SPACE-CHARGE LIMIT AND CONFINEMENT IN PARTICLE ACCELERATION

WITH HIGH-INTENSITY LIGHT WAVES

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by

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SUMMARY

From the condition of dynamical equilibrium between particles and
fields (accelerating and space-charge fields) the maximum current which
can be accelerated using optical frequencies has been calculated. This
current maximum is proportional to the product of the wavelength λ and
the accelerating field E $\left(I_{max} \approx \lambda E\right)$. The average current for this type
of optical particle accelerator is about 10^{-3} times the average current
in microwave accelerators. However, the product of the average current
and the accelerator length is approximately equal for the two types of
accelerators. Because of the high field strength in these optical accel-
erators, the equivalent dc confinement force on the particles can be
significant. This dc force in gradient fields is proportional to
$\nabla_r E(r)^2$. This force may be used for confinement of a particle beam or
even for medium-energy acceleration of the beam. Experiments to measure
and utilize this effect are proposed.

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I. INTRODUCTION

Since the electrical field strength from laser radiation may be of
the order of 10^9 V/m it is very inviting to try to use this tremendous
field strength for particle acceleration.[1,2] The proposed systems may
be feasible in theory but the technical difficulties are very serious and
even theoretical problems concerning the final design like breakdown
effects at optical frequencies, space charge effects, etc., are not known
in detail.

In this report we would like to investigate the effect of the space
charge and beam confinement problem in a medium energy accelerator. Be-
cause at fully relativistic energies $(T \gg mc^2)$ there is no space charge
problem, we would like to restrict our discussion to the energy region
where the kinetic energy of the accelerated particle is of the same order
of magnitude as or lower than its rest energy. For electron accelerators
this energy region is a few Mev but for a proton linac it is in the Bev
region.

II. SPACE CHARGE LIMIT

In the proposed laser accelerators the diameter of the accelerated
beam is of the order of magnitude of the wavelength of the emitted light
$(10^{-4}$ cm$)$.

First we would like to consider the motion of a bunch of particles
with total charge Q in an accelerating field given by:

$$\vec{E} = \left\{ 0, \ 0, \ E_0(r) \ \sin \omega \left(t - \frac{z}{v_0} \right) \right\}$$

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[1]Koichi Shimoda, "Proposal for an electron accelerator using an
optical maser," Applied Optics 1 (No. 1): January 1962.

[2] A. Lohman, IBM Internal Report, San Jose, California.

It is convenient to introduce a moving axis, moving with the velocity, v_O, of the traveling wave. If z' is the displacement of the center of the bunch with respect to these moving axes, we then have

$$z' = z - v_O t$$

The motion equation, without the space charge term is

$$\frac{dp}{dt} = QE_O \sin \omega \left(t - \frac{z}{v_O} \right)$$

which will transform to

$$\frac{dp}{dt} = - QE \sin\left(\frac{\omega z'}{v_O} \right) = - QE \sin\left(\varphi_O \right)$$

where φ_O is the equilibrium phase.

In order to take into account the space charge forces which act on bunched particles in the linear accelerator, we have to calculate the magnitude of the diverging field $\left(E_D \right)$ acting on a particle at the surface of the bunch. Now if we assume that the bunch has an angular length of φ then the motion equation along the axis for particle (1) is: (See Fig. 1)

