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MISALIGNMENT AND QUADRUPOLE ERROR EFFECTS IN A FOCUSING SYSTEM FOR THE TWO-MILE ACCELERATOR

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by

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I. INTRODUCTION

A. OBJECT AND SCOPE

In a previous report[1] the optical properties of possible focusing systems suitable for the two-mile accelerator were discussed. Ideal conditions, specifically the absence of misalignments and other perturbing effects, were assumed.

The present report will discuss some of these perturbing effects, namely various kinds of misalignments and errors in the strengths and spacing of the quadrupoles. Numerical examples will be given only for a particular focusing system (See Section B below) but the general formulation will be suitable for other types of systems.

It will be convenient to treat each of the various perturbations independently, as if all others were absent. This is reasonable because the equations of motion are essentially linear so that superposition applies.

B. RELEVANT PROPERTIES OF THE FOCUSING SYSTEM

Figure 1-1 illustrates the type of focusing system which will be assumed in numerical examples.

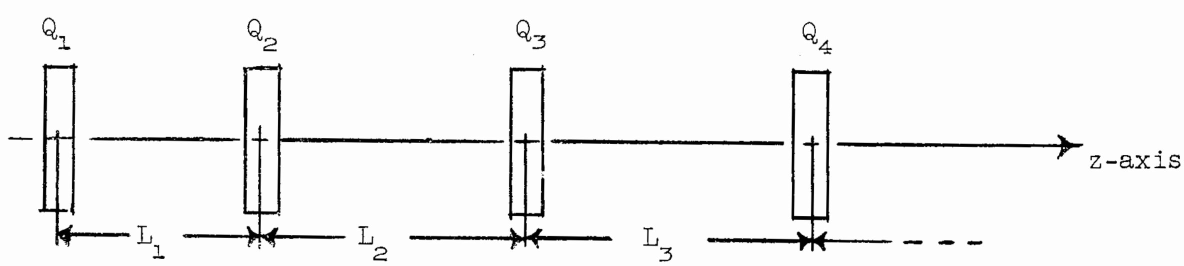

Figure 1-1--System of thin-lens, uniformly spaced quadrupoles.
(See text for explanation.)

As in Ref. 1, it will be convenient to define a quadrupole strength, Q, and a spacing parameter, ℓ, as follows:

$$Q_j \equiv \int_{quad} \left(\frac{\partial B_y}{\partial x}\right)_j dz \tag{1-1}$$

$$\ell_j \equiv \frac{1}{E} \log \left(1 + \frac{EL_j}{\gamma_j}\right) \tag{1-2}$$

where $\left(\frac{\partial B_y}{\partial x}\right)_j$ is the gradient in the j-th lens;

E is the accelerating field parameter (assumed constant);

L_j is the spacing between Q_j and Q_{j+1};

γ_j is the relativistic energy of the electrons.

(Unless otherwise noted, it will be assumed that lengths are measured in cm; that B, E, and Q in units of $mc^2/e/cm$ which for electrons is equivalent to 1703 gauss or 0.511 Mv/cm; and γ in units of rest mass = mc^2 = 0.511 Mev.)

The system in Fig. 1-1 is periodic as a function of the number of quadrupole pairs if

and
$$\left.\begin{aligned} Q_j &= Q_o(-1)^j \\ \ell_j &= \ell_1 = \text{constant} \end{aligned}\right\} \quad j = 1,2,3....$$

The second condition may be seen from Eq. (1-2) to imply an exponential increase in spacing as a function of j, if the acceleration E is constant. The situation of constant spacing ($L_j = L_1$) may be treated, as in Ref. 1, as an adiabatic deviation from strict periodicity provided $EL \ll \gamma$ (e. g., essentially constant energy).

Some relevant conclusions of Ref. 1 are:

1) In order to take full advantage of the phase-space admittance of the end-station transport system* it would be desirable to have the strength, Q, in the range

$$0.6 \lesssim |Q| \lesssim 2.4 \qquad \text{(1000 to 4000 gauss)}$$

2) A <u>low energy cut-off</u> is defined by

$$(|Q|\ell)_{c.o.} = 2$$

which is equivalent to**

$$\gamma_{c.o.} \approx \frac{|Q|L}{2} \tag{1-3}$$

3) The condition for minimum number of quadrupoles per unit length (for a given admittance) is

$$|Q|\ell = \sqrt{5} - 1 \approx 1.24$$

This may be used to define a practical low-energy band limit of

$$\gamma_{min} \gtrsim \frac{|Q|L}{1.24} \tag{1-4}$$

It will be assumed, additionally, that the fixed quadrupole spacing is 40 feet, corresponding to the maximum rigid length of accelerator

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*This is particularly important in the case of the positron beam.

**The approximation

$$\ell = \frac{1}{E} \log \left(1 + \frac{EL}{\gamma}\right) \approx \frac{L}{\gamma}$$

will usually apply because $EL \ll \gamma$ for the design assumed here.

support beam. Then if the system is to transport multiple beams (without pulsing the quadrupoles), Eq. (1-4) defines a maximum quadrupole strength in terms of the minimum beam energy. Taking $\gamma_{min} = 2 \times 10^3$ (1 Bev) and $L = 40$ ft ≈ 1200 cm,

$$|Q| \lessapprox \frac{1.24 \cdot 2 \cdot 10^3}{1200} \approx 2 \ mc^2/e/cm$$

$$\approx 3.4 \text{ kilogauss}$$

It is also of some interest to calculate the trajectory wavelength defined by

$$\lambda_t = \frac{2\pi}{\theta} \ . \ 2L$$

where for a system of equally spaced quadrupoles[1]

$$\cos \theta = 1 - \tfrac{1}{2} Q^2 \ell^2$$

or

$$\theta \approx Q\ell$$

(This approximation is within 10% even for $Q\ell = 1.24$.) Thus

$$\lambda_t \approx \frac{4\pi L}{Q\ell} \approx \frac{4\pi\gamma}{Q} \qquad (1\text{-}5)$$

For 1 Bev and $Q = 2$ (3.4 kilogauss);

$$\lambda_t \approx 400 \text{ ft}$$

For 10 Bev and $Q = 2$;

$$\lambda_t \approx 4{,}000 \text{ ft}$$

C. MODEL FOR MISALIGNMENT CALCULATIONS

The support structure of the machine presumably will consist of rigid segments each of nominal 40-foot length, each supported at one end and each pair linked together in such a way that angular bends, but no relative lateral displacements, may occur at the joints. The accelerator and quadrupoles will be prealigned on these 40-foot segments as accurately as possible--probably with a precision on the order of 0.010 inches.

Figure 1-2 illustrates the sort of misalignments associated with the 40-foot rigid support period.

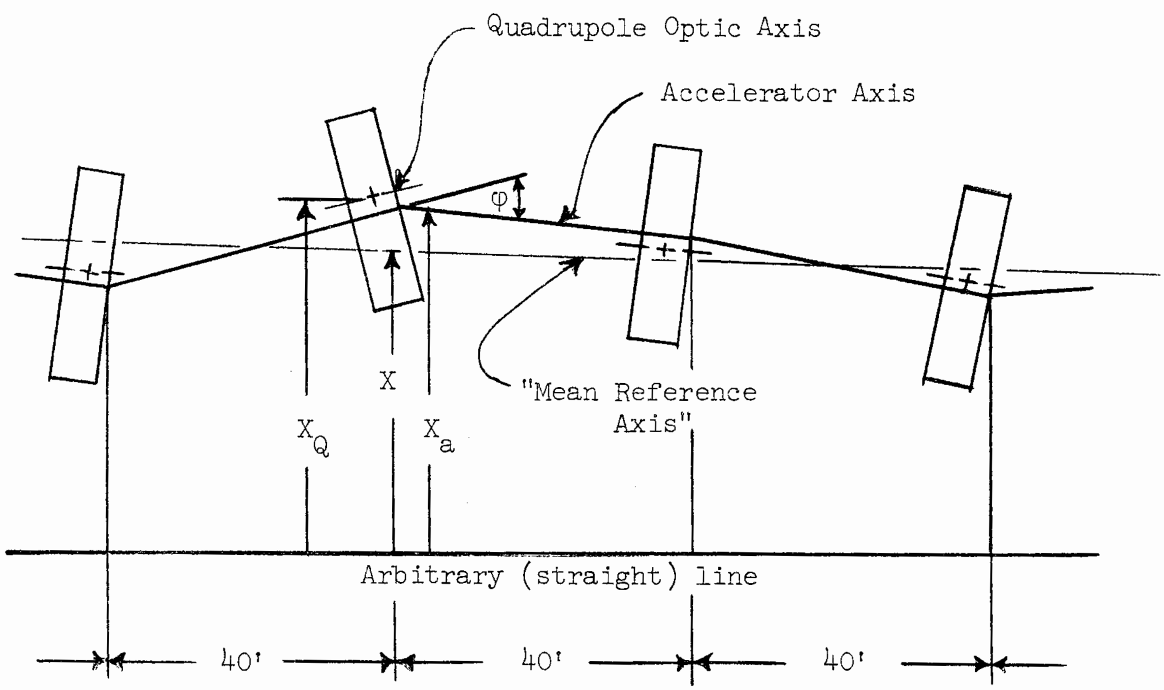

Figure 1-2--Illustrating alignment errors associated with the 40-foot support period. (See text for explanation.)

The "Mean Reference Axis" (hereafter referred to as MRA) in Fig. 1-2 is supposed to be a smooth curve which represents the local average position of the accelerator axis. The precise meaning of "local" in this definition depends on the particular problem at hand; usually it will be sufficient to say that a Fourier analysis of the MRA contains essentially no terms of wavelength shorter than the longest trajectory wavelengths which may be expected in the region under consideration. From a practical point of view, the MRA should be essentially straight over the length of a few sectors.

The MRA will be a useful reference axis provided that the transverse misalignments (relative to the MRA) may be assumed small compared to the accelerator aperture.

The following types of errors will be considered in some detail:

1) Displacements of the accelerator between the 40-foot support points.

2) Short-range misalignments; in particular, random displacements of the support points from the MRA.

3) Intermediate- and long-range misalignments;

  a) Random displacements of the support system which are correlated over a finite range.

  b) Isolated discontinuities such as bends or displaced regions.

  c) Constant curvature.

4) Rotational errors, in which the quadrupoles have small random rotations about the reference axis.*

5) Errors and periodic variations in the quadrupole strengths and spacing.

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*Rotational errors in which the quadrupoles are rotated about a transverse axis have no first-order effect in the thin-lens approximation and consequently will be considered negligible.

D. SUMMARY AND CONCLUSIONS

The results of succeeding sections may be summarized briefly here:*

1) Random, independent misalignments of the quadrupole optic axes relative to the mean reference axis should be not greater than about 0.009 inch rms in order to operate with not more than two (magnetic dipole) steering periods per sector. (Sect. II.B.)

2) The alignment of the quadrupole optic axes should be stable, to within about 0.001 inch rms relative to the mean reference axis, against vibrations and other short-term fluctuations. (Sect. II.B.)

3) For random angular bends of short or intermediate correlation range (i. e., strong positive correlations between the angular bends of adjacent quadrupoles, but negligible correlation at distances on the order of orbit wavelengths), the summation over all bends, $\sqrt{\Phi_1^2 + \Phi_2^2 + \ldots}$, should not be greater than about 10^{-5} radian. (Sect. III.A.) Some implications of observed site movements are discussed, and it is estimated that some sort of realignment might be necessary at intervals shorter than one month.

