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DISCUSSION OF FOCUSING REQUIREMENTS FOR THE STANFORD TWO-MILE ACCELERATOR

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by

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TABLE OF CONTENTS

LIST OF FIGURES

I. INTRODUCTION

A. OBJECT AND SCOPE

The object of the present paper is the discussion of the focusing requirements for the Stanford Two-Mile Linear Electron Accelerator under certain ideal conditions. The following assumptions will be made:

(1) The beam is highly relativistic (energy $>> mc^2$).

(2) All transverse forces from the rf field are ignored; hence in a region where no external focusing is provided, transverse momentum is constant.

(3) The machine itself and all focusing elements are perfectly aligned.

(4) No stray forces, such as uncompensated magnetic fields, are present.

Problems which specifically will be ignored are:

(1) Focusing in the injector.

(2) Initial focusing (up to 30 or 40 Mev) in the positron beam.

(3) Machine and quadrupole misalignment effects and tolerances.

(4) Possible residual effects of space-harmonic forces, coupler asymmetries, and other inherent non-ideal conditions.

(5) Aberrations* or nonlinear effects and all effects involving coupling between transverse and longitudinal coordinates.

Numerical examples will be used for illustrative purposes but no final design will be proposed because too many unknowns remain in the machine layout. Instead, an attempt will be made to summarize information and methods which will be useful in design of actual focusing systems. Much of the material in the present paper also will be useful in later discussions of the problems which are ignored here.

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*With the exception of chromatic aberration, which in effect is treated in Section V in considering the energy bandwidth for the transport of a positron beam.

B. PHASE-SPACE ARGUMENTS

Within the limitations set forth, the purpose of the focusing system will be the transmission of as much of the initial beam as will be useful at the end station.

A fundamental specification of the useful final beam is set by the phase-space admittance* of the beam extraction system. It is expected that the extraction system will be designed around a maximum angular divergence of $\approx (1 \text{ to } 4) \times 10^{-5}$ radian for a 1-cm-diameter beam. On the assumption that the phase space distribution is a uniformly filled right ellipse (Fig. 1),

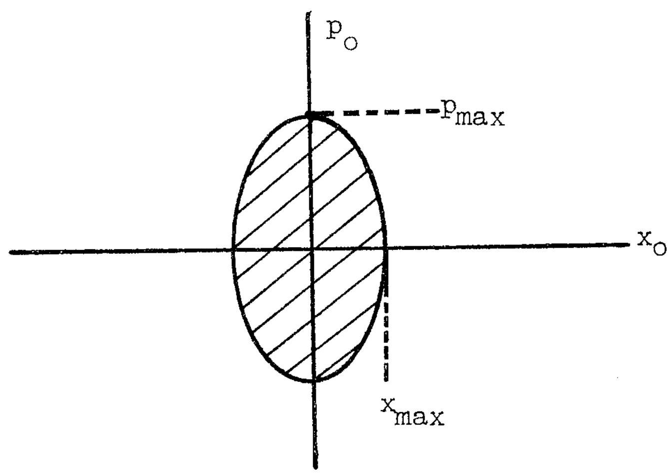

FIG. 1--Assumed initial phase space.

the implication is that the useful admittance of the machine is on the order of

$$A = \tfrac{1}{2}(1 \text{ to } 4)\pi\gamma \times 10^{-5} \quad (\text{mc-cm}) \qquad (1)$$

*The terms admittance and emittance will be used in the sense of Livingston and Blewett[1], and will refer to the appropriate phase-plane area for a typical transverse coordinate--e.g., the (x,p_x) or (y,p_y) phase planes. Liouville's Theorem tells us that for any element of transmitted phase-space, the phase-space density is preserved.

where γ is the final energy in mc^2 units (longitudinal momentum in units of mc), or

$$A = (1 \text{ to } 4) \frac{\pi V}{100} \quad (\text{mc-cm}) \qquad (1')$$

where V is final energy in Bev.