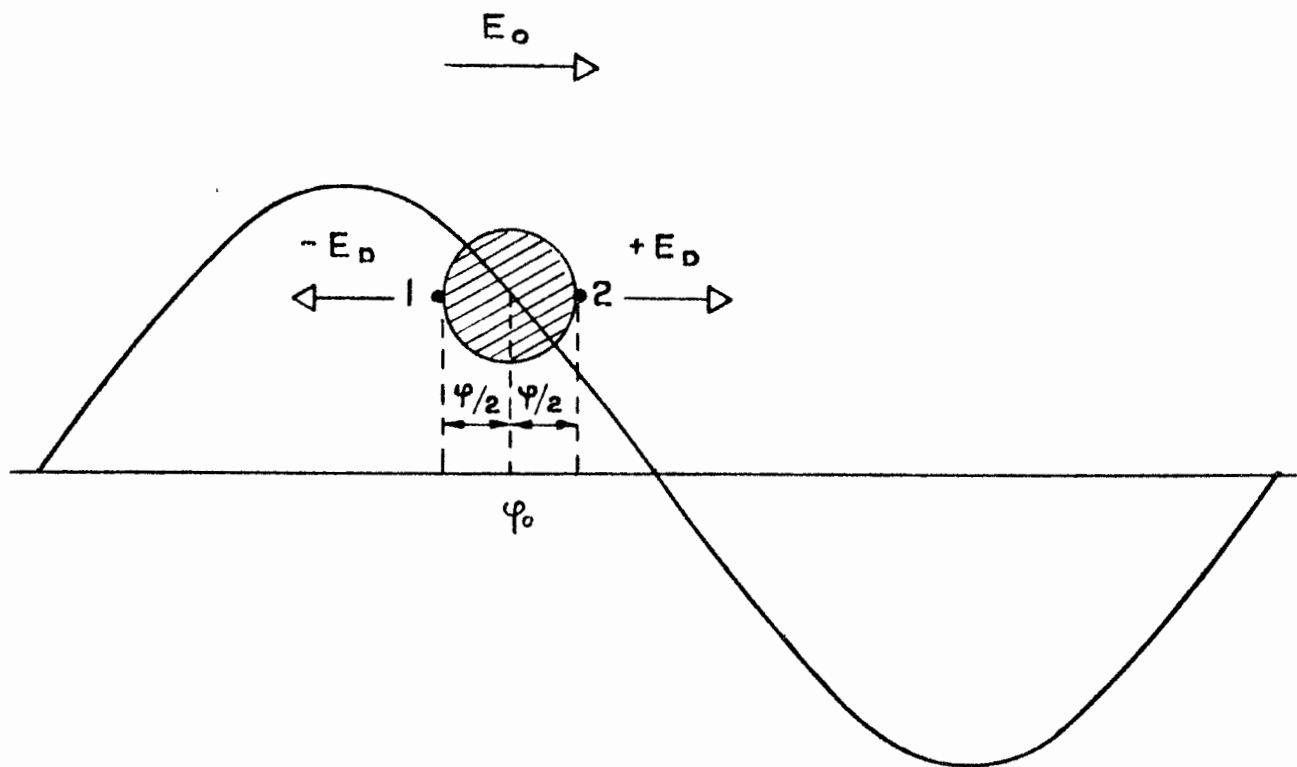$$\frac{dp_1}{dt} = - e \left[E_O \sin \left(\varphi_O + \varphi/2 \right) + E_D \right]$$

FIG. 1--Space charge forces acting on particles in a bunch.

and for particle (2):

$$\frac{dp_2}{dt} = - e \left[E_o \sin \left(\varphi_o - \varphi/2 \right) - E_D \right]$$

To calculate E_D we first would like to obtain an expression for the scalar potential for two moving point charges. Then the interaction between a particle on the surface of the bunch and the remaining charge in the bunch can be evaluated by integrating over the volume of the bunch. The Lagrangian for two charges moving with the velocity $\vec{v}$ can be written as

$$L = - mc^2 \sqrt{1 - \frac{v^2}{c^2}} - e \left(\Phi - \frac{\vec{A} \cdot \vec{v}}{c} \right)$$

where Φ and $\vec{A}$, the scalar and vector potentials established by $(Q - e)$, are the retarded potentials which are given by

$$\Phi = \int \frac{\rho(t')}{r} dV \; , \quad \vec{A} = \frac{1}{c} \int \frac{J(t')}{r} dV$$

Here the quantities ρ and J should be evaluated at the retarded times $t' = t - r/c$. However these potential expressions can be expanded in series[3] as

$$\Phi = \int \frac{\rho dV}{r} - \frac{1}{c} \frac{\partial}{\partial t} \int \rho dV + \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int r\rho dV$$

$$\vec{A} = \frac{1}{c} \int \frac{\rho \vec{v}}{r} dy$$

[3]L. Landau and E. Lifshitz, The Classical Theory of Fields (trans. M. Hamermesh), Addison-Wesley Press, Cambridge, Massachusetts, 1951; pp. 180-185.