4) For isolated "large" bends the sum $|\varphi_1| + |\varphi_2| + \ldots$ should not exceed $\approx 10^{-5}$ radian. (Sect. III.B.)

5) If the misalignment has a component of constant curvature, the maximum misalignment (relative to a chord through the ends of the machine) should not exceed ≈ 1 cm. (Sect. III.C.)

6) In terms of a harmonic analysis of the transverse misalignments, (Sect. IV.), the most important error components are in the wavelength bands of > 400 feet (coherent with orbit oscillations), and in the vicinity of 80 feet (the $\nu = \pm 1$ components); and the latter probably couple more strongly with the beam deflection.

7) Random axial rotations of the quadrupoles (Sect. V.) should not exceed about 0.2 degrees rms.

*It should be emphasized that these results apply only to the particular focusing system (40-foot spacing of equal strength quadrupoles) on which the numerical examples are based.

8) Random errors in quadrupole strength of about 0.3% may be tolerated. (Sect. VI.A.)

9) Random errors in longitudinal position of the quadrupoles of $\approx$ 1.9 inches may be tolerated. (Sect. VI.A.)

10) A Sector "superperiod" associated with an extra length of $\approx$ 10 feet every 320 feet would introduce a stopband of about 6% relative width, at a beam energy of (typically) about 1.6 Bev. (Sect. VI.B.)

Because of the extremely difficult tolerances — in particular on the short-range alignment — imposed by the quadrupole system at 40-foot spacing, it appears desirable to investigate other types of focusing systems. A design consisting of closely grouped multiplets (doublets or triplets) at Sector intervals appears promising; although the power requirement could be much greater, the short-range alignment problem should be much easier. Studies of such systems will be reported in the near future.

It would also be extremely desirable to undertake a computer study of the machine focusing problems. The computer program should be devised in such a way that it could handle non-random perturbations (e. g., observed misalignments from site surveys) and also be capable of playing games of steering and realignment.

II. SHORT-RANGE MISALIGNMENTS

A. DISPLACEMENTS BETWEEN SUPPORT POINTS

The accelerator sections between the support points will be initially aligned to high precision on the rigid support beams. However, it may happen that later realignments of the main support system will unbalance the waveguide and water supply leads, resulting in small elastic deflections.[2]

This situation has been considered in a previous report,[3] where it was recommended that such deflections should not exceed 0.020 inch, in order to keep the geometric reduction of the effective radial aperture to approximately 5%. The dynamic effect, arising from the accelerating field in the displaced sections being not exactly parallel to the mean axis, was shown to be negligible.

B. MISALIGNMENTS ASSOCIATED WITH THE 40-FOOT SUPPORT PERIOD

1. General Formulation

In general one might associate an angular bend, φ_j, and a quadrupole displacement, ϵ_j, with the j-th support point. Each of these effects is equivalent to the injection of a transverse momentum;

$$\delta p_j = - Q_j \epsilon_j + \gamma_j \varphi_j \tag{2-1}$$

where, as in Section I.2, Q is a measure of quadrupole strength and γ is the relativistic energy (longitudinal momentum) of the electron. One may consider two alternate points of view in calculating the electron motions (see Fig. 1-2):

a) Motions relative to Mean Reference Axis. In this case we take

$$\varphi_j = 0$$

$$\epsilon_j = X_{Q,j} - X_j$$

b) Motions relative to accelerator axis. In this case

$$\varphi_j = \frac{1}{L}\left(X_{a,j+1} - 2X_{a,j} + X_{a,j-1}\right)$$

$$\epsilon_j = X_{Q,j} - X_{a,j}$$

where, relative to some arbitrary straight line,

$X_{Q,j}$ is the coordinate of the quadrupole axis;

$X_{a,j}$ is the coordinate of the accelerator axis;

X_j is the coordinate of the MRA; and

L is the spacing (40 feet, nominal).

In either case the transformation of the electron coordinates (x,p) over one section of a periodic focusing system now will have the general form of a linear inhomogeneous transformation;

$$x_n = a_n + a_{11}x_{n-1} + a_{12}p_{n-1}$$

$$p_n = b_n + a_{21}x_{n-1} + a_{22}p_{n-1} \tag{2-2a}$$

where (a_{ij}) is the transformation for the unperturbed system and a_n, b_n are the perturbations depending on the accidental misalignments, but independent of the coordinates x and p.

In matrix form, the transformation is

$$\mathbf{x}_n = \mathbf{a}_n + \mathbf{A}_n\mathbf{x}_{n-1} \tag{2-2b}$$

where

$$\mathbf{x}_n \equiv \begin{pmatrix} x_n \\ p_n \end{pmatrix}$$

$$\mathbf{a}_n \equiv \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

and

$$\mathbf{A}_n \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

An important property of the transformation $\mathbf{A}_n$ is that it has unit determinant; $|\mathbf{A}_n| = a_{11}a_{22} - a_{12}a_{21} = 1$.

The general solution for the transformation over n cascaded sections is

$$\mathbf{x}_n = \mathbf{X}_n + \boldsymbol{\xi}_n \tag{2-3}$$

where $\mathbf{X}_n$ is a solution of the <u>homogeneous equation</u> (corresponding to $\mathbf{a}_n = 0$) and is given by

$$\mathbf{X}_n \equiv \begin{pmatrix} X_n \\ P_{xn} \end{pmatrix} \equiv \mathbf{A}(n|0)\mathbf{X}_o \tag{2-3a}$$

$\boldsymbol{\xi}_n$, which is a <u>particular solution</u> of the <u>inhomogeneous equation</u>, is given by

$$\boldsymbol{\xi}_n \equiv \begin{pmatrix} \xi_n \\ \rho_n \end{pmatrix} = \sum_{m=1}^{n} \mathbf{A}(n|m)\mathbf{a}_m \tag{2-3b}$$

and $\mathbf{A}(n|m)$ is the transformation from the end of the m-th section to the end of the n-th section, in the homogeneous system.

$\mathbf{A}(n|m)$ has the properties

$$\mathbf{A}^{-1}(n|m) = \mathbf{A}(m|n) \tag{2-4a}$$

$$\mathbf{A}(n|m) = \mathbf{A}_n\mathbf{A}(n-1|m) \tag{2-4b}$$

$$\mathbf{A}(n|m) = \mathbf{A}(n|m-1)\mathbf{A}_m^{-1} \tag{2-4c}$$

$$\mathbf{A}(n|m) = \mathbf{A}_n \mathbf{A}_{n-1} \cdot \cdot \cdot \mathbf{A}_{m+1} \qquad (\text{if} \quad n > m) \qquad (2\text{-}4d)$$

$$\mathbf{A}(n|m) = \mathbf{A}_{n+1}^{-1} \mathbf{A}_{n+2}^{-1} \cdot \cdot \cdot \mathbf{A}_m^{-1} \qquad (\text{if} \quad n < m) \qquad (2\text{-}4e)$$

and of course

$$\mathbf{A}(n|n) = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{A}(n|n-1) = \mathbf{A}_n$$

The particular solution $\boldsymbol{\xi}_n$ is of some interest in the present case since it represents the perturbation of the trajectories by the misalignments.

a. Periodic system. In the event that $\mathbf{A}_n$ is independent of n, the homogeneous transformation is given by[4,5]

$$\mathbf{A}(n|m) = \mathbf{I} \cos (n-m)\theta + \begin{pmatrix} \frac{1}{2}\left(a_{11} - a_{22}\right) & a_{12} \\ a_{21} & -\frac{1}{2}\left(a_{11} - a_{22}\right) \end{pmatrix} \frac{\sin(n-m)\theta}{\sin \theta} \qquad (2\text{-}5)$$

where θ is a parameter defined by

$$\cos \theta = \frac{1}{2}\left(a_{11} + a_{22}\right)$$

In such a system it is possible to define a (complex) eigenvector, w_n, such that[5]

$$w_n = w_{n-1} e^{i\theta} = w_m e^{i(n-m)\theta} \qquad (2\text{-}6)$$

A suitable representation of w_n for the present calculations is

$$w_n = X_n - \frac{i}{\sin\theta}\left[\frac{1}{2}\left(a_{11} - a_{22}\right)X_n + a_{12}P_{xn}\right] \tag{2-7}$$

which may be shown to satisfy Eqs. (2-6) and (2-5) by direct substitution.

Equation (2-7) may be represented as a vector contraction,

$$w_n = \left(\boldsymbol{\omega}, \mathbf{X}_n\right) \tag{2-7a}$$

where

$$\boldsymbol{\omega} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{i}{\sin\theta}\begin{pmatrix} \frac{1}{2}\left(a_{11} - a_{22}\right) \\ a_{12} \end{pmatrix} \tag{2-8}$$

If we make the definitions

$$\begin{aligned} v_n &\equiv \left(\boldsymbol{\omega}, \boldsymbol{\xi}_n\right) \\ \alpha_n &\equiv \left(\boldsymbol{\omega}, \mathbf{a}_n\right) \end{aligned} \tag{2-9}$$

then the equivalent of Eq. (2-3b) turns out to be

$$v_n = \sum_{m=1}^{n} \alpha_m e^{i(n-m)\theta} \tag{2-10}$$

Note that the displacement of the beam by the perturbations is

$$\mathrm{Re}\left(v_n\right) = \xi_n .$$

b. <u>Adiabatic deviation from periodicity.</u> If the basic matrices $\mathbf{A}_n$ vary slowly as a function of n, the amplitude of the homogeneous solutions varies as[6]

$$\left|X_n\right| \propto \sqrt{\frac{a_{12}}{\sin\theta}}$$

For the present discussion it will be a sufficiently good approximation to take

$$w_n \approx \left[\left(\frac{a_{12}}{\sin\theta}\right)_n \left(\frac{\sin\theta}{a_{12}}\right)_o\right]^{\frac{1}{2}} w_o e^{i\mu_n} \tag{2-11}$$

where

$$\mu_n = \sum_{m=1}^{n} \theta_m \tag{2-12}$$

and $w_n = \left(\boldsymbol{\omega}_n, \mathbf{X}_n\right)$ as before, except that now $\boldsymbol{\omega}_n$ is based on the local values of the parameters and consequently is a slowly varying function of n.

In the same approximation the trajectory perturbation corresponding to Eq. (2-10) is

$$v_n = \left(\frac{a_{12}}{\sin\theta}\right)_n^{\frac{1}{2}} \sum_{m=1}^{n} \left(\frac{\sin\theta}{a_{12}}\right)_m^{\frac{1}{2}} \alpha_m e^{i\left(\mu_n - \mu_m\right)} \tag{2-13}$$

The adiabatic invariant function,

$$\begin{aligned} \Upsilon &= \left(\frac{\sin\theta}{a_{12}}\right)_n \left|w_n\right|^2 \\ &= \frac{1}{\sin\theta_n}\left\{- a_{21} X^2 + \left(a_{11} - a_{22}\right) X P_x + a_{12} P_x^2\right\}_n \end{aligned} \tag{2-14}$$

is useful because the maximum amplitudes of X_n and P_{xn} in the vicinity of n are given by[5]

$$\left.\begin{aligned} (X_n)^2_{max} &= \left(\frac{a_{12}}{\sin\theta}\right)_n \Upsilon = |w_n|^2 \\ (P_n)^2_{max} &= \left(\frac{-a_{21}}{\sin\theta}\right)_n \Upsilon \end{aligned}\right\} \qquad (2\text{-}15)$$

If the adiabatic invariant is written in the form

$$= \frac{1}{\sin\theta_n}\left\{-a_{21}(x-\xi)^2 + (a_{11} - a_{22})\,(x-\xi)\,(p-\rho) + a_{12}(p-\rho)^2\right\}_n \qquad (2\text{-}14a)$$

then it may be seen that the characteristic admittance function[1] is the same as in the unperturbed case, only displaced by the particular solution ξ_n; see Fig. 2-1.