For a typical electron linear accelerator, the inherent phase space in the machine might be on the order of *

$$\nu = (0.01 \text{ to } 0.02)\pi \quad (\text{mc-cm}) \qquad (1\text{-}2)$$

which at a final energy of 10 Bev is well within the above specification of $\approx (0.1 \text{ to } 0.4)\pi$. Thus the focusing system should be designed to transport the entire electron beam.

For a positron beam, on the other hand, the initial phase space may be much larger. Using the approximate formulas quoted by Panofsky,[3] the effective values of x_{max} and p_{max} (Fig. 1) are on the order of 0.4 cm and 33 mc, respectively**; hence an initial phase space of

$$\nu_{total} \approx 13\pi \quad (\text{mc-cm}) \qquad (1\text{-}3)$$

is available. The maximum worthwhile strength of focusing would be determined by criterion Eq. (1) and the maximum percentage transmission, proportional to total transverse phase space, would be roughly

$$100\% \left(\frac{\text{admittance of system}}{\text{emittance of source}}\right)^2 \approx 100\% \left(\frac{0.4}{13}\right)^2 \approx 0.1\%$$

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*For the electron linac injector for the DESY Synchrotron, the figure is $\approx 0.011\pi$ mc-cm at 200 ma peak beam current.[2]

**Assuming a tantalum target about one shower-maximum in thickness.

C. SUMMARY OF SPECIFICATIONS AND PARAMETERS

The beam parameters may be summed up in terms of the phase-space specifications:

(1) Final phase space:

$$A \lessdot (1 \text{ to } 4) \frac{\pi V}{100} \quad (\text{mc-cm})$$

where V is final energy in Bev and A is the useful admittance of the machine.

(2) Initial phase space — electron beam:

$$v_{e^-} \lessdot (0.01 \text{ to } 0.02)\pi \quad (\text{mc-cm})$$

(3) Initial phase space — positron beam:

$$v_{e^+} \lessdot (0.1 \text{ to } 0.4)\pi \quad (\text{mc-cm})$$

(4) Initial energy: 30 to 40 Mev

Some pertinent machine parameters may be listed:

(1) Accelerator aperture: disc hole diameter ≈ 0.670 inch or radius = a ≈ 0.85 cm. (This corresponds to the smallest hole diameter which has been proposed as a possible design.)

(2) Accelerating field: in units of $mc^2/e/cm$, the electric field seen by a synchronous electron is

$$\alpha \approx 0.1(\text{Stage I, 1.5 Mev/ft})$$
$$\alpha \approx 0.2(\text{Stage II, 3.0 Mev/ft})$$

(3) Total length of machine: 10,000 ft.

II. TRANSMISSION OF THE ELECTRON BEAM

If we assume that the beam is injected into the machine with a phase-space distribution of the sort illustrated by Fig. 1, then the problem is the transmission of a phase-space area bounded by the ellipse

$$\frac{x_o^2}{x_{om}^2} + \frac{p_o^2}{p_{om}^2} = 1 \tag{2}$$

where (x_o, p_o) are the initial coordinates and (x_{om}, p_{om}) are the semi-axes of the ellipse.

Under the assumption of no transverse forces, the transformation to a later point in the machine is[4]

$$\left.\begin{aligned} p &= p_o \\ x &= x_o + \frac{p_o}{\alpha} \log\left(1 + \frac{\alpha z}{\gamma_o}\right) \end{aligned}\right\} \tag{3}$$

(assuming a constant accelerating field, α) where z is the longitudinal distance. It will be convenient to adopt the notation

$$\ell \equiv \frac{1}{\alpha} \log\left(1 + \frac{\alpha z}{\gamma_o}\right) \tag{4}$$

In matrix representation,

$$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ p_o \end{pmatrix} \tag{3'}$$

Transformation of the phase-space ellipse, Eq. (2), to $z = z_1$

gives

$$\frac{x_1^2}{x_{om}^2} - \frac{2\ell_1}{x_{om}^2} x_1 p_1 + \left(\frac{\ell_1^2}{x_{om}^2} + \frac{1}{p_{om}^2} \right) p_1^2 = 1 \tag{5}$$

which represents an ellipse skewed to the right (Fig. 2).

