = \frac{1}{R_1}$$

$$\alpha_2 = - n$$

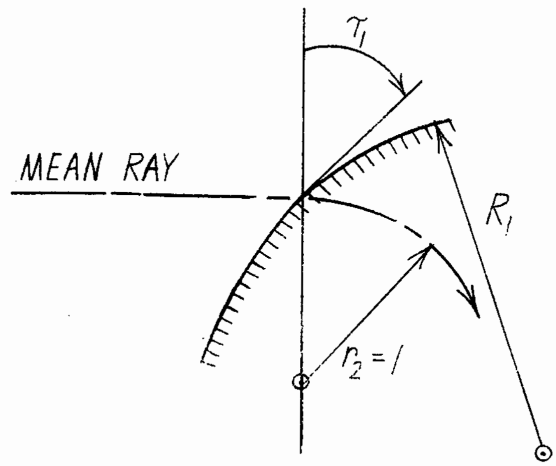

FIG. 3--Magnet entrance.

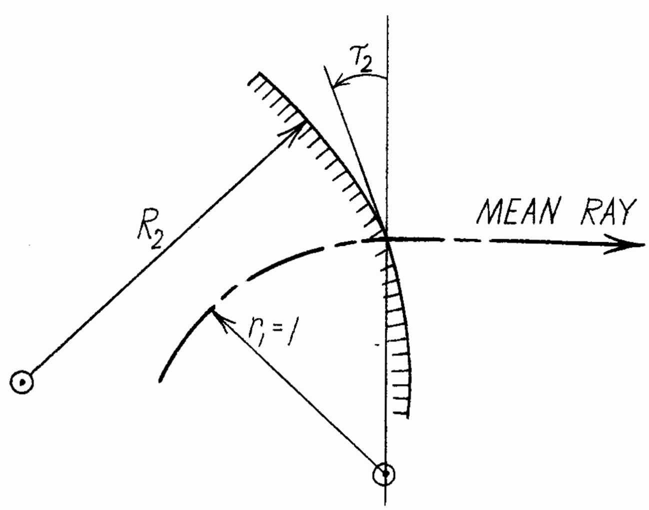

FIG. 4--Magnet exit.

The transformation then becomes (in the notation of Streib[7])*

$$(x|x_0) = (x'|x'_0) = (y|y_0) = (y'|y'_0) = 1$$

$$(x'|x_0) = \tan \tau_1 \qquad (y'|y_0) = -\tan \tau_1$$

$$(x|x_0^2) = -\tfrac{1}{2} \tan^2 \tau_1 \qquad (y|x_0 y_0) = \tan^2 \tau_1$$

$$(x|y_0^2) = \tfrac{1}{2} \sec^2 \tau_1 \qquad (y'|x_0 y_0) = -\frac{\sec^3 \tau_1}{R_1} + 2n \tan \tau_1$$

$$(x'|x_0^2) = \tfrac{1}{2} \frac{\sec^3 \tau_1}{R_1} - n \tan \tau_1 \qquad (y'|x_0 y'_0) = -\tan^2 \tau_1$$

$$(x'|x_0 x'_0) = \tan^2 \tau_1 \qquad (y'|x'_0 y_0) = -\sec^2 \tau_1$$

$$(x'|x_0 \epsilon) = -\tan \tau_1 \qquad (y'|y_0 \epsilon) = \tan \tau_1$$

$$(x'|y_0^2) = (n + \tfrac{1}{2} + \tan^2 \tau_1) \tan \tau_1 - \tfrac{1}{2} \frac{\sec^3 \tau_1}{R_1}$$

$$(x'|y_0 y'_0) = -\tan^2 \tau_1$$

(All coefficients not listed are zero.)

Example 2: Magnet Exit

The sign conventions for R and τ in the case of magnet exit, as shown in Fig. 4, are different from the entrance case.** In this case, the definitions are

$$C_1 = 1 , \quad C_2 = 0 , \quad \tau = -\tau_2 , \quad K = -\frac{1}{R_2} , \quad \alpha_1 = -n$$

---

* e.g., $\dfrac{\partial x_f}{\partial x_0} \equiv (x|x_0)$, etc.

** The usual convention is that τ is positive for positive focusing in y, and that R is positive if the field boundary is convex outward. (See, e.g., Ref. 6.)

Substitution in Eq. (29) gives the following non-zero coefficients:

$$(x|x_o) = (x'|x'_o) = (y|y_o) = (y'|y'_o) = 1$$

$$(x'|x_o) = \tan\tau_2 \qquad (y'|y_o) = -\tan\tau_2$$

$$(x|x_o^2) = \frac{1}{2}\tan^2\tau_2 \qquad (y|x_o y_o) = -\tan^2\tau_2$$

$$(x|y_o^2) = -\frac{1}{2}\sec^2\tau_2 \qquad (y'|x_o y_o) = -\frac{\sec^3\tau_2}{R_2} + (2n + \sec^2\tau_2)\tan\tau_2$$

$$(x'|x_o^2) = \frac{1}{2}\frac{\sec^3\tau_2}{R_2} - \left(n + \frac{1}{2}\tan^2\tau_2\right)\tan\tau_2$$

$$(y'|x_o y'_o) = \tan^2\tau_2$$

$$(x'|x_o x'_o) = -\tan^2\tau_2 \qquad (y'|x'_o y_o) = \sec^2\tau_2$$

$$(x'|x_o\epsilon) = -\tan\tau_2 \qquad (y'|y_o\epsilon) = \tan\tau_2$$

$$(x'|y_o^2) = \left(n - \frac{1}{2}\tan^2\tau_2\right)\tan\tau_2 - \frac{1}{2}\frac{\sec^3\tau_2}{R_2}$$

$$(x'|y_o y'_o) = \tan^2\tau_2$$

The first-order coefficients given in the preceding examples are equivalent to the usual edge-focusing effect given in numerous references.[1,2,3,6] The midplane second-order coefficients — $(x|x_o^2)$, $(x'|x_o^2)$, $(x'|x_o x')$, and $(x'|x_o\epsilon)$ — are equivalent to the results summarized by Brown[6] and in part by others.[1,3,4] Of the second-order terms, the $(x|y_o^2)$, $(x'|y_o^2)$, and $(x'|y_o y'_o)$ arise entirely from the dynamic effect of the second-order terms in the equation of motion; the $(y|x_o y_o)$, $(y'|x_o y_o)$, $(y'|y_o\epsilon)$, and $(y'|x'_o y_o)$, in addition to the midplane terms, are implicit in the first-order theory through geometric corrections or expansion of momentum dependent terms.

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May 14, 1964

To: Recipients of SLAC-24, "First and Second Order Beam Optics of a Curved, Inclined Magnetic Field Boundary in the Impulse Approximation."

From: R. H. Helm

Subject: Erratum SLAC-24

The second of Equations (23), p. 12, should read

$$x_2' = x_1' + \left[\tfrac{1}{2}(C_2 - C_1)^2 \tan^3\tau - (C_2\alpha_2 - C_1\alpha_1)\tan\tau - \tfrac{1}{2} K(C_2 - C_1)\sec^3\tau\right] y_1^2$$

$$- \left[(C_2 - C_1)\tan^2\tau\right] y_1 y_1' + \ldots .$$

The omission occurred only in Eq. (23) and does not affect any of the subsequent results of the paper.

R. H. Helm