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THE PRINCIPLE OF DESIGN OF MAGNETIC MOMENTUM SLITS

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by

E. L. Chu and J. Ballam
Stanford Linear Accelerator Center

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I. INTRODUCTION

Consider a high-energy beam of charged particles of momentum range $p_o \pm \Delta p$. This beam has emerged from a momentum-analyzing magnet and is traveling in the positive z-direction. For this beam there exists a definite correspondence between the particle momentum and one of the particle transverse coordinates. Let the particles traveling with the average momentum p_o have $x = 0$, and the particles traveling with the momentum p have $x = x(p)$. Clearly, x is an odd function of $p - p_o$ in the limit of vanishing $p - p_o$.

This beam then passes through a long magnetic device called a momentum slit. Let the total cross section of the slit opening be defined by $-b \leq x \leq b$ and $-a \leq y \leq a$ with $a \ll b$. The purpose of using this slit is to separate the particles in a very narrow momentum range ($p_o \pm \delta p$) from the rest of the beam. The function of this slit may be described most simply by assuming an ideal magnetic field as follows:

$$B_z = B_x = 0 \quad \text{everywhere} \tag{1a}$$

$$B_y(x,y) = \begin{cases} +\mu_o I & a < x \leq b, \quad -a \leq y \leq a \\ 0 & -a \leq x \leq a, \quad -a \leq y \leq a \\ -\mu_o I & -b \leq x < -a, \quad -a \leq y \leq a \end{cases} \tag{1b}$$

The particles in the desired momentum range enter the central part of the slit where the magnetic field vanishes; therefore these particles will pass through the slit without being deflected. The particles having momentum greater than $p_o + \delta p$ and less than $p_o - \delta p$ enter the slit on opposite sides of the field-free region where they experience a constant transverse field. The polarity of the field is such that they will be deflected continuously away from the central plane ($x = 0$) as they travel forward. Thus the unwanted particles may be separated from the wanted ones farther and farther by increasing the length of the momentum slit.

The component B_y of the above-described ideal field changes discontinuously with x at the plane surfaces $x = \pm a$. It is not possible to excite such an ideal field unless current sheets are used at the surfaces of discontinuity. If this were allowed, this field could then be excited by the use of four infinite sheets of current located at $x = \pm a$ and $x = \pm b$. The current in these sheets should flow in the z-direction only. Let I_z denote the surface current in units of amperes per meter. If

$$I_z = \begin{cases} + I & \text{on} \quad x = \pm a \\ \\ - I & \text{on} \quad x = \pm b \end{cases} \tag{2}$$

the field inside the region of the slit will be as specified by Eqs. (1a) and (1b).

While the magnetic field does have the ideal configuration, the slit region is not free of obstruction to the passage of the beam. There are conducting planes $x = \pm a$ inside this region. These planes would intercept part of the wanted and the unwanted beams, giving rise to heating, radioactivity and scattering.

To avoid such difficulties the slit region should be free from any obstructing material. Under this condition the magnetic field inside the slit must be an analytic function of coordinates x and y. Hence no region of non-vanishing volume inside the slit can be entirely field-free. This, however, does not preclude the possibility that the field in the central region of the slit may be made sufficiently small. In this paper we will discuss the underlying principle for designing momentum slits, in which an approximately ideal magnetic field may be excited without resorting to conducting structures inside the area of the beam.

II. REPRESENTATION OF THE APERTURE FIELD IN TERMS OF MULTIPOLES

The transverse magnetic field in the momentum slit should have the following symmetry characteristics:

$$B_x(x,y) \quad \text{is} \quad \left\{ \begin{array}{ll} \text{even in} & x \\ \text{odd in} & y \end{array} \right\}; \tag{3a}$$

$$B_y(x,y) \quad \text{is} \quad \left\{ \begin{array}{ll} \text{odd in} & x \\ \text{even in} & y \end{array} \right\}. \tag{3b}$$

The conditions on B_y follow from the requirement that the unwanted charged particles are to be deflected in the x-directions away from the central plane $x = 0$. The conditions on B_x follow from those on B_y because $\nabla \cdot \vec{B} = 0$, i.e.,

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0. \tag{4}$$

From Eq. (4) and

$$\vec{B} = \nabla \times \vec{A}, \tag{5}$$

it is clear that $\vec{B}$ is derivable from a vector potential $\vec{A}$ which has only one non-vanishing component A_z. Since $\vec{B}$ satisfies the symmetry conditions (3a) and (3b),

$$A_z(x,y) \quad \text{is} \quad \left\{ \begin{array}{ll} \text{even in} & x \\ \text{even in} & y \end{array} \right\}. \tag{6}$$

In any region of free space, any dc magnetic field is both solenoidal and irrotational. Consequently the magnetic field inside the slit should also be derivable from a scalar potential $\psi(x,y)$.

We have

$$\vec{B} = -\mu_o \vec{\nabla}\psi \tag{7}$$

and

$$\nabla^2\psi = 0. \tag{8}$$

In view of the conditions (3a) and (3b) it is found that

$$\psi(x,y) \quad \text{is} \quad \begin{Bmatrix} \text{odd in} \quad x \\ \text{odd in} \quad y \end{Bmatrix}. \tag{9}$$

Equation (5) is valid in general. The representation of $\vec{B}$ in terms of the scalar potential is possible only in current-free regions. Herein one representation or the other will be used, whichever seems to be more convenient. Note that A_z, like ψ, satisfies the Laplace equation.

To consider the field near the center of the slit it is simpler to use cylindrical coordinates (ρ,φ,z) rather than Cartesian. The most general solution of the Laplace equation for $\psi(\rho,\varphi)$ having the required symmetry may be written as

$$\psi(\rho,\varphi) = \sum_{m=1}^{\infty} S_{2m}\rho^{2m} \sin 2m\varphi. \tag{10}$$

Here the S_{2m}'s are constant coefficients.

From ψ we obtain

$$B_x = -\mu_o \sum_{m=1}^{\infty} S_{2m}\, 2m\rho^{2m-1} \sin(2m-1)\varphi; \tag{11a}$$

$$B_y = -\mu_o \sum_{m=1}^{\infty} S_{2m} 2m\rho^{2m-1} \cos(2m - 1)\varphi. \tag{11b}$$

Now denote

$$\psi_{2\ell} \equiv \sum_{n=1}^{\infty} S_{\ell(2n-1)}\, \rho^{\ell(2n-1)} \sin \ell(2n - 1)\varphi. \tag{12}$$

Equation (12) represents the scalar potential in the aperture of a <u>regular</u> 2ℓ-pole magnet. For example,

$$\psi_4 = S_2\, \rho^2 \sin 2\varphi + S_6 \rho^6 \sin 6\varphi + \ldots .$$

is the regular quadrupole potential.[1] In terms of these multipole expressions Eq. (10) may be written as

$$\psi(\rho,\varphi) = \sum_{\substack{\nu=1 \\ (\ell=2^\nu)}}^{\infty} \psi_{2\ell}. \tag{13}$$

Both ψ and ψ_4 vanish on the planes $x = 0$ and $y = 0$. The derivative $\partial\psi_4/\partial\varphi$ vanishes on the planes $x = \pm y$, but $\partial\psi/\partial\varphi$ does not vanish on these planes. It seems to be appropriate to call ψ the potential function of an <u>irregular</u> quadrupole to distinquish it from that of a <u>regular</u> quadrupole In general, if the first non-vanishing term of ψ is $\rho^\ell \sin \ell\varphi$ but $\partial\psi/\partial\varphi$ does not have the same symmetries

[1]Albert Septier, "Strong-Focusing Lenses," <u>Advances in Electronics and Electron Physics</u>, (Academic Press, New York, 1961), Vol. XIV, p. 87.

as $\partial\psi_{2\ell}/\partial\varphi$, then ψ may be called the potential function of an <u>irregular</u> 2ℓ-pole.

Let $\psi'_{2\ell}$ denote an irregular 2ℓ-pole potential.

$$\psi'_{2\ell} \equiv \sum_{q=1}^{\infty} S_{\ell q}\, \rho^{\ell q} \sin \ell q\varphi . \tag{14}$$

Here ℓ is an even integer because of the symmetry conditions. The series of $\psi'_{2\ell}$ differs from the corresponding series of $\psi_{2\ell}$ in having the non-vanishing coefficients $S_{\ell q}$ with a $q = 2n$. Evidently, each individual term in the series of ψ in Eq. (10) may be represented by a series in terms of $\psi'_{2\ell}$, namely,

$$S_{2m}\, \rho^{2m} \sin 2m\varphi = \sum_{\substack{\nu=0 \\ (\ell=2m+2\nu)}}^{\infty} \psi'_{2\ell} . \tag{15}$$

Hence, the function ψ itself may also be so represented.

$$\psi(\rho,\varphi) = \sum_{\substack{\nu=1 \\ (\ell=2\nu)}}^{\infty} \psi'_{2\ell} . \tag{16}$$

Because of less stringent symmetry requirements, the expansion of ψ given by Eq. (16) in terms of irregular multipoles is often easier to use than the corresponding expansion given by Eq. (13) in terms of regular multipoles.

III. SUPERPOSITION OF MULTIPOLE FIELDS

The series of ψ discussed in the last section converges rapidly in the central region of the slit where ρ is small. In this region only the first few multipole terms are important. The higher-order terms contribute little because the effects of many positive and negative poles are cancelled almost completely near the axis of symmetry. To obtain an approximately ideal field in the slit, it is necessary to make the first few multipole terms vanish.