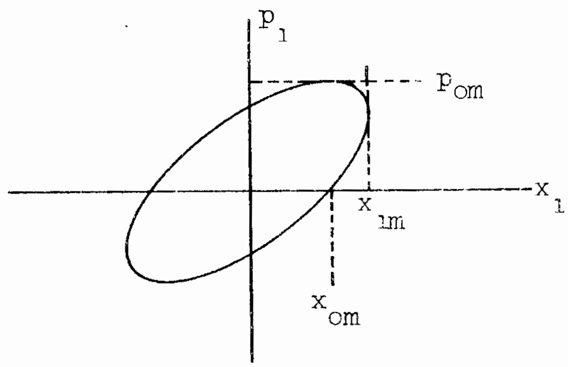

FIG. 2--Transformed phase-space ellipse with $z = z_1$.

For any ellipse of the general form

$$Ax^2 + Bxp + Cp^2 = 1 \tag{6}$$

the maximum value of x is given by

$$x_m^2 = \frac{C}{AC - \left(\frac{B}{2}\right)^2} \tag{7}$$

from which the maximum value of x_1 in Eq. (5) is

$$x_{1m}^2 = x_{om}^2 + \ell_1^2 p_{om}^2 \tag{8}$$

The maximum length of unfocused accelerator for which the transmission is 100% is given by setting x_{1m} equal to a:

$$\left.\begin{aligned} \ell_1(\max) &= \frac{\sqrt{a^2 - x_{om}^2}}{p_{om}} \\ &\text{or} \\ z_1(\max) &= \frac{\gamma_o}{\alpha}\left[\exp\left(\frac{\alpha\sqrt{a^2 - x_{om}^2}}{p_{om}}\right) - 1\right] \end{aligned}\right\} \qquad (9)$$

Using the parameters of Section I, in particular $\gamma_o = 60$ ($\approx$ 30 Mev), $x_{om} \approx 0.5$ cm, $p_{om} \approx 0.04$ mc, $a = 0.85$ cm, $\alpha = 0.1$ ($mc^2/e/cm$), we find

$$z_1(\max) \approx 90 \text{ ft}$$

$$\gamma_1 = \gamma_o + \alpha z_1 \approx 334 \text{ (171 Mev)};$$

thus at least one lens (solenoid or quadrupole multiplet) would be needed within the first 90 feet after the injector.

The situation is somewhat more favorable if the initial phase space is matched to the accelerator admittance. In Fig. 3, the parallelogram indicates the maximum phase space which can be transmitted through a length z, with a maximum beam radius of r_{max}.

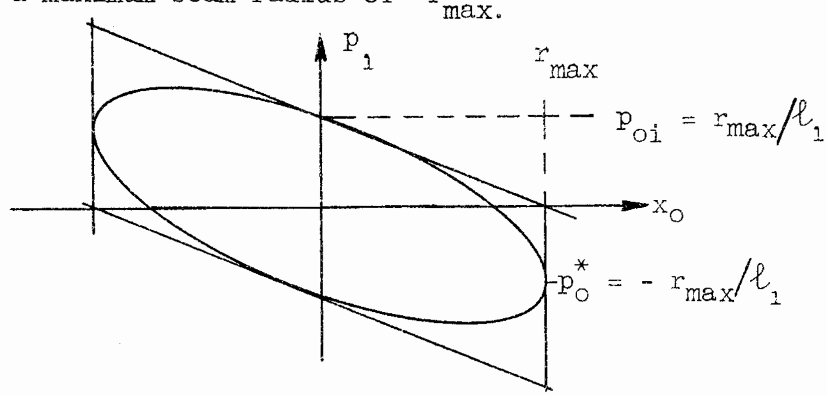

FIG. 3--Illustrating admittance and phase-space matching in an unfocused accelerator.

The inscribed ellipse represents the maximum matched beam phase space which may be transmitted. This configuration could be attained, for example, by letting the beam expand to the maximum desired diameter within the injector, and then providing a lens at the end of the injector to skew the distribution along the p-axis; the proper strength of the lens would be determined by maximization of transmission.