But $\int \rho dV = Q$ is the total charge at the bunch which is constant independent of time. Therefore the second term in the expression for Φ is zero, so that

$$\Phi = \int \frac{\rho dV}{r} + \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int r\rho dV$$

For a point charge the potential expressions can be written in the following simple form

$$\Phi = \frac{Q}{r} + \frac{Q}{2c^2} \frac{\partial^2 r}{\partial t^2}$$

$$\vec{A} = \frac{Q\vec{v}}{cr}$$

After some simple calculations with these potentials, the Lagrange function is

$$L = - mc^2 \sqrt{1 - \frac{v^2}{c^2}} - \frac{eQ}{r} \left[1 - \frac{v^2 + \left(\frac{\vec{v} \cdot \vec{r}}{r} \right)^2}{2c^2} \right]$$

From this the effective potential is

$$V = \frac{1}{4\pi\epsilon_o} \iiint \frac{\rho \left\{ 1 - \frac{1}{2c^2} \left[v^2 + \left(\frac{\vec{v} \cdot \vec{r}}{r} \right)^2 \right] \right\}}{r} dV$$

After a straightforward but tedious integration (see Appendix), one can obtain the expressions for the longitudinal and transverse field for different bunch shapes and for constant and normal charge distribution in the bunch.

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For example, for a spherical bunch shape with a uniform charge distribution, the longitudinal and the transverse fields at the surface are:

$$E_L = \frac{Q}{4\pi\epsilon_O}\frac{1}{R^2}\left(1 - \frac{2v^2}{5c^2}\right) = \frac{F_L\left(Q_1 V^2\right)}{R^2}$$

$$E_T = \frac{Q}{4\pi\epsilon_O}\frac{1}{R^2}\left(1 - \frac{4}{5}\frac{v^2}{c^2}\right) = \frac{F_T\left(Q_1 v^2\right)}{R^2}$$

Using these results one can express the condition required for keeping the surface electrons in the bunch. The accelerations for the two electrons have to be the same because otherwise one of them would leave the bunch. Therefore

$$\frac{dp_1}{dt} = \frac{dp_2}{dt}$$

is the condition for stability.[*] Or one may write that, using $E_D = E_L$,

$$E_O \sin\left(\varphi_O + \varphi/2\right) + E_L = E_O \sin\left(\varphi_O - \varphi/2\right) - E_L$$

or

$$E_O\left\{\sin\varphi_O \cos\varphi/2 + \cos\varphi_O \sin\varphi/2\right\} + E_L$$
$$- E_O\left\{\sin\varphi_O \cos\varphi/2 - \cos\varphi_O \sin\varphi/2\right\} - E_L$$
$$= E_O\left\{\cos\varphi_O \sin\varphi/2 + \cos\varphi_O \sin\varphi/2\right\} + 2E_L$$
$$= 0$$

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[*] Similarly, for a given charge distribution one can write the same type of conditions for stability.

Then

$$E_L = E_o \cos \varphi_o \sin \varphi/2$$

Because usually $\varphi/2 \ll \pi$, one may write that

$$E_L \approx \frac{E_o \varphi}{2} \cos \varphi_o$$

Using the expressions for the transported charge per wavelength and for the "half bunch angle"

$$\frac{I}{c} = \frac{Q}{\lambda}$$

$$\frac{\varphi}{2} = \frac{R}{2} \frac{\pi}{v/c\lambda}$$

one gets

$$\frac{I\lambda}{4\pi\epsilon_o c} \frac{1}{R^2}\left(1 - \frac{2v^2}{5c^2}\right) \approx E_o \frac{R}{2} \frac{\pi}{v/c\lambda} \cos \varphi_o$$

and

$$I \approx \frac{2\pi^2 \epsilon_o c^2}{\lambda^2\left(1 - \frac{2v^2}{5c^2}\right)v} E_o R^3 \cos \varphi_o$$

From this, using $R = k\lambda$, one gets

$$I \lesssim \frac{2\pi^2 \epsilon_o c^2}{\left(1 - \frac{2v^2}{5c^2}\right)v} k^3 E_o \lambda = f(v,k)E_o\lambda$$

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For an electron accelerator using an optical maser the maximum field intensity at the center is given by[4]

$$E_o^2 = \frac{8\pi^3 N}{c(1 - R)} \frac{P}{S}$$

When the radial mode number N is large, the field intensity at $r = 0$ becomes large. For example, for $N = 10^4$, $(1 - R) = 5 \times 10^{-3}$ and $P/S = 10 \text{ Kw/cm}^2$. The rms electric field is 1.2×10^9 eV per meter. The wavelengths in these accelerators are the order of magnitude of $\lambda \approx 10^{-6}$ m. Then $\lambda E \approx 10^{-6} \times 10^9 = 10^3$ eV. For microwave accelerators $\lambda \approx 10^{-1}$ meter and $E_o \approx 10^7$ eV/m; then $(\lambda E)_{microwave} = 10^{-1} \times 10^7 = 10^6$ and