The first term is a quadrupole. This gives rise to a non-vanishing derivative $\partial\vec{B}/\partial\rho$ independent of ρ. There can be no quadrupole term if $\partial\vec{B}/\partial\rho$ vanishes on the axis $\rho = 0$. This condition may be obtained by superposing two quadrupoles $\psi'_4(\rho,\varphi)$ and $\psi'_4(\rho,\varphi\pm\pi/2)$, because from Eq. (14)

$$\psi'_4(\rho,\varphi) + \psi'_4(\rho,\varphi\pm\pi/2) = 2\sum_{q=1}^{\infty} S'_{4q}\, \rho^{4q} \sin 4q\varphi. \tag{17}$$

The resultant potential is an irregular octupole ψ'_8. The corresponding derivative $\partial\vec{B}/\partial\rho$ varies as ρ^2 when $\rho \to 0$.

Now consider a different octupole ψ''_8 having the same symmetries.

$$\psi''_8 = 2\sum_{q=1}^{\infty} S''_{4q}\, \rho^{4q} \sin 4q\varphi.$$

Evidently, it is possible to make $S''_4 = S'_4$ by varying the relative strength of the two octupoles. By subtracting one from the other we obtain

$$\psi'_8 - \psi''_8 = 2\sum_{q=2}^{\infty}\left(S'_{4q} - S''_{4q}\right) \rho^{4q} \sin 4q\varphi. \tag{18}$$

The dominant term of the resulting potential is a 16-pole. In this manner the first three terms (quadrupole, octupole, and dodecapole) have been suppressed from appearing in the potential function of the slit. This is possible provided that the potentials $\psi'_4(\rho,\varphi\pm\pi/2)$ and ψ''_8 are compatible with the symmetry of the physical system.

Several multipole systems are described in the Appendix to illustrate how the different kinds of multipole fields may be excited. For the sake of simplicity, all these systems are made of plane current sheets. Formulas for calculating the field pertaining to each system are presented.

An alternative way of suppressing the first few terms proceeds as follows. First subtract $\psi'_4(\rho,\varphi\pm\pi/2)$ from $\psi'_4(\rho,\varphi)$ to obtain a regular quadrupole, namely,

$$\psi'_4(\rho,\varphi) - \psi'_4(\rho,\varphi\pm\pi/2) = 2\sum_{q=1}^{\infty} S'_{4q-2}\,\rho^{4q-2}\sin(4q-2)\varphi. \qquad (19)$$

From this subtract another regular quadrupole which has the same first term. The resulting potential $\psi(\rho,\varphi)$ may be written as

$$\psi(\rho,\varphi) = \sum_{q=1}^{\infty} S_{4q+2}\,\rho^{4q+2}\sin(4q+2)\varphi. \qquad (20)$$

The dominant term of this potential is a dodecapole. If symmetry conditions permit, the latter term may similarly be suppressed. Then the result would be a potential whose dominant term is a 16-pole, as in the previous case.

IV. SIMPLE EXAMPLES OF MOMENTUM SLITS

Again consider the system discussed in the introductory section. In order that the slit opening may be free from obstructing conductors, the two strips of current sheets located at $x = \pm a$, having $-a \leq y \leq a$, must be cut off. For the purpose of calculation, cutting off two strips carrying current I is the same as installing two strips carrying current minus I in the same location. Thus, after cutting off these strips, the magnetic field will be equal to the original field plus another field produced by the following strips of currents:

$$I_z = -I \quad \text{on } x = \pm a, \qquad -a \leq y \leq a. \tag{21a}$$

As described in the Appendix, these surface currents constitute an irregular quadrupole. If this quadrupole is rotated by $\pi/2$ radians, another quadrupole is obtained for which

$$I_z = -I \quad \text{on } y = \pm a, \qquad -a \leq x \leq a. \tag{21b}$$

By installing this rotated quadrupole with the first quadrupole, we obtain an irregular octupole according to Eq. (17). This octupole system is the same as shown in Fig. A.4 in the Appendix, with $h = a$.

Then consider an octupole specified as follows:

$$I_z = I \quad \text{on} \quad \begin{cases} x = \pm a, & a \leq y \leq a + w \\ x = \pm a, & -(a + w) \leq y \leq -a \\ y = \pm a, & -(a + w) \leq x \leq -a \\ y = \pm a, & a \leq x \leq a + w \end{cases} \tag{22}$$

This is the same as shown in Fig. A.8 with $h = a$ and $\alpha = -1$. (See Appendix.) By properly choosing the two parameters α and w, the dominant term of the potential function of this irregular octupole may certainly be made equal to that of the other octupole specified by Eqs. (21a) and (21b). Thus, according to Eq. (18), the octupole potential can be eliminated by superposing these two systems in the opposite sense.

Taking $w = a/2$ and $\alpha = -1.15$ and using the formulas given in the Appendix, the field components B_x and B_y for unit value of $\mu_o I$ have been calculated. These quantities are shown in Figs. 1a, 1b, and 1c as functions of x for constant values of y. It can be seen from these curves that the field in the central region of the slit, $\rho \lesssim a$, is indeed very small.

Now consider a simple example of an iron slit system. For this case there is difficulty in justifying the possibility of eliminating the dodecapole term from symmetry arguments alone. However, even if this were not possible, a dodecapole term should already be small enough to be negligible for practical purposes. We thus adopt the second procedure discussed in the last section by considering a regular quadrupole magnet shown in Fig. 2.

The potential in the aperture $(|x| \le a, |y| \le a)$ of this magnet may be represented by the series of Eq. (19). To eliminate the dominant term of this quadrupole potential, another regular quadrupole system may be used. This leads to the consideration of the following system of current sheets:

$$I_z = \begin{cases} +I \text{ on } x = \pm a, & a \le y \le a + w \\ +I \text{ on } x = \pm a, & -(a+w) \le y \le -a \\ -I \text{ on } y = \pm a, & -(a+w) \le x \le -a \\ -I \text{ on } y = \pm a, & a \le x \le a + w \end{cases} \qquad (23)$$

This system is the same as shown in Fig. A.7 with $h = a$.

The calculation of the aperture field in the proximity of the magnetic iron cores is complicated by the fact that not only the currents but also the magnetization in the iron can affect the field in the aperture. Furthermore the magnetization varies with the currents. Without actually calculating the field it is possible, however, to ascertain that, when the above-described current sheets are installed in position in the magnet system shown by the dotted lines in Fig. 2, these surface currents together with the attendant change in the magnetization do indeed constitute a regular quadrupole. This is obvious because the two systems, before and after the introduction of the current sheets, are both regular quadrupoles of the same symmetry. Having w fixed at some

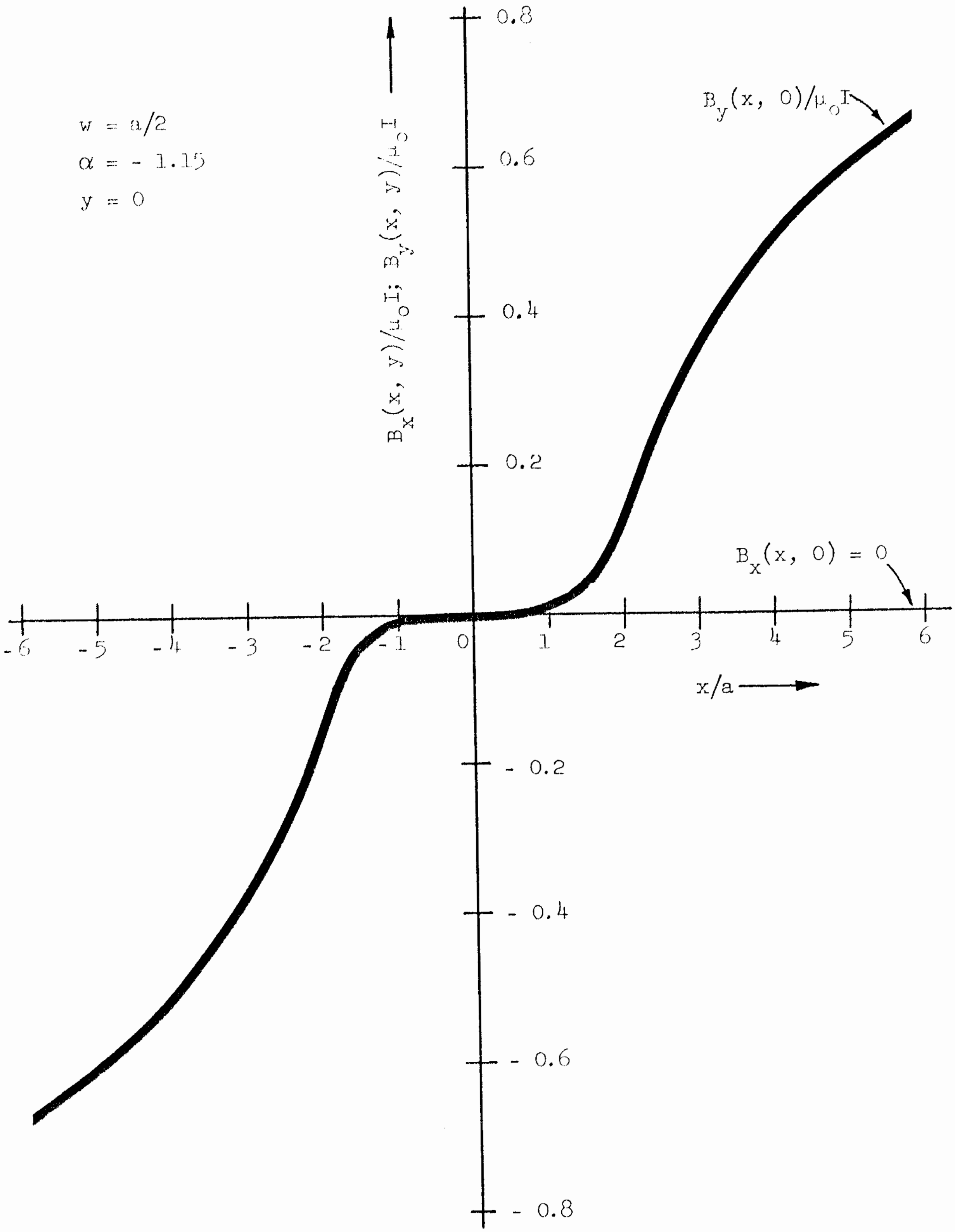

FIG. 1a--The magnetic field components shown are produced by two octupole current systems specified by Eqs. (21a), (21b) and (22) with $w = a/2$ and $\alpha = -1.15$.