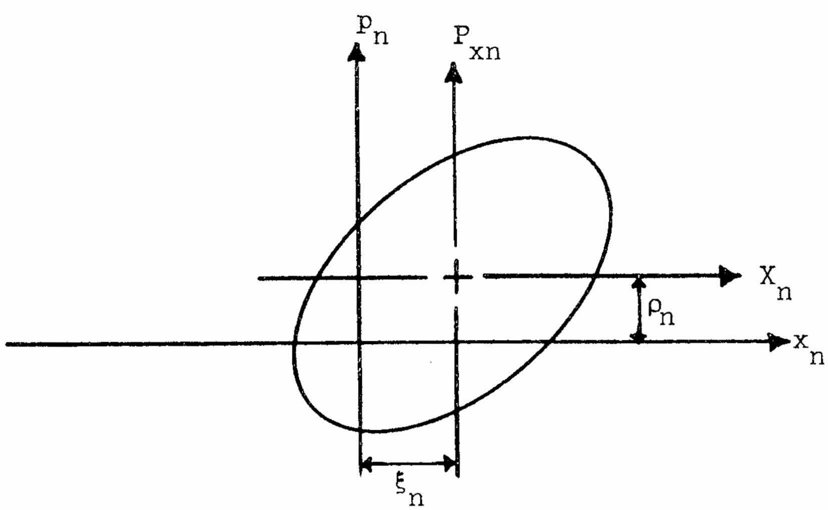

FIG. 2-1. Displacement of the characteristic admittance function in perturbed system.

Thus the effect of the perturbation is the reduction of the effective aperture of the system by $|\xi|_{max}$

2. Independent Random Errors

If the errors are random and uncorrelated, then we may write

$$\overline{\alpha_n \alpha_m} = \overline{\alpha_n^2}\, \delta_{mn}$$

$$\overline{\alpha_n \alpha_m^*} = \overline{|\alpha_n|^2}\, \delta_{mn}$$

where the superior bar denotes expectation value. In this case the expectation value of the trajectory displacement is

$$\overline{\xi_n^2} = \tfrac{1}{4}\overline{\left(v_n + v_n^*\right)^2} = \tfrac{1}{2}\left[\overline{\left|v_n\right|^2} + \mathrm{Re}\left(\overline{v_n^2}\right)\right]$$

$$\approx \tfrac{1}{2}\left(\frac{a_{12}}{\sin\theta}\right)_n \sum_{m=1}^{n} \left(\frac{\sin\theta}{a_{12}}\right)_m \left\{\overline{\left|\alpha_m\right|^2} + \mathrm{Re}\left(\overline{\alpha_m^2}\, e^{2i\left[\mu_n - \mu_m\right]}\right)\right\} \tag{2-16}$$

If μ_n ($= n\theta$ if the parameters are constant) is fairly large, the oscillatory term in Eq. (2-16) can make only a small contribution in the summation; hence a fair estimate of $\overline{\xi_n^2}$ is given by

$$\overline{\xi_n^2} \approx \tfrac{1}{2}\left(\frac{a_{12}}{\sin\theta}\right)_n \sum_{m=1}^{n} \left(\frac{\sin\theta}{a_{12}}\right)_m \overline{\left|\alpha_m\right|^2} \tag{2-16a}$$

An alternative estimate of the perturbation of the trajectory amplitude is given in terms of the adiabatic invariant function. The perturbation in the m-th focusing section may be considered as injecting an increment of transverse phase space given by

$$\delta w_m = \alpha_m$$

or

$$\delta \bar{\Upsilon}_m \approx \left(\frac{\sin\theta}{a_{12}}\right)_m \overline{|\alpha_m|^2}$$

Since the perturbations α_m have been assumed to be random and independent, the total increment of Υ after n sections is given by the summation,

$$\Delta \bar{\Upsilon}_n = \sum_{m=1}^{n} \delta \bar{\Upsilon}_m \approx \sum_{m=1}^{n} \left(\frac{\sin\theta}{a_{12}}\right)_m \overline{|\alpha_m|^2} \qquad (2\text{-}17)$$

and the expected amplitude of the trajectory perturbation is, by Eq. (2-15),

$$\overline{\left(\xi_n\right)^2_{max}} \approx \left(\frac{a_{12}}{\sin\theta}\right)_n \sum_{m=1}^{n} \left(\frac{\sin\theta}{a_{12}}\right)_m \overline{|\alpha_m|^2} \qquad (2\text{-}18)$$

Comparing Eqs. (2-16a) and (2-18) it will be noticed that

$$\overline{\left(\xi_n\right)^2_{max}} \approx 2\,\overline{\xi_n^2}$$

This simply means that the particles have accumulated transverse momentum as well as displacement, and have not quite reached their maximum expected amplitude at the n-th reference plane.

3. System of Thin-Lens Equally-Spaced Quadrupoles

Now consider the quadrupole system mentioned in Section I.B. Let the basic focusing period be represented by Fig. 2-2.
The ϵ's and φ's are the linear and angular misalignments, respectively. The quadrupole strength Q is as defined in Section I.B. To the approximation that variation in energy over one section may be ignored, the spacing parameter is

$$\ell \approx \frac{L}{\gamma}$$

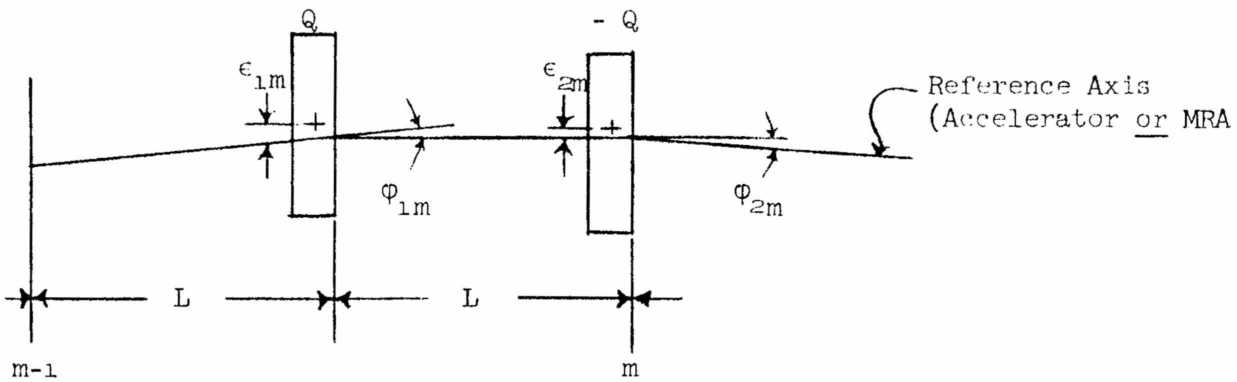

FIG. 2-2. Basic focusing section with misalignment errors. (See text.)

The transformation from m - 1 to m is

$$\mathbf{A}_m = \begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + Q\ell & 2\ell(1 + \frac{1}{2}Q\ell) \\ -Q^2\ell & 1 - Q\ell - Q^2\ell^2 \end{pmatrix} \quad (2\text{-}19)$$

(Q and ℓ are assumed tacitly to be functions of m.) The perturbation vector is

$$\mathbf{a}_m = \begin{pmatrix} 0 \\ b_{2m} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ b_{1m} \end{pmatrix} = \begin{pmatrix} \ell b_{1m} \\ \left[b_{2m} + (1 - Q\ell) b_{1m}\right] \end{pmatrix} \quad (2\text{-}20)$$

where

$$\left.\begin{aligned} b_{1m} &= -Q\epsilon_{1m} + \gamma_m \varphi_{1m} \\ b_{2m} &= Q\epsilon_{2m} + \gamma_m \varphi_{2m} \end{aligned}\right\} \quad (2\text{-}20a)$$

It follows from Eq. (2-19) that the operator $\boldsymbol{\omega}$ (Eq. 2-8) is

$$\boldsymbol{\omega} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{2i}{Q} \left(\frac{1 + \frac{1}{2}Q\ell}{1 - \frac{1}{2}Q\ell} \right)^{\frac{1}{2}} \begin{pmatrix} \frac{1}{2}Q \\ 1 \end{pmatrix}$$

from which the complex perturbation vector α_m is given by

$$\alpha_m = \left(\boldsymbol{\omega}, \mathbf{a}_m\right) = \ell b_{1m} - \frac{2i}{Q} \left(\frac{1 + \frac{1}{2}Q\ell}{1 - \frac{1}{2}Q\ell} \right)^{\frac{1}{2}} \left[\left(1 - \tfrac{1}{2}Q\ell\right) b_{1m} + b_{2m} \right] \qquad (2\text{-}21)$$

If the errors are random and independent, then application of Eq. (2-16a) gives, after considerable algebra,

$$\overline{\xi_n^2} \approx \left(\frac{4}{Q} \sqrt{\frac{1 + \frac{1}{2}Q\ell}{1 - \frac{1}{2}Q\ell}} \right)_n \sum_{m=1}^{n} \frac{\overline{b_m^2}}{\left(Q \sqrt{1 - \frac{1}{4} Q^2 \ell^2}\right)_m} \qquad (2\text{-}22)$$

where

$$\overline{b_{1m}^2} = \overline{b_{2m}^2} \equiv \overline{b_m^2}$$

4. Example: Random Linear Misalignment

Suppose that the accelerator support points have independent random displacements from a straight line. (See Fig. 2-3.) This is the sort of situation which might exist after an optical or mechnical alignment of the machine. It would apply also to any short region over which one can define an MRA having negligible curvature. It will be assumed 1) that the errors ϵ_{1m}, ϵ_{2m} have an rms expectation value which is small compared to the accelerator aperture and 2) that the quadrupoles are accurately prealigned with respect to the accelerator axis so that the zigzag line in Fig. 2-3 may be considered as coinciding with the quadrupole optic axes.