$$\frac{I_{laser}}{I_{microwave}} \approx \frac{10^3}{10^6} \approx 10^{-3}$$

The duty cycle (pulse length $\times$ repetition rate) can be larger in laser accelerators than in microwave linacs because the pulse length in laser accelerators is of the order of magnitude of 1 msec compared to 1-10 μsec in linacs. The repetition rate can be as high as 10 pulses per second in laser accelerators and 100 pps in linacs. Then the average current ratio is

$$\frac{I_{ave}}{\left(I_{linac}\right)_{ave}} \approx \frac{D}{D_{linac}} 10^{-3} = \frac{10^{-2}}{10^{-3}} \times 10^{-3} = 10^{-2}$$

If one considers that the length of a laser accelerator is about 1/100 of the length of a microwave accelerator, it is evident that at constant energies the product of the average current and the length is about the same. This fact demonstrates the usefulness of laser accelerator research, especially for low current use.

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[4]Shimoda, op.cit.

III. CONFINEMENT

Until now we have considered only one field component which travels with the electrons at the same velocity as that of the equilibrium particle. The effect of the radial acceleration in the waveguide structure was neglected completely. Now we would like to consider the effect of all radial components which in microwave linacs produce only alternating deflections of small amplitude but in laser accelerators may contribute to the beam dynamics significantly. Specifically, we would like to discuss the motion of an electron which is performing simple harmonic motion in an alternating electric field which has a gradient normal to the beam direction. Because the net outward force on the electron (from the centerline of the beam) is different from the net inward force, one might expect that the electrons perform a simple harmonic motion with the frequency of the field variation. Superimposed on this simple harmonic motion is the particle's average acceleration toward the point of minimum field. The equivalent dc force acting on the particle is

$$F_{DC} = -\frac{1}{4}\frac{e^2}{m\omega^2}\nabla\left(E_0^2\right)$$

which is very important, for example, in the case of a plasma breakdown where rf confinement is a factor.[5]

The motion of a charged particle in the moving coordinate system can be described by the following Hamiltonian when the radiation reaction can be neglected.[6]

$$H = c\sqrt{(p - eA)^2 + (m_0 c)^2} = \sqrt{c^2 p^2 - 2ep\,Ac^2 r\,e^2\,A^2 c^2 + m_0^2\,c^4}$$

[5]E. S. Weibel, Symposium on Magnetohydrodynamics, "The plasma in a magnetic field," (ed. R. Landshoff), Stanford University Press, Stanford, California, 1958.

[6]W.K.H. Panofsky and M. Phillips, Classical Electricity and Magnetism, Addison-Wesley Publishing Company, Cambridge, Massachusetts, 1955; p. 357.

Where $E = \dfrac{\partial A(r)}{\partial t} = E_o(r) \sin \omega t$, then

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{1}{2} \frac{-2jm \dfrac{\partial A(r)}{\partial r} c^2 + e^2 \dfrac{\partial A^2}{\partial r} c^2}{\sqrt{c^2 p^2 - 2jm A(r)c^2 + c^2 e^2 A^2(r) + m_o^2 c^4}}$$

$$= -\frac{-2jm c^2 \dfrac{\partial}{\partial r}\left(\dfrac{E_o(r)}{\omega} \sin \omega t\right) + e^2 c^2 \dfrac{\partial}{\partial r}\left(\dfrac{E_o^2(r)}{\omega^2} \sin^2 \omega t\right)}{\sqrt{c^2 p^2 - 2jc^2 m A(r) + e^2 A^2(r) c^2 + m_o^2 c^4}}$$

$$= -\frac{-2jm c^2 \dfrac{\partial}{\partial r}\left(\dfrac{E_o(r)}{\omega} \sin \omega t\right) + c^2 e^2 \dfrac{\partial}{\partial r}\left(\dfrac{E_o^2(r)}{\omega^2} \dfrac{1 - \cos 2\omega t}{2}\right)}{2 \sqrt{c^2 p^2 - 2jm A(r)c^2 + c^2 e^2 A^2(r) + m_o^2 c^4}}$$

Taking the time average of this expression one gets

$$\overline{\dot{p}}_r = -\frac{\dfrac{e^2}{\omega^2} \dfrac{\partial}{\partial r} E_o^2(r) c^2}{4H} = -\frac{1}{4} \frac{e^2}{m\omega^2} \frac{\partial}{\partial r} E_o^2(r) = \frac{-e^2}{4m_o \omega^2} \frac{\partial}{\partial r}\left[E_o^2(r)\right]\sqrt{1 - \beta^2}$$

or in the non-relativistic case

$$\overline{\dot{p}}_r = -\frac{1}{4} \frac{e^2}{m_o \omega^2} \frac{\partial}{\partial r} E_o^2(r)$$

Now one can show that the radiation reaction is negligible compared to the acting external force.