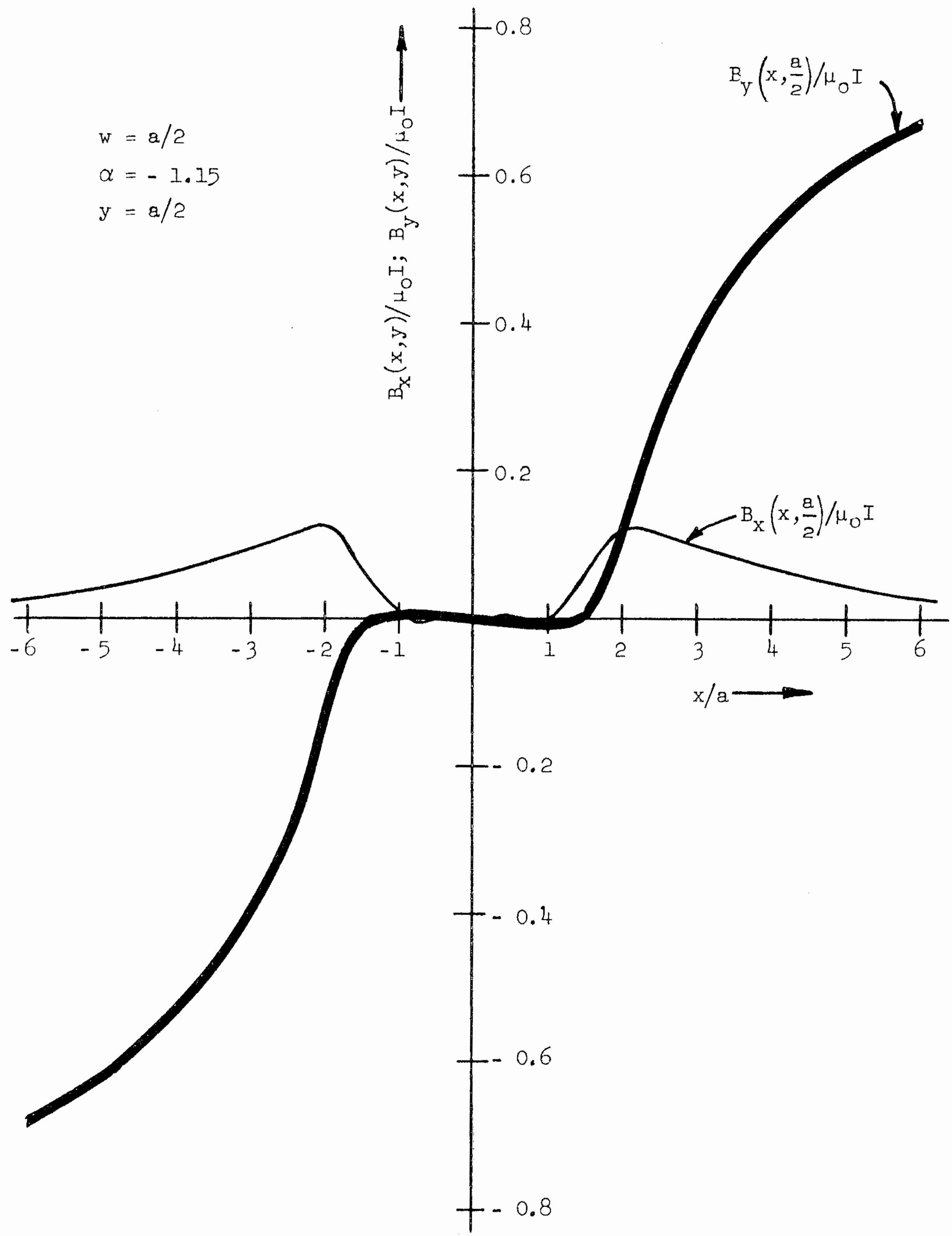

Fig. 1b--$B_x\left(x,\frac{a}{2}\right)$ and $B_y\left(x,\frac{a}{2}\right)$ are shown in units of $\mu_o I$. The source system is the same as specified in Fig. 1a.

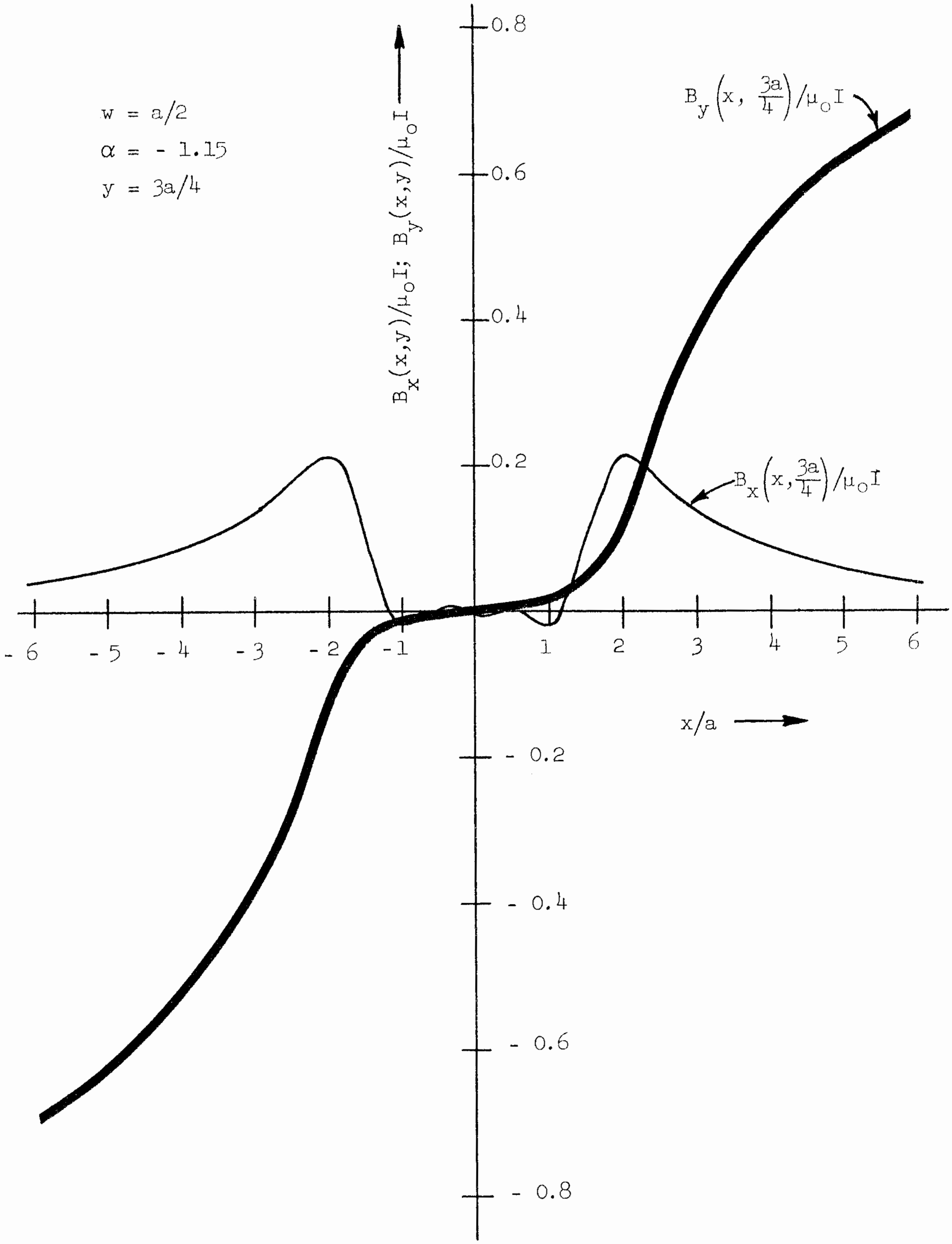

FIG. 1c--$B_x\left(x, \frac{3a}{4}\right)$ and $B_y\left(x, \frac{3a}{4}\right)$ are shown in units of $\mu_o I$. The source system is the same as specified in Fig. 1a.