Under these conditions (initial phase-space matching), the maximum length of unfocused machine is given by

$$\ell_1 \equiv \frac{1}{\alpha} \log \left(1 + \frac{\alpha z_1}{\gamma_o} \right) = \frac{r_{max}}{p_{oi}} = \frac{\pi r_{max}^2}{\nu} \qquad (10)$$

where $\nu = \pi r_{max} p_{oi}$ is the phase space of the beam. Taking $r_{max} = a =$ 0.85 cm, $\alpha = 0.1$, and $\nu = 0.02\pi$(mc-cm), as in the previous numerical example, we find

$$\ell_1 \text{(max)} = 36.1$$

$$z_1 \text{(max)} = \frac{\gamma_o}{\alpha} [\exp(\alpha \ell_1) - 1] \approx 700 \text{ ft}$$

$$\gamma_1 = \gamma_o + z_1 \alpha \approx 2190 \text{ (1.1 Bev)}$$

The phase space at z_1 would be as shown in Fig. 4.

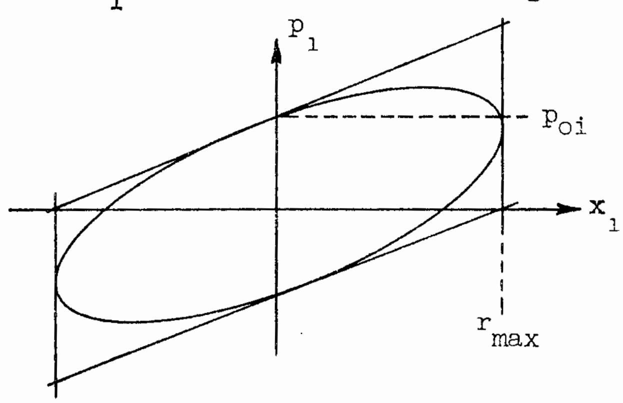

FIG. 4--Phase space at z_1 under initial phase-space matching conditions.

This would be transformed back to the form of Fig. 3 by a thin converging lens of strength

$$\frac{\gamma_1}{F_1} = \frac{2p_{oi}}{r_{max}} = \frac{2}{\ell_1} \tag{11}$$

since the transformation for a thin lens is given by

$$\begin{pmatrix} 1 & 0 \\ -\gamma_1/F_1 & 1 \end{pmatrix} \tag{12}$$

The maximum unfocused length after this lens would be given by

$$\ell_2 = \ell_1 = \frac{\pi r^2_{max}}{\nu}$$

where (13)

$$\ell_2 = \frac{1}{\alpha} \log \left(1 + \frac{z_2 - z_1}{\gamma_1/\alpha}\right)$$

In fact, the beam could be kept within the limit $|x| \leqslant r_{max}$ by a succession of lenses of strength

$$\frac{\gamma_n}{F_n} = \frac{\gamma_1}{F_1} \tag{14}$$

and spaced by

$$
\left.
\begin{aligned}
\ell_n &= \ell_1 \\
\text{where} \\
\ell_n &= \frac{1}{\alpha} \log \left(1 + \frac{z_n - z_{n-1}}{\gamma_n/\alpha} \right) \\
\text{or} \\
z_n &= \frac{\gamma_0}{\alpha} \left[\exp(n\alpha\ell_1) - 1 \right]
\end{aligned}
\right\} \qquad (15)
$$

Continuing the above numerical example with $r_{max} = a$, $\alpha = 0.1(mc^2/e/cm)$, $\nu = 0.02\pi$, $\gamma_1 = 2190$, $z_1 = 700$ ft, we find

$$F_1 \approx 1300 \text{ ft (at 1.1 Bev)}$$

$$z_2(\max) \approx 2.7 \times 10^4 \text{ ft}$$

Since this position is beyond the end of the machine, only one lens after the injector is required in this case.