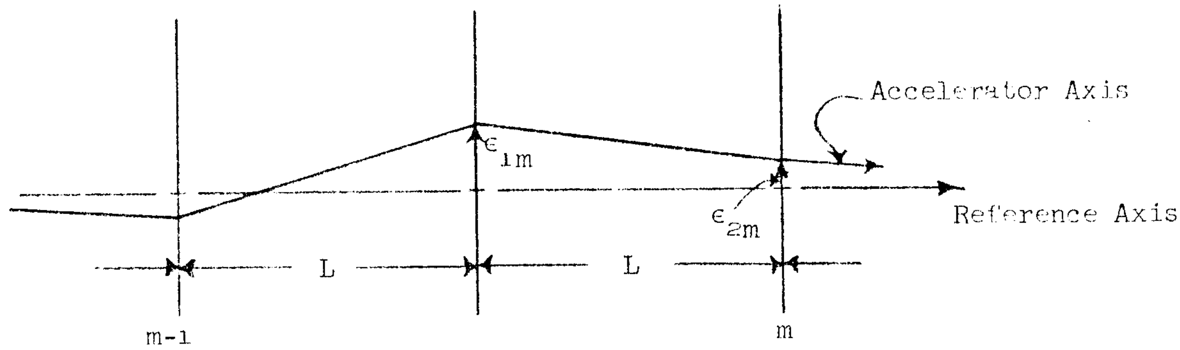

FIG. 2-3. Random linear misalignments

Since the reference axis is straight (at least locally), the φ's in Eq. (2-20a) are zero, and

$$b_{1m} = - Q \epsilon_{1m}$$

$$b_{2m} = Q \epsilon_{2m}$$

Assuming that the errors are everywhere equivalent (i.e., $\overline{\epsilon^2_{1m}} = \overline{\epsilon^2_{2m}} = \overline{\epsilon^2}$), we obtain from Eq. (2-22)

$$\overline{\xi_n^2} \approx \overline{\epsilon^2} \left(\frac{4}{Q} \sqrt{\frac{1 + \frac{1}{2}Q\ell}{1 - \frac{1}{2}Q\ell}} \right)_n \sum_{m=1}^{n} \left(\frac{Q}{\sqrt{1-\frac{1}{4}Q^2\ell^2}} \right)_m \tag{2-23}$$

If Q and ℓ are constant (constant energy, constant quadrupole strength and spacing), the result is

$$\overline{\xi_n^2} \approx \frac{4n \overline{\epsilon^2}}{1 - \frac{1}{2}Q\ell} \tag{2-23a}$$

If Q and L are constant but the energy increases linearly, then the sum may be approximated by an integral (taking $\ell_m \approx L/\gamma_m$, $\gamma_m = \gamma_o + E z_m$, and $2L \approx dz$);

$$\overline{\xi_n^2} \approx \frac{2\,\overline{\epsilon^2}}{E\,L} \sqrt{\frac{1 + \frac{1}{2}Q\ell_n}{1 - \frac{1}{2}Q\ell_n}} \int_{\gamma_o}^{\gamma_n} \frac{\gamma\, d\,\gamma}{\sqrt{\gamma^2 - \frac{1}{4}Q^2L^2}}$$

$$\approx \frac{2\,\overline{\epsilon^2}}{E\,L} \sqrt{\frac{1 + \frac{1}{2}Q\ell_n}{1 - \frac{1}{2}Q\ell_n}} \left\{ \gamma_n \sqrt{1 - \tfrac{1}{4}Q^2\ell_n^2} - \gamma_o \sqrt{1 - \tfrac{1}{4}Q^2\ell_o^2} \right\}$$

or, assuming $\gamma_n = \gamma_o + 2nEL >> \gamma_o$,

$$\left. \begin{aligned} \overline{\xi_n^2} &\approx 4n \left(1 + \tfrac{1}{2}Q\ell_n\right) \overline{\epsilon^2} \\ &\approx 4n\, \overline{\epsilon^2} \end{aligned} \right\} \qquad \text{(2-23b)}$$

Since Eq. (2-23b) agrees with Eq. (2-23a) to first order in $Q\ell$, Eq.(2-23a) will be a good estimate whether the beam is accelerated or coasting.

We now may relate a quadrupole alignment tolerance to maximum allowable beam displacement as follows:

$$\langle\epsilon\rangle \lesssim \frac{|\xi|_{max} \sqrt{1 - \frac{1}{2}Q\ell}}{\sqrt{4n}} \qquad \text{(2-24a)}$$

where $\langle\epsilon\rangle$ is the rms quadrupole alignment tolerance relative to the MRA and $|\xi|_{max}$ is the maximum allowable beam displacement. Equation (2-24) may be interpreted as relating the alignment tolerance to the maximum number of focusing periods which may be allowed before magnetic steering is required to compensate for the beam deflection. A reasonable value of $Q\ell$ (for the lowest energy beam in a multiple-beam situation) is (see Section I.B)

$$|Q|\ell \approx 1.24$$

Table 2.1 summarizes the tolerance vs number of quadrupoles for this case.

TABLE 2.1

Short-range quadrupole alignment tolerance vs number of quadrupoles per steering period for $|Q|\ell = 1.24$ and $|\xi|_{max} = 0.1$ cm = 0.04 inch.

Number of Quadrupoles 2n	Alignment Tolerance $<\epsilon>$
4	0.009 inch
8	0.006
16	0.004
80	0.002
240	0.001

Several interpretive remarks apply to these results:

a) In order to operate with not more than one or two steering periods per sector, it is necessary to impose an alignment precision of 0.006 to 0.009 inch on the quadrupoles relative to a local mean reference axis.

b) This does not necessarily imply physical alignment to this tolerance by optical or mechanical means. The position of the quadrupole optic axes may in principle be adjusted to sufficient precision by using electron beam-deflection information* and dipole correcting magnets or dipole biasing of the quadrupoles.

c) Given an initially satisfactory alignment of the quadrupoles, the above numbers (0.006 to 0.009 inch) indicate the amount of random, short-range misalignment due to earth movements, etc., which might be tolerated (i.e., corrected by steering alone) before short-range realignment is necessary.

d) The last two entries in Table 2.1, for 80 and 240 quadrupole spacings define the sort of stability required on the quadrupole positions. For example, random vibrations of the quadrupoles must not exceed 0.001 to 0.002 inch in amplitude.

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*Methods of achieving the required alignment precision are outside the scope of the present discussion.

III. INTERMEDIATE-AND LONG-RANGE MISALIGNMENTS

A. RANDOM MISALIGNMENTS WITH CORRELATION

Misalignments arising from such effects as earth movements and settlement will in general not be completely independent but will be correlated over a finite range. It is convenient to assume that we may define a <u>correlation function</u> $F(k)$ such that

$$\overline{\alpha_m \alpha_k^*} = \overline{|\alpha_m|^2} \, F(m - k) \tag{3-1}$$

where α_m is as defined previously. The correlation period $\overline{\Delta N}$ is defined by

$$\overline{\Delta N} = \sum_{-\infty}^{\infty} F(k) \tag{3-2}$$

The expectation value of the amplitude of the orbit perturbation now is given by

$$\overline{\left(\xi_n\right)^2_{max}} = \overline{\left|v_n\right|^2}$$

$$\cong \left(\frac{a_{12}}{\sin\theta}\right)_n \sum_{m=1}^{n} \sum_{k=1}^{n} \left(\frac{\sin\theta}{a_{12}}\right)_m^{\frac{1}{2}} \left(\frac{\sin\theta}{a_{12}}\right)_k^{\frac{1}{2}} \overline{\alpha_m \alpha_k^*} \, e^{i(\mu_k - \mu_m)}$$

On the assumption that the parameters vary only slightly in the range over which correlations are important, this becomes

$$\overline{\left|v_n\right|^2} \approx \left(\frac{a_{12}}{\sin\theta}\right)_n \sum_{m=1}^{n} \left(\frac{\sin\theta}{a_{12}}\right)_m \overline{\left|\alpha_m\right|^2} \sum_{k=1}^{n} F(m - k) e^{i(k - m)\theta_m}$$

Finally, if both n and the orbit period $2\pi/\theta$ are fairly large compared to $\overline{\Delta N}$, the last sum in the equation may be approximated by Eq. (3-2), giving

$$\overline{\left|v_n\right|^2} \approx \left(\frac{a_{12}}{\sin\theta}\right)_n \sum_{m=1}^{n} \left(\frac{\sin\theta}{a_{12}}\right)_m \overline{\left|\alpha_m\right|^2}\ \overline{\Delta N} \tag{3-3}$$

Example. Random bends in the accelerator axis: Suppose that misalignments have set in because of earth movements, etc. If we take the accelerator axis as the reference axis and assume that quadrupole misalignments are negligible, then,

$$b_{1m} = \gamma_m \varphi_{1m}$$

$$b_{2m} = \gamma_m \varphi_{2m}$$

If the correlation period is fairly long compared to the quadrupole spacing, so that the machine axis has essentially constant curvature over a short range, then

$$\varphi_{1m} \approx \varphi_{2m}$$

and Eq. (2-21) gives

$$\overline{\left|\alpha_m\right|^2} \approx \left(\frac{16\left(1 - \frac{1}{8}Q^2\ell^2\right)}{Q^2\left(1 - \frac{1}{2}Q\ell\right)}\right)_m \gamma_m^2\ \overline{\varphi^2} \tag{3-4}$$

Because of the γ^2 dependence, the perturbation is stronger at high energy; hence the most interesting result will be the high energy limit

($Q\ell \ll 1$). In this case substitution of Eq. (3-4) in Eq. (3-3) gives (assuming Q, $\overline{\varphi^2}$, and $\overline{\Delta N}$ constant),

$$\overline{\left(\xi_n\right)^2_{max}} = \overline{\left|v_n\right|^2} \approx \frac{16\ \overline{\varphi^2}\ \overline{\Delta N}}{Q^2} \sum_{m=1}^{n} \gamma_m^2 \tag{3-5}$$

In the case of constant acceleration, the sum may be approximated by an integral as was done in Eq. (2-23-b), giving

$$\overline{\left(\xi_n\right)^2_{max}} \approx \frac{8\ \overline{\varphi^2}\ \overline{\Delta N}}{3\ ELQ^2} \left(\gamma_n^3 - \gamma_o^3\right)$$

or, assuming $\gamma_n \approx EZ \gg \gamma_o$,

$$\overline{\left(\xi_n\right)^2_{max}} \approx \frac{8}{3}\frac{E^2}{Q^2}\frac{Z^3}{L}\ \overline{\varphi^2}\ \overline{\Delta N} \tag{3-5a}$$

where

$$Z \equiv 2nL$$

It should be recalled that ξ is measured relative to the <u>accelerator axis</u> in the above calculation.

The quantity $\overline{\varphi^2}\ \overline{\Delta N}$ may be related to an experimentally observable quantity in the following way: Suppose we have a partial survey of the misalignments as illustrated in Fig. 3-1. The quantities Φ_1, Φ_2, ... are interpreted as the net angular bends over the regions n_o to n_1, n_1 to n_2, ..., where $n_1 - n_o$ is the number of focusing periods from n_o to n_1, etc.

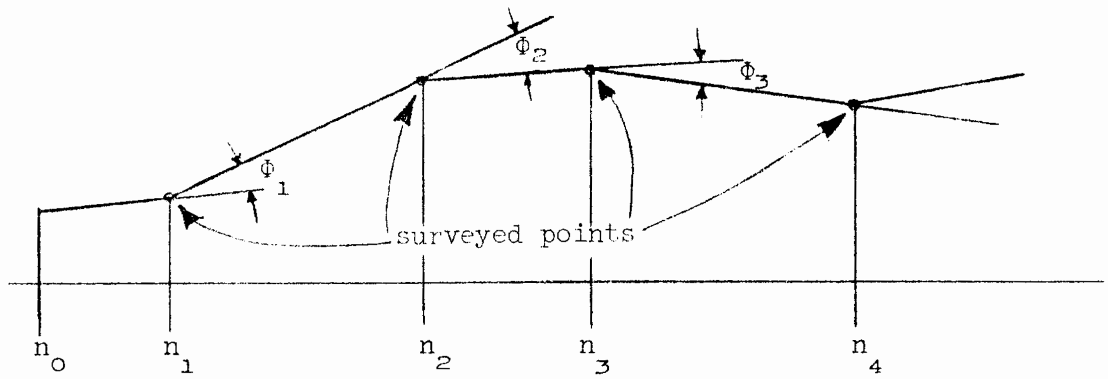

Fig. 3-1--Illustrating "random walk" of angular misalignments. (See text for explanation.)