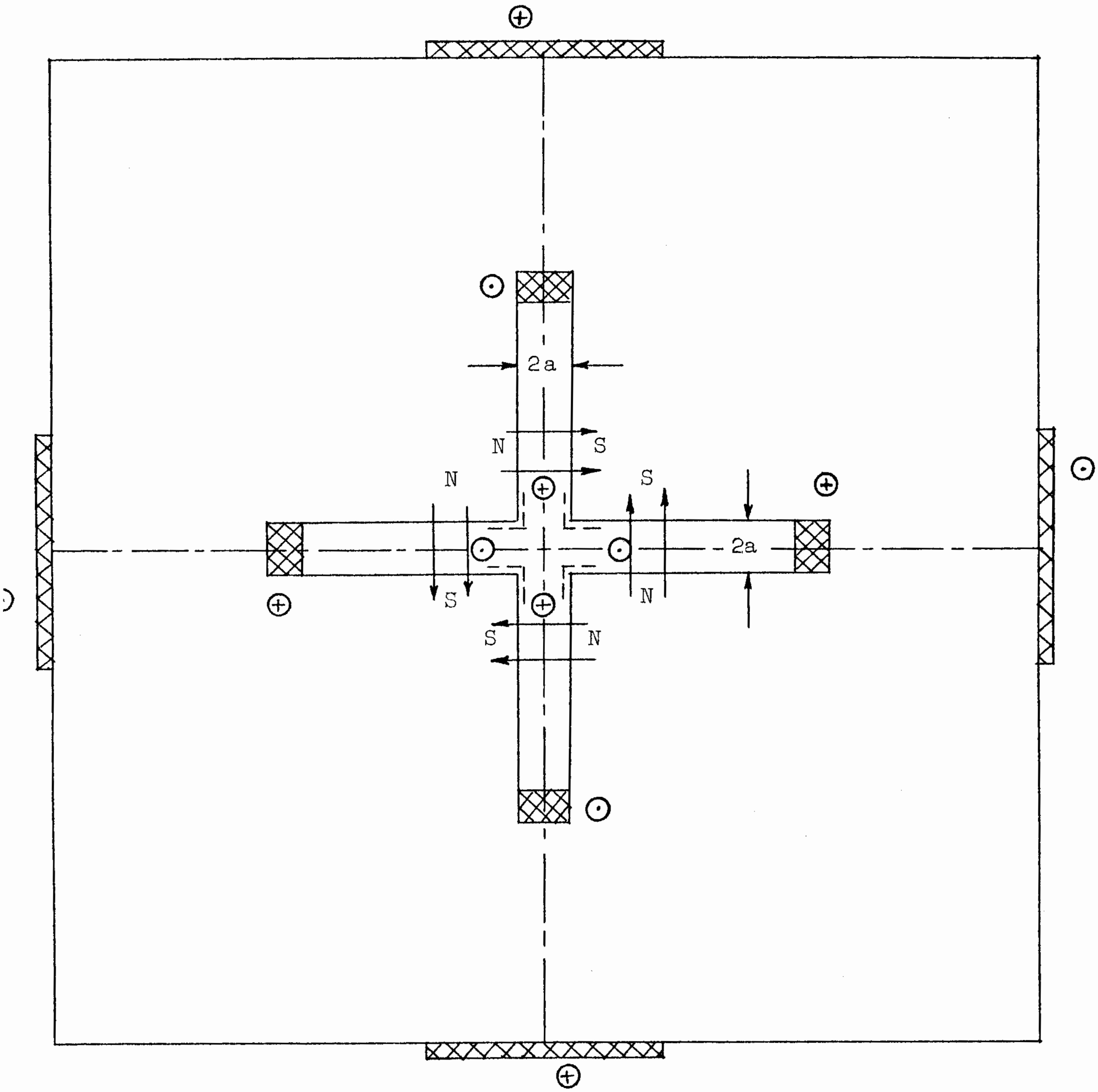

FIG. 2--The dotted lines around the pole tips represent the current sheets to be installed for correcting the aperture field. In the absence of these current sheets the iron magnet system is a regular quadrupole.

suitable value, the current I may be varied so that the first term in the series of ψ_4 vanishes.

We thus obtain a magnetic slit system for which the potential function in the region $\rho < a$ may be written as

$$\psi(\rho, \varphi) = S_6 \rho^6 \sin 6\varphi + S_{10} \rho^{10} \sin 10\varphi + \ldots\ldots \qquad (24)$$

The first term of this function is a dodecapole. It seems plausible that this term may also be eliminated by properly shaping the pole tips and by using suitable corrective currents. To determine whether this is true, detailed analysis is needed. The solution for the general case of the magnetostatic problem will be discussed in the following section.

V. THE MAGNETOSTATIC PROBLEM OF IRON SLITS

In the system of an iron slit, the vector potential $\vec{A}$ at a certain point (x', y', z') is given by[2]

$$4\pi\vec{A}(x') = \mu_o \int_{V(J)} \frac{\vec{J}}{r}\, d\tau + \mu_o \int_{V(M)} \vec{M} \times \vec{\nabla} \frac{1}{r}\, d\tau . \qquad (25)$$

[2]See, e.g., J. A. Stratton, "Electromagnetic Theory," Chap. IV, McGraw-Hill Co., Inc., New York (1941); W.K.H. Panofsky and M. Phillips, "Classical Electricity and Magnetism," Chaps. 7 and 8, Addison-Wesley Publishing Co., Inc., Cambridge, Massachusetts (1955).

Here, r is the distance between the field point (x') and the source point (x):

$$r = \left\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\right\}^{\frac{1}{2}}; \tag{26a}$$

$$\nabla^2 (1/4\pi r) = -\delta(x' - x) \; \delta(y' - y) \, \delta(z' - z). \tag{26b}$$

The first integral of Eq. (25) is taken over the total volume $V(J)$ of current carrying conductors, having permeability μ_o and volume current density $\vec{J}$. The second integral is taken over the total volume $V(M)$ of iron cores, having permeability μ and magnetization intensity $\vec{M}$. These two regions $V(J)$ and $V(M)$ are assumed to have no common parts.[3] The surfaces enclosing $V(J)$ and $V(M)$ will be denoted, respectively, by $S(J)$ and $S(M)$. Both regions are surrounded by free space, which constitutes the third region, having volume $V(0)$ and surface $S(0)$. The region of the slit opening is contained in $V(0)$.

Equation (25) is valid for any point (x') in all three regions. The vector function $\vec{A}$ is continuous everywhere and can be differentiated piecewise. Equation (25) tells us how to calculate $\vec{A}$ when $\vec{J}$ and $\vec{M}$ are known. But the magnetization $\vec{M}$ is induced by $\vec{J}$; we must first determine $\vec{M}$ in order to solve for $\vec{A}$. Instead, both $\vec{A}$ and $\vec{M}$ may be expressed in terms of one single unknown vector. Thus Eq. (25) is differentiated to obtain $\nabla \times \vec{A}$,

$$\frac{4\pi}{\mu_o} \nabla' \times \vec{A}\,(x') = \int_{V(J)} \vec{J} \times \vec{\nabla} \frac{1}{r} \, d\tau + \int_{V(M)} \left\{ (\vec{M} \cdot \vec{\nabla}) \, \vec{\nabla} \frac{1}{r} - \vec{M} \, \nabla^2 \frac{1}{r} \right\} d\tau,$$

[3] A finite separation between $V(J)$ and $V(M)$ is assumed to exist for the sake of simplicity. This entails no essential restriction and, if desired, may be dispensed with by going to the limit of zero separation.

i.e.,

$$\frac{4\pi}{\mu_o} \vec{B}(x') = 4\pi\vec{M}(x') + \int_{V(J)} \vec{J} \times \vec{\nabla} \frac{1}{r} d\tau + \int_{V(M)} (\vec{M} \cdot \vec{\nabla}) \vec{\nabla} \frac{1}{r} d\tau . \qquad (27)$$

Here it may be noted that $\vec{M}(x') = 0$ unless (x') is inside $V(M)$. Equation (27) is valid everywhere except on $S(M)$ where $\vec{B}$ is discontinuous. Since $\nabla \cdot \vec{B} = 0$, $\vec{B} = \mu\vec{H}$, and $\vec{M} = \left\{ (\mu/\mu_o) - 1 \right\} \vec{H}$, Eq. (27) may be rewritten as

$$4\pi\vec{H}(x') = \int_{V(J)} \vec{J} \times \vec{\nabla} \frac{1}{r} d\tau + \int_{V(M)} \left(\frac{\mu}{\mu_o} - 1\right) (\vec{H} \cdot \vec{\nabla}) \vec{\nabla} \frac{1}{r} d\tau . \qquad (28)$$

This integral equation enables us to solve for $\vec{H}$ from given $\vec{J}$ under the following boundary conditions:

$$\begin{aligned} &\vec{H} \text{ is continuous across } S(J); \\ &\vec{n} \times (\vec{H}_- - \vec{H}_+) = 0 \text{ on } S(M) ; \\ &\vec{n} \cdot (\mu\vec{H}_- - \mu_o\vec{H}_+) = 0 \text{ on } S(M). \end{aligned} \qquad (29)$$

In the last two equations $\vec{n}$ is the unit normal pointing outward from the closed surface $S(M)$, $\vec{H}_-$ is the magnetic field intensity just inside $S(M)$, and $\vec{H}_+$ is the magnetic field intensity just outside $S(M)$.