As a second example, suppose we wish to keep the beam radius to $\leqslant$ 0.5 cm. Taking as before $\gamma_o = 60$, $\alpha = 0.1$, $\nu = 0.02\pi$, we find in this case

$$\ell_1 = 12.5$$

The locations and focal lengths of the lenses are listed in Fig. 5.

n	z_n	Beam Energy	Focal Length
1	49 ft	0.107 Bev	43 ft
2	222	0.374	150
3	820	1.31	527
4	2,920	4.65	1,870
5	10,100 (not needed)		

FIG.5--Table of lens locations and focal lengths.

This example shows that in order to gain a modest reduction in beam size, we need an appreciable number of lenses. There may be other reasons for wanting focusing considerably stronger than the minimal requirement for electron transmission; for example, to allow for perturbing effects leading to non-conservatism of transverse phase space, to increase the yield in the positron beam, or to take advantage of the beam-guiding properties of a focusing system. The use of periodic strong focusing magnetic quadrupoles, of alternating sign and more or less equal spacing, is the most efficient known method of providing such focusing.

To facilitate the discussion of design parameters, some of the properties of such systems will be summarized in the next section.

III. PERIODIC STRONG-FOCUSING SYSTEMS*

A. GENERAL PROPERTIES OF THE PERIODIC SYSTEM[5]

Consider a system in which the focusing elements repeat periodically with a period Λ along the length. The transfer matrix for x and p_x between any pair of consecutive reference planes z, $z + \Lambda$ is

$$\begin{pmatrix} x_n \\ p_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{n-1} \\ p_{n-1} \end{pmatrix} \tag{16}$$

with the periodicity defined by

$$a_{ij}(z) = a_{ij}(z + \Lambda) \tag{17}$$

The determinant of the matrix is unity. The transfer matrix for n cascaded sections is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^n = \begin{pmatrix} A_{11}(n) & A_{21}(n) \\ A_{21}(n) & A_{22}(n) \end{pmatrix}$$

where

$$\left.\begin{aligned} A_{11}(n) &= \cos n\theta + \frac{\left(a_{11} - a_{22}\right)}{2} \frac{\sin n\theta}{\sin \theta} \\ A_{12}(n) &= a_{12} \frac{\sin n\theta}{\sin \theta} \\ A_{21}(n) &= a_{21} \frac{\sin n\theta}{\sin \theta} \\ A_{22}(n) &= \cos n\theta - \frac{\left(a_{11} - a_{22}\right)}{2} \frac{\sin n\theta}{\sin \theta} \end{aligned}\right\} \tag{18}$$

---

*The material in Sections III.A and III.B is mainly from Bell, Reference 5.

and the parameter θ is defined by

$$\cos \theta = \tfrac{1}{2}\left(a_{11} + a_{22}\right) \tag{19}$$

For such a system the function U, defined by

$$U = - a_{21}x^2 + \left(a_{11} - a_{22}\right) xp + a_{12}p^2 \tag{20}$$

is invariant under any change in reference planes (not necessarily by an integral number of periods). That is, an orbit whose phase space coordinates initially satisfy a given value of U will lie on a curve having the same value of U at any reference plane further along in the system. Equation (20) defines a family of ellipses in the (x,p) plane for U = constant, provided that the parameter θ, defined by Eq. (19), is real (i.e., $\left|a_{11} + a_{22}\right| \leqslant 2$). ($\theta$ also is invariant under a translation of reference planes.)

The area enclosed by the ellipse, U = constant, is

$$\nu = \int_U dxdp = \frac{\pi U}{\sin \theta} \tag{21}$$

The curve, U = constant, may be said to define a characteristic admittance surface since any phase-space element within this curve at one reference plane in the system will also be within it at any other reference plane.

The maximum amplitude of a given orbit may be found from the invariance of U; it is

$$x^2_{max} = \frac{a^*_{12}U}{\sin^2 \theta} \tag{22}$$

where $\left|a^*_{12}\right|$ is the maximum value of $\left|a_{12}(z)\right|$.