Then

$$\Phi_1 = \sum_{m=n_o+1}^{n_1} \left(\varphi_{1m} + \varphi_{2m}\right) \approx \sum_{m=n_o+1}^{n_1} 2\varphi_m$$

where the φ's as defined previously are the angular bends at the support points. The mean square expectation value of Φ_1 is

$$\overline{\Phi_1^2} \approx 4\sum_{n_o+1}^{n_1} \overline{\varphi_m^2} \sum_{n_o+1}^{n_1} F(m - k)$$

$$\approx 4\sum_{n_o+1}^{n_1} \overline{\varphi^2}\ \overline{\Delta N}$$

$$\approx 4\left(n_1 - n_o\right) \overline{\varphi^2}\ \overline{\Delta N}$$

Hence

$$\overline{\Phi_1^2} + \overline{\Phi_2^2} + \ldots \approx 4n \, \overline{\varphi^2} \, \overline{\Delta N} \qquad (3\text{-}6)$$

where the sum is over all the measured angles and n is the total number of focusing periods included. A fair estimate of the expectation value is given by

$$\Phi_1^2 + \Phi_2^2 + \ldots \approx \overline{\Phi_1^2} + \overline{\Phi_2^2} + \ldots$$

which gives $\overline{\varphi^2} \, \overline{\Delta N}$ in terms of the measured misalignments.

It is perhaps interesting to note that a site survey[7] for the period March - September, 1962 shows apparent misalignments, resembling a random walk, over the western end of the site from Station 0 to about Station 40 (4000 feet). The numerically evaluated quantity $4n\overline{\varphi^2}\overline{\Delta N}$ turns out to be about 1.5×10^{-8} (radian)^2. If we assume $Q = 2$, $E = 0.1$, and $Z = 1.2 \times 10^5$ cm (4000 feet), then Eq. (3-5a) gives for the expected beam displacement

$$\left(\overline{\xi^2}\right)^{\frac{1}{2}}_{max} \approx 0.85 \text{ cm}$$

Assuming that this misalignment builds up linearly as a function of time, a realignment would be necessary about every 3 weeks, unless pulsed steering were used.

It may be inferred that we should require

$$\sqrt{\Phi_1^2 + \Phi_2^2 + \ldots} \;\tilde{<}\; 1.5 \times 10^{-5} \text{ radian}$$

in order to insure $\left(\overline{\xi^2}\right)^{\frac{1}{2}}_{max} \;\tilde{<}\; 0.1$ cm with $Q = 2$. On the other hand, if $Q = 0.6$ (1000 gauss) and the other parameters are the same as in the above example,

$$\left(\xi^2\right)^{\frac{1}{2}}_{max} \approx 2.8 \text{ cm}$$

and realignment would be necessary about once a week.

These numbers of course are not to be taken literally because the site survey does not give the fine structure of the misalignments, and it is not clear that the statistical picture is valid. When more detailed surveys of site motion become available, numerical ray tracing should be performed to get a more accurate picture of how the beam dynamics would be affected.

B. ISOLATED PERTURBATIONS

The accelerator might happen to have a few relatively large misalignments of a localized nature; for example, isolated bends, or possibly a short region which is displaced with respect to the rest of the machine.

In this case the orbit perturbation for a misalignment in the vicinity of the n-th focusing section could be approximated by

$$v_n \approx \left(\frac{a_{12}}{\sin\theta}\right)^{\frac{1}{2}}_n \left(\frac{\sin\theta}{a_{12}}\right)^{\frac{1}{2}}_m \overline{\alpha_m}\, \Delta m\, e^{i\left(\mu_n - \mu_m\right)} \tag{3-7}$$

where $\overline{\alpha_m}$ and Δm are the mean value and effective range, respectively, of the perturbation vector; it is assumed that Δm is short compared to the orbit period, $2\pi/\theta_m$, and that $n > m$.

Example 1. Isolated bend. In this case we have

$$\varphi_{1m} \approx \varphi_{2m} \approx \frac{1}{2}\frac{\Delta\varphi}{\Delta m}$$

or

$$b_{1m} \approx b_{2m} \approx \frac{1}{2} \gamma_m \frac{\Delta\varphi}{\Delta m}$$

where $\Delta\varphi$ is the total bend. Equation (2-21) then gives

$$\bar{\alpha}_m \Delta m \approx \frac{1}{2} \gamma_m \Delta\varphi \left[\ell - \frac{4i}{Q} \left(\frac{1 + \frac{1}{2} Q\ell}{1 - \frac{1}{2} Q\ell} \right)^{\frac{1}{2}} \left(1 - \frac{1}{4} Q\ell\right) \right]_m$$

from which we may calculate, from Eq. (3-7),

$$\left|\xi_n\right|_{max} = \left|v_n\right| \approx \left| \frac{2\gamma_m \Delta\varphi}{Q} \right| \left(\frac{1 + \frac{1}{2} Q\ell}{1 - \frac{1}{2} Q\ell} \right)_n^{\frac{1}{4}} \left(\frac{1 - \frac{1}{8} Q^2\ell^2}{\sqrt{1 - \frac{1}{4} Q^2\ell^2}} \right)_m^{\frac{1}{2}} \qquad (3\text{-}8)$$

where it has been assumed that Q is constant.

In the high energy limit, $Q\ell \ll 1$, this is simply

$$\left|\xi_n\right|_{max} \approx \left| \frac{2\gamma_m \Delta\varphi}{Q} \right| \qquad (3\text{-}8a)$$

As an example, take $Q = 2$ ($\approx$ 3400 gauss) and $\gamma_m = 10^4$ ($\approx$ 5 Bev). Then

$$\left|\xi_n\right|_{max} \approx 10^4 \left|\Delta\varphi\right|$$

so that for $\left|\xi\right|_{max} < 0.1$ cm, we would have to require $\left|\Delta\varphi\right| < 10^{-5}$ radian (or else apply pulsed dipole steering).

Example 2. Displaced section of accelerator. Suppose that a short region of the machine, consisting of Δm focusing sections, is displaced by an amount $\bar{\epsilon}$ from the rest of the machine. (assume $\bar{\epsilon} \ll$ the accelerator aperture). In this case,

$$b_{2m} \approx - b_{1m} \approx Q\bar{\epsilon}$$

and Eq. (2-21) gives

$$\overline{\alpha_m} \approx - Q\ell_m \bar{\epsilon} \left[1 + i \left(\frac{1 + \frac{1}{2} Q\ell}{1 - \frac{1}{2} Q\ell} \right)_m^{\frac{1}{2}} \right]$$

From Eq. (3-7), we find

$$\left|\xi_n\right|_{max} = \left|v_n\right| \approx \left|Q\ell_m \bar{\epsilon} \Delta m\right| \left(\frac{1 + \frac{1}{2} Q\ell}{1 - \frac{1}{2} Q\ell} \right)_n^{\frac{1}{4}} \left(\frac{2}{\sqrt{1 - \frac{1}{4}Q^2\ell^2}} \right)_m^{\frac{1}{2}} \quad (3\text{-}9)$$

In this case the effect is strongest at low energy where $Q\ell$ might be of order unity.

For constant parameters,

$$\left|\xi_n\right|_{max} \approx \frac{\left|\sqrt{2}\, Q\ell \bar{\epsilon} \Delta m\right|}{\sqrt{1 - \frac{1}{2} Q\ell}} \quad (3\text{-}9a)$$

For example, if $Q\ell \approx 1.24$,

$$\left|\xi_n\right|_{max} \approx 3.5 \left|\bar{\epsilon} \Delta m\right|$$

To keep $|\xi|_{max} < 0.1$ cm, we would have to require $\bar{c}\,\Delta m < 0.011$ inch.

Example 3. Several isolated perturbations. If discontinuities occur at several points $m_1, m_2, \ldots$, then the net orbit perturbation is given by

$$v_n \approx \left(\frac{a_{12}}{\sin\theta}\right)_n^{\frac{1}{2}} e^{i\mu_n}\left(\delta_1 e^{-i\mu_{m1}} + \delta_2 e^{-i\mu_{m2}} + \ldots\right)$$

where

$$\delta_1 \equiv \left(\frac{\sin\theta}{a_{12}}\right)_{m_1}^{\frac{1}{2}} \bar{\alpha}_{m_1}\,\Delta m_1,\ \text{etc.};$$

it is assumed that $n > m_1$, $n > n_2$, ...

Because of the energy-dependence of the phase angle μ_m, several of the terms of the sum might reinforce one another at particular energies. Hence in order to avoid stop bands, the quantity $\left(|\delta_1| + |\delta_2| + \ldots\right)$, where the sum is over all large "non-random" perturbations, should be kept small enough so that $|v_n| \ll$ accelerator aperture.

C. CONSTANT CURVATURE

Some sort of systematic error in the alignment system might result in a component of constant curvature in the misalignment. In this case we presumably would have an equal bend at each support point, so that

$$b_{1m} = b_{2m} = \gamma_m \varphi$$

Because the effect is strongest at high energy, it will be of the most interest to go immediately to the high energy limit $(Q\ell \approx QL/\gamma \ll 1)$.

Equation (2-21) then gives

$$\alpha_m \approx - \frac{4i\varphi\gamma_m}{Q}$$

Hence the orbit displacement is (assuming Q constant),

$$v_n \approx - \frac{4i\varphi}{Q} \sum_{m=1}^{n} \gamma_m e^{i\left(\mu_n - \mu_m\right)}$$

where

$$\mu_m = \sum_{k=1}^{m} \theta_k \approx \sum_{k=1}^{m} Q\ell_k \approx QL \sum_{k=1}^{m} \frac{1}{\gamma_k}$$

The sums may be approximated by integrals, by identifying $2L$ as dz; then

$$v(z) \approx - \frac{2i\varphi}{QL} e^{i\mu} \int_0^z \gamma_1 e^{-i\mu_1} \, dz_1$$

$$\mu(z) \approx \tfrac{1}{2} Q \int_0^z \frac{dz_1}{\gamma_1}$$

where the notation $\gamma(z_1) \equiv \gamma_1$, etc., is used. For constant acceleration, $\frac{d\gamma}{dz} = E$, straightforward integration gives

$$\mu \approx \frac{Q}{2E} \log(\gamma/\gamma_o)$$

$$v \approx \frac{4\varphi}{L}\left(1 - \frac{4iE}{Q}\right)\frac{\gamma^2 - \gamma_o^2 e^{i\mu}}{Q^2 + (4E)^2}$$

Hence if $\gamma_o \ll \gamma \approx Ez$,

$$\xi = \mathrm{Re}(v) \approx \frac{z^2}{R}\,\frac{4E^2}{Q^2 + (4E)^2}$$

where $R = L/\varphi$ = radius of curvature of the accelerator axis.

The maximum misalignment relative to a chord through the ends of the machine is given in this case by

$$\Delta X = \frac{1}{8}\frac{Z^2}{R}$$

where Z is the total length; hence

$$\xi(Z) = \xi_{max} \approx \frac{32\ E^2}{Q^2 + (4E)^2}\,\Delta X$$

For a maximum misalignment of 1 cm, with $E = 0.1$ and $Q = 2$, we would have

$$\xi_{max} \approx 0.08 \text{ cm}$$

which is about the maximum tolerable deflection.

IV. HARMONIC ANALYSIS IN PERIODIC SYSTEM

Fourier analysis of the errors is not quite as natural in a linear accelerator as in a circular machine, because

a) the independent variable (z) is not cyclic and

b) many modes of operation are possible and hence there is no well-defined "phase" variable.