The interpretation of Eq. (28) is very simple. The first integral in Eq. (28) gives the direct contribution of all the currents to the magnetic field, as required by the Biot-Savart law. The second integral of Eq. (28) accounts for the effects arising from the magnetization induced by the currents. The latter effects may be described more clearly by transforming the second integral as follows:

$$\int_{V(M)} \left(\frac{\mu}{\mu_o} - 1\right) (\vec{H} \cdot \vec{\nabla}) \vec{\nabla} \frac{1}{r} \, d\tau$$

$$= \int_{V(M)} (\nabla \cdot \vec{H}) \vec{\nabla} \frac{1}{r} \, d\tau + \int_{S(M)} \left(\frac{\mu}{\mu_o} - 1\right) (\vec{n} \cdot \vec{H}_-) \vec{\nabla} \frac{1}{r} \, d\sigma. \tag{30}$$

The volume integral on the right side of Eq. (30) gives the magnetic field at point (x') due to the magnetic poles of volume density $\nabla \cdot \vec{H}$ located in $V(M)$. The surface integral in Eq. (30) gives the magnetic field at point (x') due to the magnetic poles of surface density $\vec{n} \cdot \vec{M}$ on the surface $S(M)$. Since there exists no isolated magnetic pole, the sum of the magnetic poles in $V(M)$ and on $S(M)$ must vanish. Indeed, we find that

$$\int_{V(M)} \nabla \cdot \vec{H} \, d\tau + \int_{S(M)} \vec{n} \cdot \vec{M} \, d\sigma = \frac{1}{\mu_o} \int_{V(M)} \nabla \cdot \vec{B} \, d\tau = 0.$$

Let us denote

$$\overleftrightarrow{G}(x, x') \equiv \frac{1}{4\pi} \vec{\nabla} \vec{\nabla} \frac{1}{r}. \tag{31}$$

This is a symmetric dyadic, because $\overleftrightarrow{G}(x, x') = \overleftrightarrow{G}(x', x)$ follows from $\vec{\nabla}\vec{\nabla}(1/r) = \vec{\nabla}'\vec{\nabla}'(1/r)$. Further denote

$$\vec{f}(x') \equiv \frac{1}{4\pi} \int_{V(J)} \vec{J} \times \vec{\nabla} \frac{1}{r} \, d\tau. \tag{32}$$

On introducing these notations, Eq. (28) becomes

$$\vec{H}(x') = \vec{f}(x') + \int_{V(M)} \left(\frac{\mu}{\mu_o} - 1\right) \vec{H}(x) \cdot \overleftrightarrow{G}(x, x') \, d\tau. \tag{33}$$

This is the vector form of Hilbert's polar integral equation[4]. There are several approximate methods for solving this equation. In particular, the variational method seems to offer special advantages. These methods will not be discussed here because they are quite involved.

One particular case, however, is relatively simple. This is the case where the permeability μ can be considered constant. This case is important because in many instances the assumption of constant μ is applicable. Under this assumption $\nabla \cdot \vec{H} = 0$. Thus Eq. (33) may be written as

$$\vec{H}(x') = \vec{f}(x') + \left(\frac{\mu}{\mu_o} - 1\right) \int_{S(M)} (\vec{n} \cdot \vec{H}_-) \vec{\nabla} \frac{1}{4\pi r} \, d\sigma .$$

Now let the point (x') approach (ξ') on the surface $S(M)$ through the point $(\xi' + \epsilon)$ from outside $S(M)$, ϵ being arbitrarily small, and let $\vec{n}'$ be the unit normal at (ξ') pointing outward from $S(M)$ and passing through $(\xi' + \epsilon)$. Then we obtain from the foregoing equation

$$\frac{1}{\mu_o} \vec{n}' \cdot \vec{B}(\xi') = \vec{n}' \cdot \vec{f}(\xi') + \left(\frac{1}{\mu_o} - \frac{1}{\mu}\right) \int_{S(M)} \vec{n} \cdot \vec{B}(\xi) \, \vec{n}' \cdot \vec{\nabla} \frac{1}{4\pi r(\xi, \xi' + \epsilon)} d\sigma . \tag{34}$$

Since

$$\int_{S(M)} \vec{n} \cdot \vec{B}(\xi) \, \vec{n}' \cdot \vec{\nabla} \frac{1}{4\pi r(\xi, \xi' + \epsilon)} d\sigma = \frac{1}{2} \vec{n}' \cdot \vec{B}(\xi') + \int_{S(M)} \vec{n} \cdot \vec{B}(\xi) \, \vec{n}' \cdot \vec{\nabla} \frac{1}{4\pi r(\xi, \xi')} \, d\sigma ,$$

[4]D. Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," Kap. XV, Chelsea Publishing Co., New York (1953); W. V. Lovitt, "Linear Integral Equations," pp. 187-192, McGraw-Hill Book Co., Inc., New York (1924).

Eq. (34) becomes

$$\frac{1}{2}\left(\frac{1}{\mu_o} + \frac{1}{\mu}\right)\vec{n}' \cdot \vec{B}(\xi') = \vec{n}' \cdot \vec{f}(\xi') + \left(\frac{1}{\mu_o} - \frac{1}{\mu}\right)\int_{S(M)} \vec{n} \cdot \vec{B}(\xi)\vec{n}' \cdot \vec{\nabla} \frac{1}{4\pi r(\xi,\xi')} \, d\sigma. \tag{35}$$

By further denoting

$$\left.\begin{aligned} K(\xi,\xi') &\equiv \vec{n}' \cdot \vec{\nabla}' \left\{1/r(\xi,\xi')\right\}, \\ \lambda &\equiv -\frac{1}{2\pi} \cdot \frac{\mu - \mu_o}{\mu + \mu_o}, \\ g(\xi') &\equiv \vec{n}' \cdot \vec{f}\,(\xi'), \end{aligned}\right\} \tag{36}$$

and

$$h(\xi') \equiv \frac{1}{2}\left(\frac{1}{\mu_o} + \frac{1}{\mu}\right) \vec{n}' \cdot \vec{B}(\xi'),$$

we obtain from Eq. (35)

$$h(\xi') = g(\xi') + \lambda \int_{S(M)} h(\xi)\, K(\xi,\xi')\, d\sigma. \tag{37}$$

This is an integral equation of <u>the second kind</u>,[5] which was first solved rigorously by Fredholm. Before Fredholm, the solution of this equation obtained by the method of successive substitutions could only be considered tentative.

From the asymmetric kernel $K(\xi,\xi')$ an infinite number of <u>iterated</u> functions may be derived. Let these functions be denoted by

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[5]E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis," Chap. XI, Cambridge University Press, London (1945); M.V. Lovitt, loc. cit. footnote 4.

$K_m(\xi,\xi')$, m = 1, 2, 3 . . . .

$$K_1(\xi,\xi') = K(\xi,\xi'), \tag{38a}$$

$$K_m(\xi,\xi') = \int_{S(M)} K_{m-1}(\xi,\xi_1)\, K(\xi_1,\xi')\, d\sigma_1. \tag{38b}$$

In terms of these functions Fredholm's solution of Eq. (37) may be written as

$$h(\xi') = g(\xi') + \sum_{m=1}^{\infty} \lambda^m \int_{S(M)} g(\xi)\, K_m(\xi,\xi')\, d\sigma. \tag{39}$$

Note that

$$\int_{S(M)} g(\xi')\, d\sigma' = 0 \tag{40}$$

and

$$\int_{S(M)} h(\xi')\, d\sigma' = 0. \tag{41}$$

Equation (40) is obvious. Equation (41) states the fact that there can be no isolated magnetic poles in existence. The latter equation can be proved by integrating Eq. (37) and using Eq. (40).

VI. SUMMARY AND DISCUSSION

The magnetic field in the central region of a momentum slit may be expanded into a series of either regular or irregular 2ℓ-poles, ℓ being even integers ≥ 2. For regular 2ℓ-poles, $\ell = 2^{\nu}$; for irregular 2ℓ-poles, $\ell = 2\nu$. The first few multipoles are the dominant terms. To design an effective momentum slit it is necessary to eliminate several lower-order terms. From pure symmetry arguments it has been shown that in the case of slit systems having no iron cores the first three terms ($\ell = 2$, 4, and 6) may be eliminated, and that in the case of iron slit systems at least the first two terms may be eliminated.

An example was given in Section IV illustrating how the aperture field in an air-core slit system can be made small by using simple corrective systems of surface currents. The return conductors of the corrective systems are supposed to be far removed from the axis of symmetry. By neglecting the effects of these return conductors, we have actually calculated the field distribution inside the slit opening. The maximum field in the region $\rho \lesssim a$ is only about 3.6×10^{-3} of the maximum field $\mu_0 I$ outside this region. If $\mu_0 I$ is taken to be 10 kilogauss, then $I \approx 8{,}000$ amp/cm and the current in the corrective current sheets is of the same order of magnitude as I. This current is too large to be actually used, unless the current is pulsed with a duty cycle of about 0.002 or smaller. It is also possible to use superconducting current sheets, though it seems at the present time to be not quite practical.

If iron cores are used and the maximum gap field is again 10 kilogauss, the main exciting current will be much smaller, reduced by a factor of about μ_0/μ. The corrective currents, however, remain of the same order of magnitude as 8,000 amp/cm. This follows from an approximate evaluation of the line integral of $\vec{H}$ over a closed contour, which encircles one strip of the corrective current and encloses an area, say, $(a - \epsilon) \leq x \leq (a + w + \epsilon)$ and $0 \leq y \leq (a + \epsilon)$ (see Fig. 2). The corrective system carries the equal and opposite currents, so there is no need of special return conductors.