The motion equation of the electron, with the radiation term, can be written as

$$F_{ext} + \frac{e^2\,\dddot{r}}{6\pi\epsilon_o c^3} = m\,\ddot{r}$$

where $F_{ext} = eE_o(r) \sin \omega t$

In first approximation

$$\dddot{r} = \frac{e\omega}{m} E_0(r) \cos \omega t$$

and the equation of motion is

$$m \ddot{r} = eE_0(r) \sin \omega t + \frac{e^3 \omega}{6\pi\epsilon_0 c^3 m} E_0(r) \cos \omega t$$

But because

$$\frac{F_{rad}}{F_{ext}} = \frac{\frac{1}{2} \frac{e^2}{m\omega^2} \frac{\partial E_0(r)}{\partial r} E_0}{eE_0(r)} \approx 4.5 \times 10^{-3}$$

is small, using $\nu = 3 \times 10^{12} \text{ sec}^{-1}$ and $\frac{\partial E}{\partial r} = 10^{12}$ V/cm^2, one can neglect the radiation term in the motion equation. The time averaging process is justified because the damping and the frequency change are small.

This equivalent dc force in gradient fields, like the radial modes in a linac, can be used for beam confinement if the field increases with increasing r. (Alternatively, the equivalent dc force might be useful for beam acceleration when the particle beam is traveling perpendicular to the light beam where the field decreases with increasing r.) Just to estimate the equivalent field in laser accelerators corresponding to this force, one can make an order of magnitude calculation:

$$E_{DC} = - \frac{e}{4m\omega^2} \frac{d}{dr} E_0^2$$

$$= \frac{1.6 \times 10^{-19}}{4 \times 0.9 \times 10^{-30} \times 4\pi^2 \times 9 \times 10^{24}} \frac{d}{dr} E_0^2$$

$$= 9.45 \times 10^{-17} \frac{d}{dr} E_0^2$$

Now when the beam thickness is 10^{-4} m and the field is 10^9 V/m then $E_{DC} \approx 1$ Mev/m which is an order of magnitude less than the acceleration field achievable in microwave linacs. However, if the field strength can be increased by a factor of 10, then E_{DC} would be of the order of 100 Mev/m. But in this energy region the beam is relativistic and E_{DC} has to be corrected by a factor of $\sqrt{1 - \beta^2}$ which reduces the accelerating field considerably. Using multiple light beams and a drift section as shown in Fig. 2, one might use this gradient field for beam acceleration in the non-relativistic energy region. One other possible application is to release electrons from photo-cathodes by modulated laser beams[7] and to inject them into microwave tubes to demodulate the transmitted signal.

Acknowledgments

The author wishes to express his appreciation to Charles Moore for helping to perform the calculations for the field quantities.