Nevertheless considerable insight may be gained by considering either a beam coasting at constant energy, or a short section of machine over which relative energy change is small.

Assume that the function α_n which describes the trajectory perturbation per focusing section, is given by

$$\alpha_n = \alpha(2nL) \tag{4-1}$$

where $\alpha(z)$ is an appropriate continuous function. In terms of a Fourier analysis,

$$\alpha(z) = \int_{-\infty}^{\infty} \hat{\alpha}(k)\, e^{ikz}\, dk \tag{4-2}$$

The error spectrum $\hat{\alpha}(k)$ will be specified uniquely if we make some assumption as to the form of $\alpha(z)$ outside the range $0 \le z \le Z$; for example,

$$\hat{\alpha}(k) = \frac{1}{2\pi} \int_{0}^{Z} \alpha(z)\, e^{-ikz}\, dz \tag{4-3}$$

if we assume $\alpha(z) = 0$ for $z < 0$ and $z > Z$. Equation (2-10) with substitution of Eq. (4-2) becomes

$$v_n = e^{in\theta} \int_{-\infty}^{\infty} \hat{\alpha}(k)\ dk \sum_{m=1}^{n} e^{im(2kL-\theta)}$$

$$= e^{in\theta} \int_{-\infty}^{\infty} \hat{\alpha}(k)\ e^{i(n+1)(kL-\frac{1}{2}\theta)} \frac{\sin n(kL - \frac{1}{2}\theta)}{\sin (kL - \frac{1}{2}\theta)}\ dk$$

$$= e^{in\theta} \int_{-\infty}^{\infty} \hat{\alpha}\left(\kappa + \frac{\theta}{2L}\right) e^{i(n+1)\kappa L} \frac{\sin n\kappa L}{\sin \kappa L}\ d\kappa \qquad (4\text{-}4)$$

where $\kappa = k - \dfrac{\theta}{2L}$.

If n is large, the function $\sin n\kappa L/\sin \kappa L$ has a sharp resonance near $\kappa L = \nu\pi$, where ν is any integer; hence Eq. (4-4) may be approximated by a sum over the various resonances;

$$v_n \approx e^{in\theta} \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\alpha}(k_\nu + \kappa)\ e^{i(n+1)\kappa L} \frac{\sin n\kappa L}{\kappa L}\ dk \qquad (4\text{-}5)$$

where the substitution $k_\nu = \frac{1}{L}\left(\nu\pi + \frac{1}{2}\theta\right)$ has been made and $\sin \kappa L$ has been replaced by κL. Since the resonance function $(\sin n\kappa L)/\kappa L$ has an effective range of $-\pi/2 \lesssim n\kappa L \lesssim \pi/2$, Eq. (4-5) says that for a given error spectrum $\hat{\alpha}(k)$, the important contributions to the beam deflection are for the argument in the range

$$k_\nu - \Delta k \lesssim k \lesssim k_\nu + \Delta k,$$

where

$$k_\nu \pm \Delta k \approx \frac{1}{L}\left(\nu\pi + \frac{1}{2}\theta \pm \frac{\pi}{2n}\right),\ \nu = 0, \pm 1, \ldots . \qquad (4\text{-}6)$$

The width of these resonances is thus

$$\Delta\theta \approx \frac{2\pi}{n}$$

or since $\theta \approx \frac{QL}{\gamma}$, the energy band width is

$$\frac{\Delta\gamma}{\gamma} \approx \frac{2\pi\gamma}{nQL} \approx \frac{2\pi}{nQ\ell} \tag{4-7}$$

This means that in order to transmit a given energy, we must "tune out" (e.g., by a suitable steering or alignment procedure) the band of error components in the vicinity of

$$k_\nu = \frac{\nu\pi}{L} + \frac{Q}{2\gamma}, \ \nu = 0, \pm 1 \ldots.;$$

and that tuning at a given energy only guarantees transmission over an energy band given by Eq. (4-7).

As an example, suppose we wish to transport beams, in the energy range of 1 to 10 Bev, over one-third of the machine. With $L = 1.2 \times 10^3$ cm (40 feet), $Q = 2$ (3400 gauss), and $2n = 80$,

$$\frac{\Delta\gamma}{\gamma} = 0.13 \text{ at 1 Bev}$$

$$= 0.78 \text{ at 6 Bev}$$

The "wavelengths" of the important error components would be typically

$$\begin{aligned} \frac{2\pi}{\left|k_\nu\right|} = \frac{2L}{\left|\nu + \frac{QL}{2\pi\gamma}\right|} &= 10.5\ L && (1 \text{ Bev}, \ \nu = 0) \\ &= 1.68\ L && (1 \text{ Bev}, \ \nu = 1) \\ &= 2.47\ L && (1 \text{ Bev}, \ \nu = -1) \\ &= 63\ L && (6 \text{ Bev}, \ \nu = 0) \\ &= (2 \pm 0.064)L && (6 \text{ Bev}, \ \nu = \pm 1) \end{aligned}$$

Resonances higher than $|\nu| = 1$ probably are relatively unimportant because the error spectrum would be expected to fall off rapidly for wavelengths shorter than 2L.

Suppose we consider the previous example of equally spaced quadrupoles which have linear misalignments relative to a straight reference axis. (See Fig. 2-3.) Then b_{1m} and b_{2m} in Eq. (2-21) are given by

$$\left. \begin{aligned} b_{1m} &= -Q\epsilon_{1m} = -Q\int \hat{\epsilon}(k)\, e^{(2m-1)ikL}\, dk \\ b_{2m} &= Q\epsilon_{2m} = Q\int \hat{\epsilon}(k)\, e^{2mikL}\, dk \end{aligned} \right\} \qquad (4\text{-}8)$$

where $\hat{\epsilon}(k)$ is the Fourier transform of the misalignment error $\epsilon(z)$. In this case, one finds

$$\hat{\alpha}(k) = 2i\left\{\sqrt{\frac{1+\frac{1}{2}Q\ell}{1-\frac{1}{2}Q\ell}} - e^{i(\frac{1}{2}\theta - kL)}\right\}\hat{\epsilon}(k) \qquad (4\text{-}9a)$$

or in the vicinity of the ν-th resonance,

$$\hat{\alpha}(k_\nu + \kappa) \approx 2i\left\{\sqrt{\frac{1+\frac{1}{2}Q\ell}{1-\frac{1}{2}Q\ell}} - (-1)^\nu\right\}\hat{\epsilon}(k_\nu + \kappa);\ (\kappa L \ll \pi) \qquad (4\text{-}9b)$$

According to the latter equation, error components of wavelengths in the vicinity of $2L(\nu = \pm 1)$ couple more strongly to the transverse motion than do the $\nu = 0$ components which are coherent with the transverse motions, of trajectory wavelengths $4\pi L/\theta$. Thus even at high energies where the trajectory wavelengths are long, the short-range misalignment effects are predominant.

V. ROTATIONAL MISALIGNMENTS

Rotation of a quadrupole about the reference axis introduces coupling between the x and y components of motion. In a system in which the principal axes of the quadrupole are the x and y axes, the quadrupole may be represented by a 4×4 matrix;

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -Q & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & Q & 1 \end{pmatrix} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}_0 = \mathbf{A}_Q \mathbf{x}_0 \qquad (5\text{-}1)$$

The rotation matrix which transforms the reference system to the rotated quadrupole system is

$$\mathbf{R} = \begin{pmatrix} \cos\psi & 0 & \sin\psi & 0 \\ 0 & \cos\psi & 0 & \sin\psi \\ -\sin\psi & 0 & \cos\psi & 0 \\ 0 & -\sin\psi & 0 & \cos\psi \end{pmatrix} \qquad (5\text{-}2)$$

where ψ is the angle by which the quadrupole is rotated from the reference system. The matrix for the rotated quadrupole, then, is given by

$$\mathbf{A}_{Q,R} = \mathbf{R}^{-1} \mathbf{A}_Q \mathbf{R}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -Q\cos 2\psi & 1 & -Q\sin 2\psi & 0 \\ 0 & 0 & 1 & 0 \\ -Q\sin 2\psi & 0 & Q\cos 2\psi & 1 \end{pmatrix} \qquad (5\text{-}3)$$

Hence, if ψ is a small angle, the net first-order effect of the rotational error is to introduce a transverse momentum impulse given by

$$\begin{aligned} \delta p_x &= -\ 2Q\psi y \\ \delta p_y &= -\ 2Q\psi x \end{aligned} \tag{5-4}$$

(The convention is that Q is positive if the quadrupole is focusing in the x-direction and defocusing in the y-direction.)

The transformation over a focusing period, analogous to Eq. (2-19), now is given by

$$\mathbf{T}_n = (\mathbf{A}_{Q,R_2}\mathbf{L}\mathbf{A}_{-Q,R_1}\mathbf{L})_n \tag{5-5}$$

where

$$\mathbf{A}_{\mp Q,R_{1,2}} \cong \begin{pmatrix} 1 & 0 & 0 & 0 \\ \pm Q & 1 & \pm 2Q\psi_{1,2} & 0 \\ 0 & 0 & 0 & 0 \\ \pm 2Q\psi_{1,2} & 0 & \mp Q & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{L} \equiv \begin{pmatrix} 1 & \ell & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \ell \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To first order in the rotations ψ_{1n} and ψ_{2n}, the product turns out to be of the form

$$\mathbf{T}_n = \begin{pmatrix} \mathbf{A} & \mathbf{E} \\ \mathbf{F} & \mathbf{B} \end{pmatrix}_n \tag{5-6}$$

where $\mathbf{A}_n$ is the 2×2 matrix given by Eq. (2-19); $\mathbf{B}_n$ is the analogous transformation in the y plane;* $\mathbf{E}_n$ and $\mathbf{F}_n$ are 2×2 matrices of first order in the ψ's (the explicit forms of $\mathbf{E}$ and $\mathbf{F}$ will not be required.)

We now make the substitution

$$\begin{aligned} \mathbf{x}_n &= \mathbf{X}_n + \boldsymbol{\xi}_n \\ \mathbf{y}_n &= \mathbf{Y}_n + \boldsymbol{\eta}_n \end{aligned} \tag{5-7}$$

where

$$\mathbf{x}_n = \begin{pmatrix} x \\ p_x \end{pmatrix}_n \quad \text{, etc.;}$$

$\mathbf{X}_n$, $\mathbf{Y}_n$ are solutions of the unperturbed system (i. e., all ψ's set equal to zero);

$\boldsymbol{\xi}_n, \boldsymbol{\eta}_n$, the orbit perturbations, are assumed to be small.