Current sheets are not practical systems. They are used for the sole purpose of simplifying the calculation. Actual conducting planes may have a thickness varying from 0.2 cm to 1 cm. The aperture of the slit system may not always have a square cross section. It may be required that the aperture width should be adjustable in order to vary the chosen range of momentum. If the slit system is not sufficiently long in the z-direction, the end effects may become important. The actual design of an iron-core slit system will depend largely on such practical matters as mentioned above. It may turn out that, because of some practical restrictions, the dodecapole term of the aperture field is difficult to eliminate. As pointed out before, this possible limitation should not be of practical importance.

Section V was intended primarily for making detailed calculations for designing iron momentum slits. Two integral equations were derived: Equation (33) is valid in general; Equation (37) is valid for constant μ only. These equations should also be useful for calculating the fringing fields in other kinds of dc iron magnets.

APPENDIX

MULTIPOLE SYSTEMS AND FIELDS

As discussed in Section II, the magnetic field in the slit opening of a momentum analyzer must satisfy the symmetry conditions (3a) and (3b). Several multipole systems which satisfy this requirement are described below. To simplify the calculation of the magnetic field, all these systems are assumed to consist of only plane sheets of currents.

1. IRREGULAR QUADRUPOLE (TYPE I)

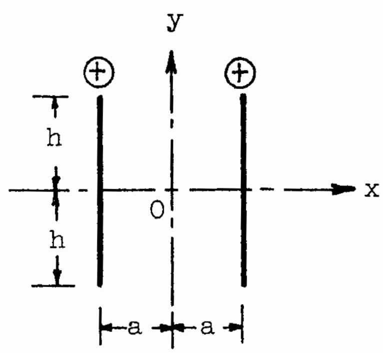

Fig. A.1. Two conducting planes are shown located at $x = \pm a$, $-h \leq y \leq h$, and $-\infty \leq z \leq \infty$. Each plane carries a current $I_z = - I$ amp/m. They constitute an irregular quadrupole.

$$B_x = \frac{\mu_o I}{4\pi} \left\{ \log \frac{(a+x)^2 + (h+y)^2}{(a+x)^2 + (h-y)^2} + \log \frac{(a-x)^2 + (h+y)^2}{(a-x)^2 + (h-y)^2} \right\}. \qquad (A.1a)$$

$$B_y = - \frac{\mu_o I}{2\pi} \left(\tan^{-1} \frac{h+y}{a+x} + \tan^{-1} \frac{h-y}{a+x} - \tan^{-1} \frac{h+y}{a-x} - \tan^{-1} \frac{h-y}{a-x} \right). \qquad (A.1b)$$

The arctangents in Eq. (A.1b) are to be less than or equal to $\pi/2$ radians in magnitude.

Using cylindrical coordinates (ρ, φ, z) we obtain the multipole expansions of the field components as follows:

$$B_x = \begin{cases} \dfrac{2\mu_o I}{\pi} \displaystyle\sum_{m=0}^{\infty} f_{2m+1}(\theta) \dfrac{\rho^{2m+1}}{a^{2m+1}} \sin(2m+1)\varphi, & \rho < a; \\ \\ \dfrac{2\mu_o I}{\pi} \displaystyle\sum_{m=0}^{\infty} F_{2m+1}(\theta) \dfrac{a^{2m+1}}{\rho^{2m+1}} \sin(2m+1)\varphi, & \rho > a \sec\theta. \end{cases} \tag{A.1c}$$

$$B_y = \begin{cases} \dfrac{2\mu_o I}{\pi} \displaystyle\sum_{m=0}^{\infty} f_{2m+1}(\theta) \dfrac{\rho^{2m+1}}{a^{2m+1}} \cos(2m+1)\varphi, & \rho < a; \\ \\ \dfrac{-2\mu_o I}{\pi} \displaystyle\sum_{m=0}^{\infty} F_{2m+1}(\theta) \dfrac{a^{2m+1}}{\rho^{2m+1}} \cos(2m+1)\varphi, & \rho > a \sec\theta. \end{cases} \tag{A.1d}$$

Here

$$\theta = \tan^{-1}(h/a), \tag{A.1e}$$

$$f_{2m+1}(\theta) = \int_0^{\theta} \cos^{2m}\varphi' \cos(2m+2)\varphi' \, d\varphi', \tag{A.1f}$$

and

$$F_{2m+1}(\theta) = \int_0^{\theta} \sec^{2m+2}\varphi' \cos 2m\varphi' \, d\varphi'. \tag{A.1g}$$

The first five multipole coefficients $f_{2m+1}(\theta)$ for the region $\rho < a$ are shown in Fig. A. 9 as functions of θ, $\theta \leq \pi/2$. The coefficients

$F_{2m+1}(\theta)$ for $\rho > a \sec\theta$ are related to $f_{2m+1}(\theta)$ by the simple equation

$$F_{2m+1}(\theta)\cos^{4m+2}\theta = f_{2m+1}(\theta). \qquad \text{(A.1h)}$$

2. IRREGULAR QUADRUPOLE (TYPE II)

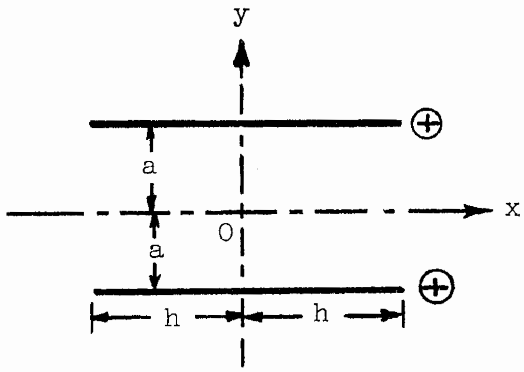

Fig. A.2. The system shown in Fig. A.1 is rotated about the z-axis by an angle of $\pi/2$ radians.

$$B_x = \frac{\mu_o I}{2\pi}\left(\tan^{-1}\frac{h-x}{a+y} + \tan^{-1}\frac{h+x}{a+y} - \tan^{-1}\frac{h-x}{a-y} - \tan^{-1}\frac{h+x}{a-y}\right). \qquad \text{(A.2a)}$$

$$B_y = \frac{\mu_o I}{4\pi}\left\{\log\frac{(a+y)^2+(h-x)^2}{(a+y)^2+(h+x)^2} + \log\frac{(a-y)^2+(h-x)^2}{(a-y)^2+(h+x)^2}\right\}. \qquad \text{(A.2b)}$$

$$B_x = \begin{cases} \dfrac{-2\mu_o I}{\pi}\displaystyle\sum_{m=0}^{\infty}(-)^m f_{2m+1}(\theta)\frac{\rho^{2m+1}}{a^{2m+1}}\sin(2m+1)\varphi, & \rho < a; \\[2ex] \dfrac{2\mu_o I}{\pi}\displaystyle\sum_{m=0}^{\infty}(-)^m F_{2m+1}(\theta)\frac{a^{2m+1}}{\rho^{2m+1}}\sin(2m+1)\varphi, & \rho > a\sec\theta. \end{cases} \qquad \text{(A.2c)}$$

$$B_y = \begin{cases} \dfrac{-2\mu_o I}{\pi} \displaystyle\sum_{m=0}^{\infty} (-)^m f_{2m+1}(\theta) \dfrac{\rho^{2m+1}}{a^{2m+1}} \cos(2m+1)\varphi, \ \rho < a; \\ \\ \dfrac{-2\mu_o I}{\pi} \displaystyle\sum_{m=0}^{\infty} (-)^m F_{2m+1}(\theta) \dfrac{a^{2m+1}}{\rho^{2m+1}} \cos(2m+1)\varphi, \ \rho > a \sec\theta \end{cases} \quad (A.2d)$$

3. REGULAR QUADRUPOLE

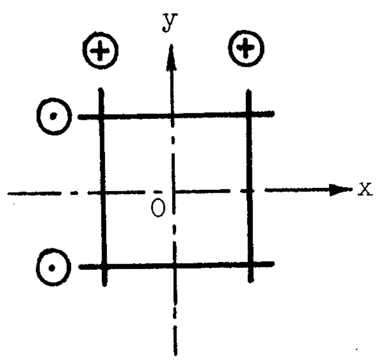

Fig. A.3. Two irregular quadrupoles shown in Figs. A.1 and A.2 are superposed in the opposite sense to form a regular quadrupole. $I_z = -I$ at $x = \pm a$, $-h \leq y \leq h$, and $-\infty \leq z \leq \infty$; $I_z = +I$ at $y = \pm a$, $-h \leq x \leq h$, and $-\infty \leq z \leq \infty$.

$$B_x = \text{R.H.S. of } \left\{ \text{Eq. (A.1a) - Eq. (A.2a)} \right\}. \quad (A.3a)$$

$$B_y = \text{R.H.S. of } \left\{ \text{Eq. (A.1b) - Eq. (A.2b)} \right\}. \quad (A.3b)$$

$$B_x = \begin{cases} \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n)}}^{\infty} f_{2m+1}(\theta) \dfrac{\rho^{2m+1}}{a^{2m+1}} \sin(2m+1)\varphi, & \rho < a; \\ \\ \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n+1)}}^{\infty} F_{2m+1}(\theta) \dfrac{a^{2m+1}}{\rho^{2m+1}} \sin(2m+1)\varphi, & \rho > a \sec\theta . \end{cases} \tag{A.3c}$$

$$B_y = \begin{cases} \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n)}}^{\infty} f_{2m+1}(\theta) \dfrac{\rho^{2m+1}}{a^{2m+1}} \cos(2m+1)\varphi, & \rho < a; \\ \\ -\dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n+1)}}^{\infty} F_{2m+1}(\theta) \dfrac{a^{2m+1}}{\rho^{2m+1}} \cos(2m+1)\varphi, & \rho > a \sec\theta . \end{cases} \tag{A.3d}$$