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[7]S. E. Harris and A. E. Siegman, IRE Transactions on Electron Devices, "A proposed fm phototube for demodulating microwave-frequency-modulated light signals," July 1962.
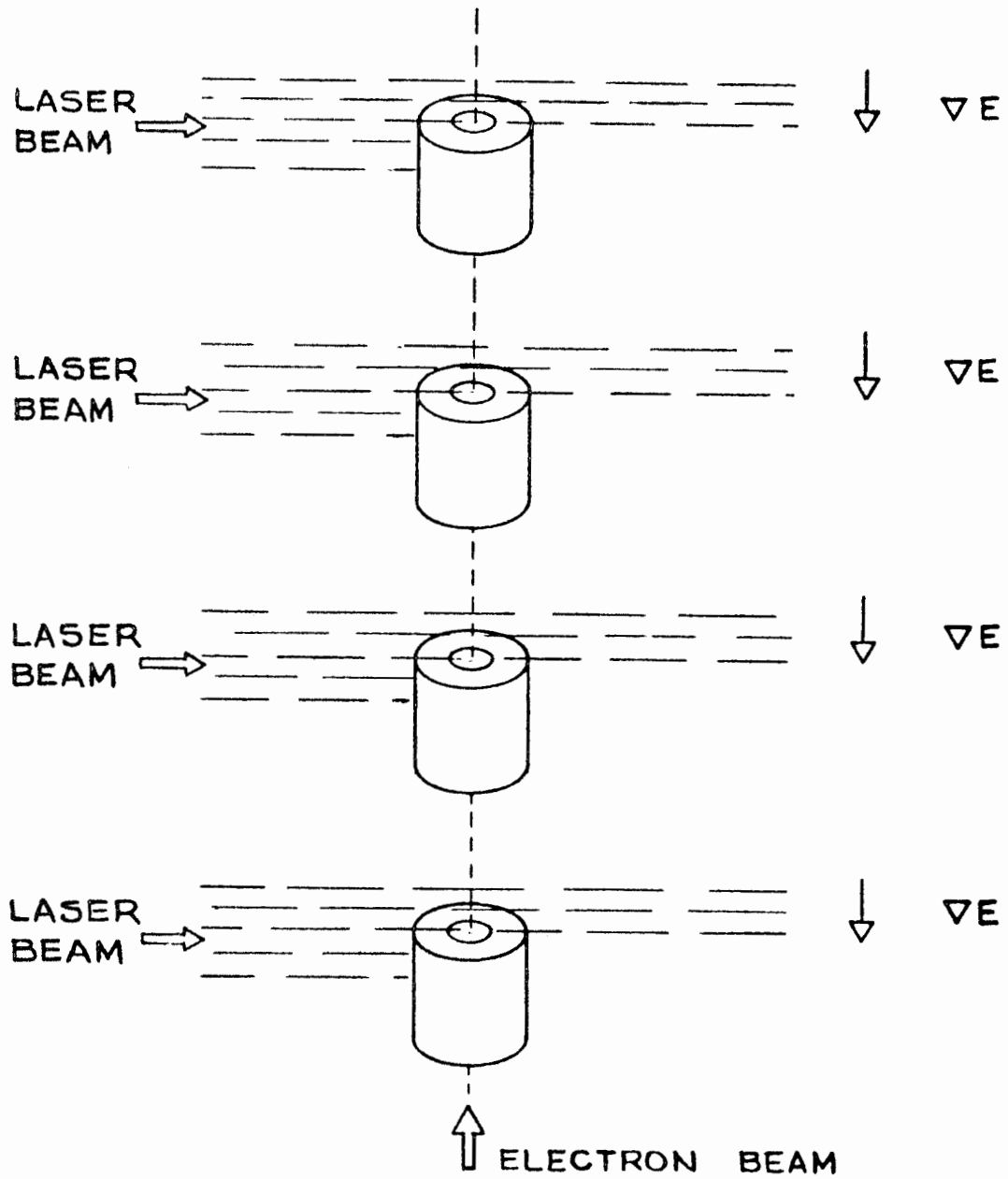
FIG. 2--Use of multiple light beams and a drift section for beam
acceleration in the non-relativistic energy region.

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APPENDIX

Calculation of the Field Quantities

The potential of the charge distribution in the bunch can be expanded very conveniently in rectangular coordinates. Terminating the expansion at second order presents the potential in a form readily applicable to a generalized distribution.

The vector notation convention employed is shown below:

$$x^T \equiv \text{transpose of } x$$

$$x^T y = y^T x \equiv \text{scalar product of } x \text{ and } y$$

$$xy^T = yx^T \equiv \text{direct product of } x \text{ and } y$$

The potential represented by

$$V = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho}{r}\left(1 - \frac{1}{2c^2}\left\{ v^2 + \left(\frac{\vec{v} \cdot \vec{r}}{r} \right)^2 \right\} \right) dV$$

can be restated more precisely as

$$V(\xi) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(x)}{|\xi - x|}\left\{ \left(1 - \frac{1}{2}\frac{v^T v}{c^2} \right) - \frac{1}{2c^2}\frac{\left(v^T [\xi - x] \right)^2}{|\xi - x|^2} \right\} dV$$

$$= \frac{1}{4\pi\epsilon_0}\left(1 - \frac{1}{2}\frac{v^T v}{c^2} \right) \iiint \frac{\rho(x)}{|\xi - x|} dV - \frac{1}{8\pi c^2 \epsilon_0} \iiint \rho(x) \frac{v^T(\xi - x)(\xi - x)^T v}{|\xi - x|^3} dV$$

$$(1)$$

With the conventions that

$$\iiint \rho\,(x)\; dV = Q$$

$$\iiint x\, x^T\, \rho(x)\; dV = R$$

any distribution may be represented to second order with the coordinate axes chosen so that R is diagonal.