It then follows that

$$\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix}_n = \begin{pmatrix} [\mathbf{A}_n \boldsymbol{\xi}_{n-1} + \mathbf{E}_n \mathbf{Y}_{n-1}] \\ [\mathbf{B}_n \boldsymbol{\eta}_{n-1} + \mathbf{F}_n \mathbf{X}_{n-1}] \end{pmatrix} + \begin{pmatrix} \mathbf{E}_n \boldsymbol{\eta}_{n-1} \\ \mathbf{F}_n \boldsymbol{\xi}_{n-1} \end{pmatrix} \tag{5-8}$$

But the last term in Eq. (5-8) is second-order small and hence will be dropped. Thus we have

$$\boldsymbol{\xi}_n \approx \mathbf{A}_n \boldsymbol{\xi}_{n-1} + \mathbf{E}_n \mathbf{Y}_{n-1} \tag{5-9}$$

---

*In the present system, where the focusing period consists of two quadrupoles of equal strength and opposite sign, $\mathbf{B}$ is derived from $\mathbf{A}$ by changing the sign of Q.

and an analogous expression for η_n. Since Eq. (5-9) is of the same form as Eq. (2-2), the solution is given by Eq. (2-13);

$$v_{xn} \equiv (\boldsymbol{\omega}_x, \boldsymbol{\xi})_n$$

$$= \left(\frac{a_{12}}{\sin \theta_x}\right)_n^{\frac{1}{2}} \sum_{m=1}^{n} \left(\frac{\sin \theta_x}{a_{12}}\right)_m^{\frac{1}{2}} \left(\boldsymbol{\omega}_x, \mathbf{E}_m \mathbf{Y}_{m-1}\right) e^{i\,\mu_{xn} - \mu_{xm}} \qquad (5\text{-}10)$$

(Subscripts x and y are used to indicate that in general the various quantities may be different in the **x** and **y** planes.)

Rather than evaluate this expression explicitly it will be convenient to calculate the mean increment of the adiabatic invariant function. (The results through Eq. (5-9) are still important in showing that the orbit perturbations are decoupled to first order.)

From Eqs. (2-14) and (5-4) the rotational error of the second quadrupole in the m-th period contributes an increment to the invariant function given by

$$\delta\overline{\Upsilon}_{2,m} = \left(\frac{a_{12}}{\sin \theta_x}\right)_m \overline{\left(\delta p_{2,x}\right)_m^2}$$

$$= \left(\frac{a_{12}}{\sin \theta_x}\right)_m \cdot 4\, Q_m^2\, \overline{\psi_{2,m}^2}\, Y_m^2$$

The contribution from the quadrupole in the middle of the period is

$$\delta\overline{\Upsilon}_{1,m} = \left(\frac{\hat{a}_{12}}{\sin \theta_x}\right)_m \cdot 4\, Q_m^2\, \overline{\psi_{1,m}^2}\, \hat{Y}_m^2$$

where $\hat{a}_{12}$, $\hat{Y}_m$ refer to the matrix element and coordinate, respectively, evaluated at reference planes at the mid-period. In the present system of equally-spaced quadrupoles of equal strength, $\hat{\mathbf{A}}_m$ is found simply by changing the sign of Q in $\mathbf{A}_m$.

The net increment of Υ after n periods is

$$\Delta\bar{\Upsilon}_{xn} = \sum_{m=1}^{n} \left(\delta\bar{\Upsilon}_{1,m} + \delta\bar{\Upsilon}_{2,m}\right)$$

which is related to the maximum expected orbit perturbation by

$$\left(\overline{\xi_n^2}\right)_{max} = \left(\frac{a_{12}}{\sin\theta}\right) \Delta\bar{\Upsilon}_{xn}$$

Because of the quasi-periodic form of Y_m, we may replace Y_m^2 by $\frac{1}{2}\left(Y_m\right)^2_{max}$ in a summation over many orbit wavelengths;

$$\sum_{m=1}^{n} \delta\bar{\Upsilon}_{2,m} = \sum_{m=1}^{n} \left(\frac{a_{12}}{\sin\theta_x}\right)_m \cdot 4\, Q_m^2\, \psi_{2,m}^2\, Y_m^2$$

$$\approx 2 \sum_{m=1}^{n} \left(\frac{a_{12}}{\sin\theta_x}\right)_m Q_m^2\, \psi_{2,m}^2 \left(Y_m\right)^2_{max}$$

with a similar result for the contributions from the mid-period quadrupoles. Finally, using the identity analogous to Eq. (2-15), namely

$$\Upsilon_Y = \left(\frac{\sin\theta_y}{b_{12}}\right)_m \left(Y_m\right)^2_{max} = \left(\frac{\sin\theta_y}{\hat{b}_{12}}\right)_m \left(\hat{Y}_m\right)^2_{max}$$

we obtain

$$\overline{\left(\xi_n\right)^2_{max}} \approx 2\left(\frac{a_{12}}{\sin\theta_x}\right)_n \left(\frac{\sin\theta_y}{\hat{b}_{12}}\right)_n \left(\hat{Y}_n\right)^2_{max} \overline{\Psi^2} \sum_{m=1}^{n} \left(\frac{a_{12}b_{12} + \hat{a}_{12}\hat{b}_{12}}{\sin\theta_x \sin\theta_y}\right)_m Q^2_m \quad (5\text{-}11)$$

where it has been assumed that the rotational errors Ψ_{1m}, Ψ_{2m}, have the same mean square expectation value everywhere.

In the present system (equally spaced, equal strength quadrupoles) we find from results given in Section II and above, that

$$a_{12} = \hat{b}_{12} = 2\ell(1 + \tfrac{1}{2}Q\ell)$$

$$\hat{a}_{12} = b_{12} = 2\ell(1 - \tfrac{1}{2}Q\ell)$$

and

$$\sin\theta_x = \sin\theta_y = Q\ell\sqrt{1 - \tfrac{1}{4}Q^2\ell^2}$$

from which comes the simple result

$$\overline{\left(\xi_n\right)^2_{max}} \approx 16n\left(\hat{Y}_n\right)^2_{max} \overline{\Psi^2} \quad (5\text{-}11a)$$

The result has been expressed in terms of $\left|\hat{Y}_n\right|^2_{max}$ because in the present system Y has its maximum amplitude at the mid-period (i.e., the quadrupole in the middle of the period is focusing in the **y** plane).

As a numerical example, suppose that $|Y|_{max} \approx 1$ cm; 2n = 240 = total number of quadrupoles; and that we require $|\xi|_{max} \leq 0.1$ cm. Then the tolerance on quadrupole rotational error is given by

$$\left(\overline{\Psi^2}\right)^{\frac{1}{2}} < \frac{|\xi|_{max}}{4\sqrt{n}\,|Y|_{max}} \approx 2.3 \times 10^{-3} \text{ radian}$$

$$\approx 0.13 \text{ degrees}$$

VI. NON-ADIABATIC LONGITUDINAL VARIATIONS IN THE PARAMETERS

A. RANDOM ERRORS IN QUADRUPOLE STRENGTH AND SPACING

If an error occurs in the strength or position of a quadrupole, it will always be possible to write the perturbed matrix for the quadrupole in the form

$$\mathbf{Q}' = \mathbf{Q} + \mathbf{\Delta}_Q \tag{6-1}$$

where $\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix}$ is the unperturbed matrix, and $\mathbf{\Delta}_Q$ contains the perturbation terms. For instance we have:

Case (1): Error in quadrupole strength. In this case

$$\mathbf{Q}' = \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta Q & 0 \end{pmatrix} \tag{6-2}$$

where δQ is the error term.

Case (2): Error in quadrupole position. In this case the perturbed matrix, transformed to the "correct" reference plane, is

$$\mathbf{Q}' = \begin{pmatrix} 1 & -\delta\ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \delta\ell \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} + \begin{pmatrix} -Q\delta\ell & -Qd\ell^2 \\ 0 & Q\delta\ell \end{pmatrix} \tag{6-3}$$

or to first order in the error,

$$\mathbf{\Delta}_Q = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} Q\delta\ell \tag{6-4}$$

where $\delta\ell = \frac{\delta L}{\gamma}$ and δL is the error in position.

In either case, the transformation over the n-th focusing period may be written

$$\mathbf{A}_n' = \left[\left(\mathbf{Q}^{-1} + \boldsymbol{\Delta}_{Q2}\right) \mathbf{L} \left(\mathbf{Q} + \boldsymbol{\Delta}_{Q1}\right) \mathbf{L}\right]_n$$

where $\mathbf{L} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$. Expanding and keeping only first-order error terms, we find

$$\mathbf{A}_n' = \mathbf{A}_n + \boldsymbol{\Delta}_n \tag{6-5}$$

where A_n is as given by Eq. (2-19);

$$\boldsymbol{\Delta}_n \approx \left(\mathbf{Q}^{-1} \mathbf{L} \boldsymbol{\Delta}_{Q1} \mathbf{L}\right)_n + \left(\boldsymbol{\Delta}_{Q2} \mathbf{L} \mathbf{Q} \mathbf{L}\right)_n \tag{6-6}$$

and $\boldsymbol{\Delta}_{Q1n}, \boldsymbol{\Delta}_{Q2n}$ are the appropriate quadrupole perturbation matrices defined by Eq. (6-2) or (6-4). The explicit form of $\boldsymbol{\Delta}_n$ will not be required.

It will be noticed that $\cos\theta$ is on the average unchanged to first order in the errors;

$$\overline{\cos\theta'} \equiv \tfrac{1}{2}\left(a_{11} + a_{22} + \overline{\Delta_{11} + \Delta_{22}}\right)$$

$$= \tfrac{1}{2}\left(a_{11} + a_{22}\right) = \cos\theta,$$

because the errors are assumed randomly distributed about the correct values. Furthermore, the second-order error terms can only be important when the system is near the short-wavelength cut-off ($\cos\theta \approx -1$ or $Q\ell \approx \frac{QL}{\gamma} \approx 2$) which will never be the case in practice.

If we now make the substitution

$$\mathbf{x}_n = \mathbf{X}_n + \boldsymbol{\xi}_n$$

where $\mathbf{X}_n$ is a solution of unperturbed system, and the orbit perturbation $\boldsymbol{\xi}_n$ is presumed small, then to first order,

$$\boldsymbol{\xi}_n \approx \mathbf{A}_n \boldsymbol{\xi}_{n-1} + \boldsymbol{\Delta}_n \mathbf{X}_{n-1} \tag{6-7}$$

(A second-order term, $\boldsymbol{\Delta}_n \boldsymbol{\xi}_{n-1}$, is neglected.)

Thus as in Section V the problem is reduced to the form of the linear inhomogeneous transformation, Eq. (2-2), and the solution is given formally by Eq. (2-13). However, it will again be more convenient to calculate the expected mean square orbit perturbation by considering the increment of the adiabatic invariant function.