4. IRREGULAR OCTUPOLE

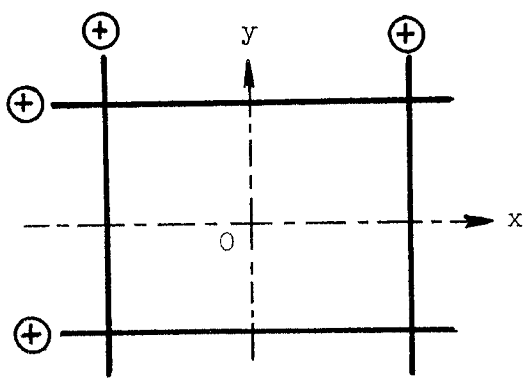

Fig. A.4. Two irregular quadrupoles shown in Figs. A.1 and A.2 are superposed in the same sense to form an irregular octupole. $I_z = -I$ at $x = \pm a$, $-h \le y \le h$, and $-\infty \le z \le \infty$; $I_z = -I$ at $y = \pm a$, $-h \le x \le h$, and $-\infty \le z \le \infty$.

$$B_x = \text{R.H.S. of Eq.}\left\{(A.1a) + \text{Eq. } (A.2a)\right\}. \qquad (A.4a)$$

$$B_y = \text{R.H.S. of Eq.}\left\{(A.1b) + \text{Eq. } (A.2b)\right\}. \qquad (A.4b)$$

$$B_x = \begin{cases} \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n+1)}}^{\infty} f_{2m+1}(\theta) \dfrac{\rho^{2m+1}}{a^{2m+1}} \sin(2m+1)\varphi, \; \rho < a ; \\[2ex] \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n)}}^{\infty} F_{2m+1}(\theta) \dfrac{a^{2m+1}}{\rho^{2m+1}} \sin(2m+1)\varphi, \; \rho > a \sec\theta . \end{cases} \qquad (A.4c)$$

$$B_y = \begin{cases} \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n+1)}}^{\infty} f_{2m+1}(\theta) \dfrac{\rho^{2m+1}}{a^{2m+1}} \cos(2m+1)\varphi, \; \rho < a ; \\[2ex] -\dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n)}}^{\infty} F_{2m+1}(\theta) \dfrac{a^{2m+1}}{\rho^{2m+1}} \cos(2m+1)\varphi, \; \rho > a \sec\theta . \end{cases} \qquad (A.4d)$$

5. IRREGULAR QUADRUPOLE (TYPE III)

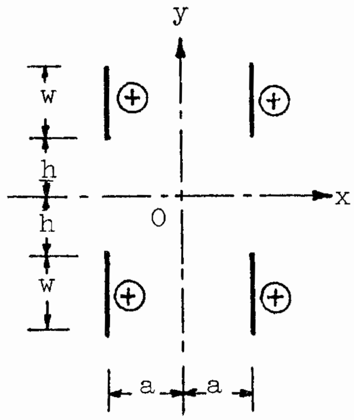

Fig. A.5. Four conducting plane strips are shown, located at $x = \pm a$, $h \leq |y| \leq h + w$, and $-\infty \leq z \leq \infty$. Each strip carries the current $I_z = -I$. They constitute an irregular quadrupole.

$$B_x = \frac{\mu_o I}{4\pi} \left\{ \log \frac{(a+x)^2+(h+w+y)^2}{(a+x)^2+(h+w-y)^2} + \log \frac{(a-x)^2+(h+w+y)^2}{(a-x)^2+(h+w-y)^2} \right\}$$

$$- \text{R.H.S. of Eq. (A.1a).} \qquad \text{(A.5a)}$$

$$B_y = -\frac{\mu_o I}{2\pi} \left(\tan^{-1} \frac{h+w+y}{a+x} + \tan^{-1} \frac{h+w-y}{a+x} - \tan^{-1} \frac{h+w+y}{a-x} - \tan^{-1} \frac{h+w-y}{a-x}\right)$$

$$- \text{R.H.S. of Eq. (A.1b).} \qquad \text{(A.5b)}$$

$$B_x = \begin{cases} \dfrac{2\mu_o I}{\pi} \displaystyle\sum_{m=0}^{\infty} \left\{ f_{2m+1}(\xi) - f_{2m+1}(\theta) \right\} \dfrac{\rho^{2m+1}}{a^{2m+1}} \sin(2m+1)\varphi, & \rho < a; \\ \\ \dfrac{2\mu_o I}{\pi} \displaystyle\sum_{m=0}^{\infty} \left\{ F_{2m+1}(\xi) - F_{2m+1}(\theta) \right\} \dfrac{a^{2m+1}}{\rho^{2m+1}} \sin(2m+1)\varphi, & \rho > a \sec \xi. \end{cases} \quad \text{(A.5c)}$$

$$B_y = \begin{cases} \dfrac{2\mu_o I}{\pi} \displaystyle\sum_{m=0}^{\infty} \left\{ f_{2m+1}(\xi) - f_{2m+1}(\theta) \right\} \dfrac{\rho^{2m+1}}{a^{2m+1}} \cos(2m+1)\varphi, & \rho < a; \\ \\ -\dfrac{2\mu_o I}{\pi} \displaystyle\sum_{m=0}^{\infty} \left\{ F_{2m+1}(\xi) - F_{2m+1}(\theta) \right\} \dfrac{a^{2m+1}}{\rho^{2m+1}} \cos(2m+1)\varphi, & \rho > a \sec \xi. \end{cases} \quad \text{(A.5d)}$$

Here, $\theta = \tan^{-1}(h/a)$ as before and

$$\xi = \tan^{-1}[(h+w)/a]. \quad \text{(A.5e)}$$

6. IRREGULAR QUADRUPOLE (TYPE IV)

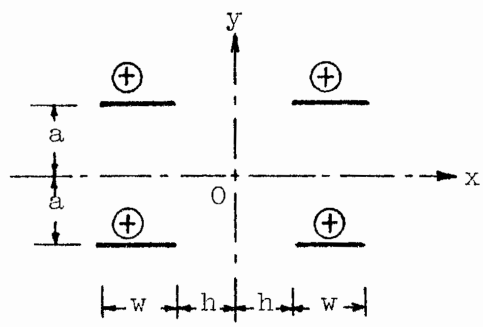

Fig. A.6. The system shown in Fig. A.5 is rotated about the z-axis by an angle of $\pi/2$ radians.

$$B_x = \frac{\mu_o I}{2\pi}\left(\tan^{-1}\frac{h+w-x}{a+y} + \tan^{-1}\frac{h+w+x}{a+y} - \tan^{-1}\frac{h+w-x}{a-y} - \tan^{-1}\frac{h+w+x}{a-y}\right)$$

$$- \text{R.H.S. of Eq. (A.2a).} \qquad \text{(A.6a)}$$

$$B_y = \frac{\mu_o I}{4\pi}\left\{\log\frac{(a+y)^2 + (h+w-x)^2}{(a+y)^2 + (h+w+x)^2} + \log\frac{(a-y)^2 + (h+w-x)^2}{(a-y)^2 + (h+w+x)^2}\right\}$$

$$- \text{R.H.S. of Eq. (A.2b).} \qquad \text{(A.6b)}$$

$$B_x = \begin{cases} -\dfrac{2\mu_o I}{\pi}\displaystyle\sum_{m=0}^{\infty}(-)^m\left\{f_{2m+1}(\xi) - f_{2m+1}(\theta)\right\}\dfrac{\rho^{2m+1}}{a^{2m+1}}\sin(2m+1)\varphi, & \rho < a; \\ \dfrac{2\mu_o I}{\pi}\displaystyle\sum_{m=0}^{\infty}(-)^m\left\{F_{2m+1}(\xi) - F_{2m+1}(\theta)\right\}\dfrac{a^{2m+1}}{\rho^{2m+1}}\sin(2m+1)\varphi, & \rho > a\sec\xi. \end{cases} \qquad \text{(A.6c)}$$

$$B_y = \begin{cases} -\dfrac{2\mu_o I}{\pi} \sum_{m=0}^{\infty} (-)^m \left\{ f_{2m+1}(\xi) - f_{2m+1}(\theta) \right\} \dfrac{\rho^{2m+1}}{a^{2m+1}} \cos(2m+1)\varphi, \ \rho < a ; \\ \\ -\dfrac{2\mu_o I}{\pi} \sum_{m=0}^{\infty} (-)^m \left\{ F_{2m+1}(\xi) - F_{2m+1}(\theta) \right\} \dfrac{a^{2m+1}}{\rho^{2m+1}} \cos(2m+1)\varphi, \ \rho > a \sec \xi. \end{cases} \quad (A.6d)$$