To calculate the integral, two expansions are required for $|x| < |\xi|$. First,

$$
\begin{aligned}
\frac{1}{|\xi - x|} &= \left(\xi^T \xi - 2\,\xi^T x + x^T x\right)^{-\frac{1}{2}} \\[2mm]
&= \left(\xi^T \xi\right)^{-\frac{1}{2}} \left(1 - 2\,\frac{\xi^T x}{\xi^T \xi} + \frac{x^T x}{\xi^T \xi}\right)^{-\frac{1}{2}} \\[2mm]
&= \left(\xi^T \xi\right)^{-\frac{1}{2}} \left(1 + \frac{\xi^T x}{\xi^T \xi} - \frac{1}{2}\frac{x^T x}{\xi^T \xi} + \frac{3}{8}\cdot 4\,\frac{\left(\xi^T x\right)^2}{\left(\xi^T \xi\right)^2} + \cdots\right) \\[2mm]
&= \left(\xi^T \xi\right)^{-\frac{1}{2}} \left(1 + \frac{\xi^T x}{\xi^T \xi} - \frac{1}{2}\frac{1}{\xi^T \xi}\left(I - 3\,\frac{\xi \xi^T}{\xi^T \xi}\right)\cdot xx^T + \cdots\right)
\end{aligned}
$$

where the remaining terms are of the order of $\left(\dfrac{|x|}{|\xi|}\right)^3$.

Likewise

$$
\frac{1}{|\xi - x|^3} = \left(\xi^T \xi\right)^{-3/2} \left(1 + 3\,\frac{\xi^T x}{\xi^T \xi} - \frac{3}{2}\frac{1}{\xi^T \xi}\left(I - 5\,\frac{\xi \xi^T}{\xi^T \xi}\right)\cdot xx^T + \cdots\right)
$$

Now substituting these expansions in the integral (1) for the potential

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expression and leaving out the higher order terms, one obtains the following for the potential

$$V = \frac{Q}{4\pi\epsilon_0 (\xi^T \xi)^{\frac{1}{2}}} \left[\left(\left(1 - \frac{1}{2} \frac{v^T v}{c^2} \right) \left(1 - \frac{1}{2} \frac{1}{\xi^T \xi} \left\{ I - 3 \frac{\xi\xi^T}{\xi^T \xi} \right\} \right) \right) \cdot R \right.$$

$$\left. - \frac{1}{2} \frac{v^T}{c^2} \left\{ \frac{1}{\xi^T \xi} \left(I - 6 \frac{\xi\xi^T}{\xi^T \xi} \right) R + \frac{\xi\xi^T}{\xi^T \xi} \left(1 - \frac{3}{2} \frac{1}{\xi^T \xi} \right) \left(I - 5 \frac{\xi\xi^T}{\xi^T \xi} \right) \cdot R \right\} v \right]$$

As an example of the application of this expression let

$$v = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \qquad\qquad R = \begin{pmatrix} a^2 \\ & b^2 \\ & & c^2 \end{pmatrix}$$

For

$$\xi = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \qquad\qquad \frac{\xi\xi^T}{\xi^T \xi} = \begin{pmatrix} 1 \\ & 0 \\ & & 0 \end{pmatrix}$$

$$I - 3 \frac{\xi\xi^T}{\xi^T \xi} = \begin{pmatrix} -2 \\ & 1 \\ & & 1 \end{pmatrix}$$

$$I - 6 \frac{\xi\xi^T}{\xi^T \xi} = \begin{pmatrix} -5 \\ & 1 \\ & & 1 \end{pmatrix}$$

$$I - 5 \cdot \frac{\xi\xi^T}{\xi^T \xi} = \begin{pmatrix} -4 \\ & 1 \\ & & 1 \end{pmatrix}$$

Then, with these, the "longitudinal" potential is

$$V_L = \frac{Q}{4\pi\epsilon_o x}\left[\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)\left\{1 - \frac{1}{2x^2}(-2)(a^2 - b^2)\right\}\right]$$

$$- \frac{1}{2c^2}v^T\left\{\frac{1}{x^2}\begin{pmatrix} -5a^2 & & \\ & b^2 & \\ & & b^2 \end{pmatrix} + \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}\left[1 - \frac{3}{2x^2}(-2)(2a^2 - b^2)\right]\right\}v$$

$$= \frac{Q}{4\pi\epsilon_o x}\left[\left(1 - \frac{v^2}{c^2}\right) + \frac{1}{x^2}\left\{a^2\left(1 - \frac{v^2}{c^2}\right) - b^2\left(1 - \frac{2v^2}{c^2}\right)\right\}\right]$$