1. <u>Error in Quadrupole Strength</u>. In this case the quadrupole injects a transverse momentum which by Eq. (6-2) is

$$\delta p = X \delta Q$$

Thus the quadrupole at the end of the m-th focusing period increases the invariant function by

$$\overline{\delta \Upsilon_{2,m}} = \left(\frac{a_{12}}{\sin\theta}\right)_m \overline{\delta Q^2_{2,m}} X^2_m$$

and the effect of the midperiod quadrupole is

$$\overline{\delta \Upsilon_{1,m}} = \left(\frac{\hat{a}_{12}}{\sin\theta}\right)_m \overline{\delta Q^2_{1,m}} \hat{X}^2_m$$

where as in Section V, $\hat{a}_{12}$ and $\hat{X}$ refer to the quantities as evaluated at the midperiod reference planes. Summing over n periods, assuming $\overline{\delta Q^2_{1m}} = \overline{\delta Q^2_{2m}}$, and using Eq. (2-15), we find

$$\Delta \overline{\Upsilon}_n \approx \frac{1}{2} \sum_{m=1}^{n} \left(\frac{a^2_{12} + \hat{a}^2_{12}}{\sin^2\theta}\right)_m \overline{\delta Q^2_m}$$

(The oscillatory part of the expression

$$X_m^2 \approx \tfrac{1}{2}\left[\,|w_m|^2 + \left(\frac{a_{12}}{\sin\theta}\right)_m \left(\frac{\sin\theta}{a_{12}}\right)_o \mathrm{Re}\left(w_o^2\, e^{2i\mu_m}\right)\right]$$

has as usual been ignored in the summation over several orbit wavelengths.) Thus we have

$$\overline{\left(\xi_n\right)^2_{max}} = \tfrac{1}{2}\left(X_n\right)^2_{max} \sum \left(\frac{a_{12}^2 + \hat{a}_{12}^2}{\sin^2\theta}\right)_m \overline{\delta Q_m^2} \qquad (6\text{-}8)$$

Using the parameters for the equi-spaced quadrupole systems, we find

$$\overline{\left(\xi_n\right)^2_{max}} = 4(X_n)^2_{max} \sum_{m=1}^{n} \left[\frac{\delta Q^2}{Q^2}\;\frac{1 + \frac{1}{4}Q^2\ell^2}{1 - \frac{1}{4}Q^2\ell^2}\right]_m \qquad (6\text{-}9)$$

Since the effect is strongest for low energy ($Q\ell$ of order unity), it will suffice to evaluate the sum for <u>constant parameters</u>, whence the tolerance on $\frac{\delta Q}{Q}$ is given by

$$\frac{\langle \delta Q \rangle_{rms}}{|Q|} \leq \frac{|\xi_n|_{max}}{|X|_{max}} \left(\frac{1}{4n}\;\frac{1 - \frac{1}{4}Q^2\ell^2}{1 + \frac{1}{4}Q^2\ell^2}\right)^{\frac{1}{2}} \qquad (6\text{-}10)$$

For $|\xi|_{max} = 0.1$ cm, $X_{max} = 1$ cm, $Q\ell = 1.24$, and $2n = 240$, this gives

$$\frac{\langle \delta Q \rangle_{rms}}{|Q|} \leq 0.30\%$$

which seems a not unreasonable requirement on quadrupole uniformity.

2. Errors in Quadrupole Spacing. According to Eq. (6-4) the effect of an error in quadrupole position is

$$\delta x = - XQ\delta\ell$$

$$\delta p = P_x Q\delta\ell$$

where $\delta\ell = \frac{\delta L}{\gamma}$ is the longitudinal displacement of the quadrupole from its correct position. The increment of the adiabatic invariant from the lens at the end of the period thus is

$$\delta\overline{\Upsilon}_{2,m} = \frac{1}{\sin\theta_m}\left\{- a_{21}\,\overline{\delta x^2} + \left(a_{11} - a_{22}\right)\overline{\delta x\,\delta p} + a_{12}\,\overline{\delta p^2}\right\}_m$$

$$= \left(\frac{Q^2\overline{\delta\ell^2}}{\sin\theta}\right)_m \left\{\left(- 2a_{21}X^2 + 2a_{12}P_x^2\right)_m - \Upsilon\sin\theta_m\right\}$$

where Υ is the adiabatic invariant in the unperturbed system. Proceeding as in the previous example, one finds

$$\overline{\Delta\Upsilon_n} \approx \tfrac{1}{4}\Upsilon\sum_{m=1}^{n}\left[Q^2\overline{\delta\ell^2}\;\frac{\left(a_{11} - a_{22}\right)^2 + \left(\hat{a}_{11} - \hat{a}_{22}\right)^2}{\sin^2\theta}\right]_m \qquad (6\text{-}11)$$

whence for the present system,

$$\overline{\left(\xi_n\right)^2_{max}} \approx 2(X_n)^2_{max}\;\frac{\overline{\delta L^2}}{L^2}\sum_{m=1}^{n}\left(Q^2\ell^2\,\frac{1 + \frac{1}{2}Q\ell}{1 - \frac{1}{2}Q\ell}\right)_m \qquad (6\text{-}12)$$

Since the effect is again largest for low energy, the longitudinal position tolerance may be based on the constant-parameter case:

$$\frac{<\delta L>_{rms}}{L} \leq Q\ell\;\frac{|\xi|_{max}}{X_{max}}\left(\frac{1}{2n}\;\frac{1 - \frac{1}{2}Q\ell}{1 + \frac{1}{2}Q\ell}\right)^{\frac{1}{2}} \qquad (6\text{-}13)$$

Using $Q\ell = 1.24$, $|\xi|_{max} = 0.1$ cm, $X_{max} = 1$ cm, and $2n = 240$, we find

$$\frac{<\delta L>}{L} \leq 0.40\%$$

or, if $L = 480$ inches,

$$<\delta L> \leq 1.9 \text{ inch}$$

Thus it may be concluded that no particular pains need be taken in longitudinal positioning of the quadrupoles.

B. PERIODIC VARIATION; SUPERPERIOD STOP-BAND

A periodically recurring perturbation of parameters of the focusing system can introduce stop-bands when the perturbation period ("superperiod") has an approximately rational relationship with the orbit half-wavelength.

The most likely such effect in the present type of system would consist of one abnormally large quadrupole spacing per sector (eight 40-foot intervals). It will be of some interest to treat this particular example.

Consider the case where one extra-long spacing occurs at the beginning of ΔN regular sections. The transformation over the superperiod then is given, with the help of Eq. (2-5), by*

$$\mathbf{B} = \mathbf{A}(\Delta N|0)\begin{pmatrix}1 & \delta\ell\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & \delta\ell\\ 0 & 1\end{pmatrix}\cos\Delta N\theta + \begin{pmatrix}\eta & a_{12} + \eta\delta\ell\\ a_{21} & -\eta + a_{21}\delta\ell\end{pmatrix}\frac{\sin\Delta N\theta}{\sin\theta} \tag{6-14}$$

*The effect will be important only if the system is strictly periodic; consequently parameters of the transformations are considered constant.

where $\eta = \frac{1}{2}\left(a_{11} - a_{22}\right)$; a_{ij} are the elements of the transformation over a regular period; and $\cos\theta = \frac{1}{2}\left(a_{11} + a_{22}\right)$.

The characteristic phase angle for the superperiod thus is given by

$$\cos\Theta = \frac{1}{2}\left(b_{11} + b_{22}\right) = \cos\Delta N\theta + \frac{1}{2}a_{21}\,\delta\ell\,\frac{\sin\Delta N\theta}{\sin\theta} \qquad (6\text{-}15)$$

Since the last term in Eq. (6-15) is in general small, we expect that stopbands, for which

$$\left|\cos\Theta\right| > 1$$

will occur in the vicinity of

$$\left|\cos\Delta N\theta\right| \approx 1,$$

or

$$\Delta N\theta \approx \nu\pi$$

where ν is a positive integer.

In order to investigate the stopband in detail, it will be necessary to consider a specific system. For the present system,

$$\cos\theta = 1 - \frac{1}{2}\,\frac{Q^2L^2}{\gamma^2}$$

or

$$\sin\tfrac{1}{2}\theta = \frac{1}{2}\,\frac{QL}{\gamma}$$

and

$$a_{21}\,\delta\ell = -\,\frac{Q^2L^2}{\gamma^2}\;\frac{\delta L}{L}$$

from which Eq. (6-15) becomes

$$\cos \Theta = \cos \Delta N\theta - \frac{\delta L}{L} \tan \tfrac{1}{2}\theta \sin \Delta N\theta \qquad (6\text{-}16)$$

If we make the substitution

$$\Delta N\theta = \nu\pi + \delta\theta \qquad (6\text{-}17)$$

where ν is a positive integer, then Eq. (6-16) becomes (keeping only terms up to second order smallness in δL and $\delta\theta$)

$$\cos \Theta \approx (-1)^{\nu} \left[1 - \tfrac{1}{2}\Delta N\delta\theta \left(\Delta N\delta\theta + 2\,\frac{\delta L}{L} \tan \tfrac{1}{2}\theta_{\nu} \right) \right] \qquad (6\text{-}18)$$

where $\theta_{\nu} = \frac{\pi\nu}{\Delta N}$.

The extent of the stopband (for which $|\cos \Theta| > 1$) is

$$-\frac{2}{\Delta N}\,\frac{\delta L}{L} \tan \tfrac{1}{2}\theta_{\nu} < \delta\theta < 0 \quad (\text{if } \delta L > 0)$$

$$0 < \delta\theta < \frac{2}{\Delta N}\,\frac{|\delta L|}{L} \tan \tfrac{1}{2}\theta_{\nu} \quad (\text{if } \delta L < 0)$$

Hence the stopband width is in either case

$$\Delta\theta_{\nu} = \frac{2}{\Delta N}\,\frac{|\delta L|}{L} \tan \tfrac{1}{2}\theta_{\nu}$$

or, using the identity $\sin \tfrac{1}{2}\theta = \tfrac{1}{2}\,\frac{QL}{\gamma}$,

$$\left(\frac{\Delta\gamma}{\gamma}\right)_{\nu} = \tfrac{1}{2} \cot \tfrac{1}{2}\theta_{\nu}\,\Delta\theta_{\nu} = \frac{1}{\Delta N}\,\frac{|\delta L|}{L} \qquad (6\text{-}19)$$

The extreme value of $\cos \Theta$, which we define as $(-1)^{\nu} \cosh \Gamma_{\nu}$ within the stopband, is from Eq. (6-18)

$$\cosh \Gamma_{\nu} = 1 + \tfrac{1}{2}\left(\frac{\delta L}{L}\right)^2 \tan^2 \tfrac{1}{2}\theta_{\nu}$$

or

$$\Gamma_{\nu} \approx \frac{|\delta L|}{L} \tan \tfrac{1}{2}\theta_{\nu} \qquad (6\text{-}20)$$

This quantity is of some interest because within the stopband we expect the amplitudes to vary as

$$\left|X\right|_{max} \propto e^{N\Gamma_\nu} \qquad (6\text{-}21)$$

where N is the number of superperiods.

The important values of ν will be $\nu = 1,2,\ldots\Delta N - 1$; the cases of $\nu = 0$ and $\nu \geq \Delta N$ are uninteresting because the former defines the long-wavelength cutoff ($\theta_\nu = 0$) and the latter are within the short-wavelength stopband ($\theta_\nu \geq \pi$).

In the present case we have $\Delta N = 4$ (i.e., four sections of period $2L = 80$ feet per 320-foot sector). Of the possible values of $\nu(\nu = 1,2$ or $3)$, only $\nu = 1$ is likely to give trouble because

$$\theta_2 = \frac{\pi}{2} > 1.34$$

where 1.34 is the value of θ corresponding to $Q\ell = 1.24$. If $Q = 2$, the energy of the $(\nu = 1)$ stopband would be given by

$$\gamma_1 = \frac{1}{2} \frac{QL}{\sin^{-1} \frac{1}{2}\theta_1} = \frac{1}{2} \frac{2 \cdot 1.2 \cdot 10^3}{\sin^{-1} \pi/8} = 3.1 \times 10^3$$

or <u>1.6 Bev</u>.

It is likely that there will be an extra length increment of ≈ 10 feet occurring at sector intervals; hence the stopband width would be given by Eq. (6-19)

$$\frac{\Delta\gamma}{\gamma} \approx \frac{1}{4} \cdot \frac{10}{40} \approx 6\%$$

The quantity Γ_1 [Eq. (6-20)] is

$$\Gamma_1 \approx \frac{1}{4} \tan \frac{\pi}{8} = 0.10$$

so that, as a result of Eq. (6-21) one would want to avoid transporting energies within the stopband over more than one or two sectors. This could of course be accomplished either by a slight change in quadrupole strength or by initially accelerating the beam to higher energy and back-phasing a few sections just before extracting the beam.

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