7. REGULAR QUADRUPOLE

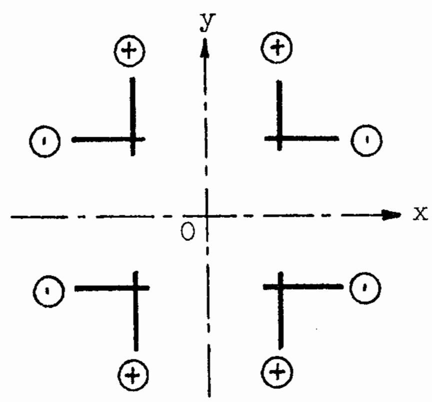

Fig. A.7. Two irregular quadrupoles shown in Figs. A.5 and A.6 are superposed in the opposite sense to form a regular quadrupole. $I_z = -I$ at $x = \pm a$, $-h \leq |y| \leq h + w$, and $-\infty \leq z \leq \infty$; $I_z = + I$ at $y = \pm a$, $h \leq |x| \leq h + w$, and $-\infty \leq z \leq \infty$.

$$B_x = \text{R.H.S. of} \left\{ \text{Eq. (A.5a) - Eq. (A.6a)} \right\} . \quad (A.7a)$$

$$B_y = \text{R.H.S. of} \left\{ \text{Eq. (A.5b) - Eq. (A.6b)} \right\} . \quad (A.7b)$$

$$B_x = \begin{cases} \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n)}}^{\infty} \left\{ f_{2m+1}(\xi) - f_{2m+1}(\theta) \right\} \dfrac{\rho^{2m+1}}{a^{2m+1}} \sin(2m + 1)\varphi, \ \rho < a; \\ \\ \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n+1)}}^{\infty} \left\{ F_{2m+1}(\xi) - F_{2m+1}(\theta) \right\} \dfrac{a^{2m+1}}{\rho^{2m+1}} \sin(2m + 1)\varphi, \ \rho > a \sec \xi. \end{cases} \quad (A.7c)$$

$$B_y = \begin{cases} \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n)}}^{\infty} \left\{ f_{2m+1}(\xi) - f_{2m+1}(\theta) \right\} \dfrac{\rho^{2m+1}}{a^{2m+1}} \cos(2m + 1)\varphi, \ \rho < a; \\ \\ -\dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n+1)}}^{\infty} \left\{ F_{2m+1}(\xi) - F_{2m+1}(\theta) \right\} \dfrac{a^{2m+1}}{\rho^{2m+1}} \cos(2m + 1)\varphi, \ \rho > a \sec \xi. \end{cases} \quad (A.7d)$$

8. IRREGULAR OCTUPOLE

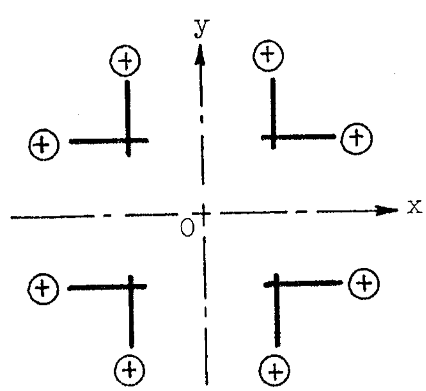

Fig. A.8 Two irregular quadrupoles shown in Figs. A.5 and A.6 are superposed in the same sense to form an irregular octupole. $I_z = -I$ at $x \pm a$, $h \leq |y| \leq h + w$, and $-\infty \leq z \leq \infty$; $I_z = -I$ at $y = \pm a$, $h \leq |x| \leq h + w$, and $-\infty \leq z \leq \infty$.

$$B_x = \text{R.H.S. of } \left\{ \text{Eq. (A.5a)} + \text{Eq. (A.6a)} \right\}. \qquad \text{(A.8a)}$$

$$B_y = \text{R.H.S. of } \left\{ \text{Eq. (A.5b)} + \text{Eq. (A.6b)} \right\}. \qquad \text{(A.8b)}$$

$$B_x = \begin{cases} \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n+1)}}^{\infty} \left\{ f_{2m+1}(\xi) - f_{2m+1}(\theta) \right\} \dfrac{\rho^{2m+1}}{a^{2m+1}} \sin(2m+1)\varphi, \; \rho < a ; \\[2ex] \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n)}}^{\infty} \left\{ F_{2m+1}(\xi) - F_{2m+1}(\theta) \right\} \dfrac{a^{2m+1}}{\rho^{2m+1}} \sin(2m+1)\varphi, \; \rho > a \sec \xi. \end{cases} \qquad \text{(A.8c)}$$

$$B_y = \begin{cases} \dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n+1)}}^{\infty} \left\{ f_{2m+1}(\xi) - f_{2m+1}(\theta) \right\} \dfrac{\rho^{2m+1}}{a^{2m+1}} \cos(2m+1)\varphi, \; \rho < a ; \\[2ex] -\dfrac{4\mu_o I}{\pi} \displaystyle\sum_{\substack{n=0 \\ (m=2n)}}^{\infty} \left\{ F_{2m+1}(\xi) - F_{2m+1}(\theta) \right\} \dfrac{a^{2m+1}}{\rho^{2m+1}} \cos(2m+1)\varphi, \; \rho > a \sec \xi. \end{cases} \qquad \text{(A.8d)}$$

9. MULTIPOLE COEFFICIENTS

Shown in Fig. A.9 are the multipole coefficients $f_{2m+1}(\theta)$ for the first five integral values of m. The other multipole coefficients $F_{2m+1}(\theta)$ may be calculated from $f_{2m+1}(\theta)$ according to Eq. (A.1h).

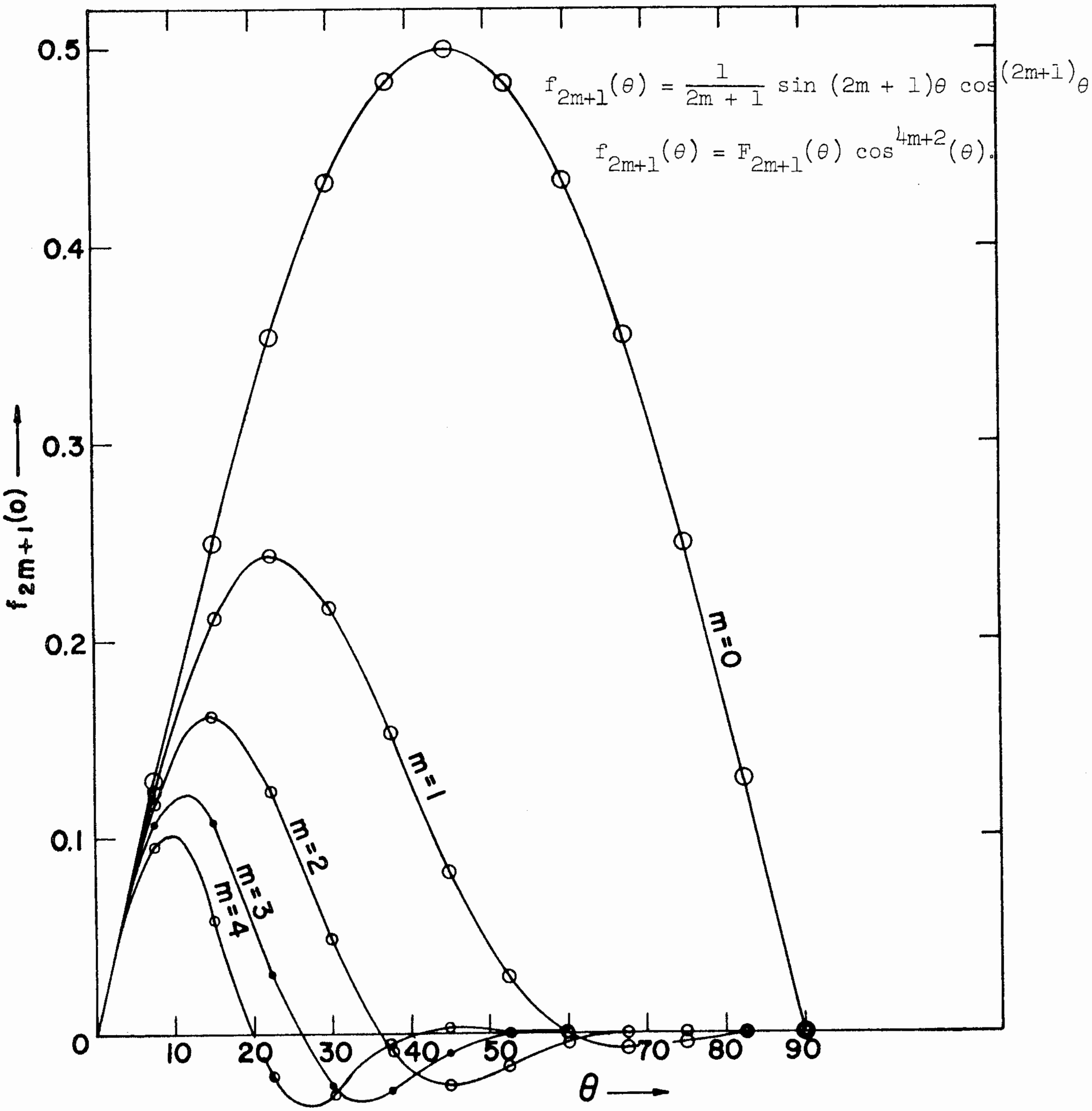

FIG. A.9--Multipole coefficients $f_{2m+1}(\theta)$ for the region $\xi < a$ shown as functions of θ, $\theta = \tan^{-1}(h/a)$.