For $\xi = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ $\frac{\xi\xi^T}{\xi^T\xi} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}$, $R = \begin{pmatrix} a^2 & & \\ & b^2 & \\ & & b^2 \end{pmatrix}$

the "transverse" potential is

$$V_T = \frac{Q}{4\pi\epsilon_o y}\left\{\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) - \frac{1}{2y^2}\left[a^2\left(1 + \frac{v^2}{2c^2}\right) - b^2\left(1 - \frac{v^2}{2c^2}\right)\right]\right\}$$

If the charge distribution is uniform, then for sphere bunch $a = b = \frac{R}{5}$, the average of r^2, $<r^2> \doteq \frac{1}{5}R^2$ where R is the radius of the charge cloud. Then

$$V_L = \frac{Q}{4\pi\epsilon_o}\frac{1}{x}\left[\left(1 - \frac{v^2}{c^2}\right) + \frac{1}{x^2}\left(a^2\frac{v^2}{c^2}\right)\right]$$

$$= \frac{Q}{4\pi\epsilon_o}\frac{1}{x}\left[1 - \frac{v^2}{c^2}\left(1 - \frac{1}{5}\frac{R^2}{x^2}\right)\right]$$

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and

$$V_T = \frac{Q}{4\pi\epsilon_o} \; \frac{1}{y} \; \left[\left(1 - \tfrac{1}{2} \frac{v^2}{c^2} \right) - \frac{1}{2y^2} \left(a^2 \frac{v^2}{c^2} \right) \right]$$

$$= \frac{Q}{4\pi\epsilon_o} \; \frac{1}{y} \; \left[1 - \tfrac{1}{2} \frac{v^2}{c^2} \left(1 + \frac{R^2}{5y^2} \right) \right]$$

From these, the field quantities are:

$$E_L = - \frac{\partial V_L}{\partial x} = \frac{Q}{4\pi\epsilon_o} \frac{1}{x^2} \left\{ 1 - \frac{v^2}{c^2} \left(1 - \frac{3}{5} \frac{R^2}{x^2} \right) \right\}$$

$$E_T = - \frac{\partial V_T}{\partial y} = \frac{Q}{4\pi\epsilon_o} \frac{1}{y^2} \left\{ 1 - \frac{v^2}{2c^2} \; 1 + \frac{3}{5} \frac{R^2}{y^2} \right\}$$

and the fields at the surface $x = y = R$ are:

$$E_L = \frac{Q}{4\pi\epsilon_o} \frac{1}{R^2} \left(1 - \frac{2}{5} \frac{v^2}{c^2} \right)$$

$$E_T = \frac{Q}{4\pi\epsilon_o} \frac{1}{R^2} \left(1 - \frac{4}{5} \frac{v^2}{c^2} \right)$$

Similarly one can calculate the fields in the x and y directions for any charge configuration. For a sphere, an ellipsoid and a cylinder (supposing normal and uniform charge distribution in the bunch), the values for a and b are as shown in Fig. A.1.
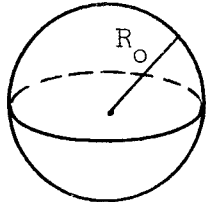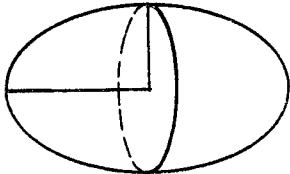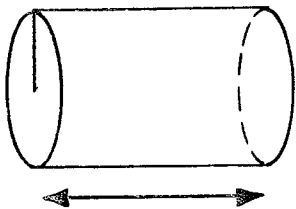
		Distribution
Sphere	$a = b = R_o$	Normal
	$a = b = \dfrac{R_o}{5}$	Uniform
Ellipsoid	$a = \alpha \qquad b = \beta$	Normal
	$a = \dfrac{\alpha}{5} \qquad b = \dfrac{\beta}{5}$	Uniform
Cylindrical	$a = \ell \qquad b = R_o$	Normal
	$a = \tfrac{1}{3}\ell \qquad b = \tfrac{1}{2} R_o$	Uniform

FIG. A.1--Values of a and b for a sphere, an ellipsoid, and a cylinder.