In a system of finite aperture, the maximum value of $|U|$ is given by setting $x_{max} = a$:

$$U^* = \frac{a^2 \sin^2 \theta}{a_{12}^*} \tag{23}$$

where a is the radial aperture. Hence the admittance, or maximum transmitted area of phase space is, by Eqs. (21) and (23),

$$A = \int_{U^*} dxdp = \frac{\pi a^2 \sin \theta}{a_{12}^*} \tag{24}$$

B. ADIABATIC VARIATION OF THE PARAMETERS[5]

Consider a system which is approximately periodic in the sense that Λ and a_{ij} vary <u>slowly</u> with z; i.e.,

$$\left.\begin{aligned} &\left|\frac{\Lambda(z + \Lambda) - \Lambda(z)}{\Lambda(z)}\right| \ll 1 \\ &\text{and} \\ &\left|\frac{a_{ij}(z + \Lambda) - a_{ij}(z)}{a_{ij}(z)}\right| \ll 1 \end{aligned}\right\} \tag{25}$$

For such a system, U is no longer invariant but the area enclosed by U is;

$$\left.\begin{aligned} V &= \frac{\pi U}{\sin \theta} = \frac{\pi}{\sin \theta}\left[- a_{21}x^2 + \left(a_{11} - a_{22}\right) xp + a_{12}p^2\right] \\ &= \text{invariant under translation.} \end{aligned}\right\} \tag{26}$$

The amplitude of a given orbit now is

$$x^2_{max} = \frac{a^*_{12}}{\sin\,\theta} \cdot \frac{\nu}{\pi} = \frac{a^*_{12}}{U^*} \frac{\nu^2}{\pi^2} \tag{27}$$

Hence the admittance of a system of constant aperture, a, is

$$A = \pi a^2 \left(\frac{\sin\,\theta}{a^*_{12}}\right)_{min} = \pi a \sqrt{\left(\frac{U^*}{a^*_{12}}\right)_{min}} \tag{28}$$

NOTE: a^*_{12} refers to the maximum within a period, while $\left(\sin\,\theta/a^*_{12}\right)_{min}$ refers to the minimum over the whole length.

C. PROPERTIES OF A SYSTEM OF THIN-LENS ALTERNATING-GRADIENT QUADRUPOLES

A suitable strong-focusing system might consist of thin quadrupole lenses of alternating sign, equally spaced with a separation of $L = \frac{1}{2}\Lambda$.

The transfer matrix for a thin quadrupole in one of its two principal planes of symmetry is

$$\begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix}$$

where

$$Q = \int_{quad} \frac{\partial B_y}{\partial x}\, dz \tag{29}$$

The matrix in the other principal plane is the same except the sign of Q is reversed. [B and Q are presumed to be measured in units of (mc^2/e)/unit length = 1703 gauss if lengths are in cm; momenta are in units of mc.]

The matrix for a drift space is

$$\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \tag{30}$$

where

$$\ell = \frac{1}{\alpha} \log \left(1 + \frac{\alpha L}{\gamma_1}\right);$$

α = accelerating field (assumed constant) in the same units as B and Q;

γ_1 = the initial energy in mc^2 units ($\gamma_1 \gg 1$ is assumed);

L = distance between quadrupoles.

In the limit $\alpha L \ll \gamma_1$ (negligible energy change),

$$\ell \approx \frac{L}{\gamma} \qquad (30')$$

It will be interesting to consider two choices of the periodic reference planes:

(1) at the center of a quadrupole, and

(2) at the center of a drift space.

1. <u>Case (1): Reference Planes at Centers of Quadrupoles</u>

In this case the basic section is* as shown in Fig. 6.

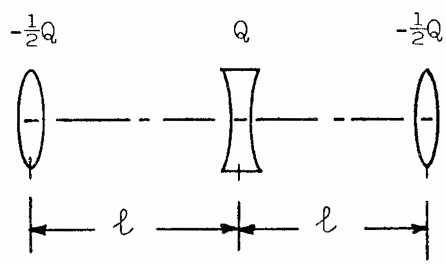

FIG. 6--Basic section of thin-lens alternating-gradient quadrupole system with reference planes at centers of quadrupoles.

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*To make the system periodic in n it is assumed that $\ell \propto \log(1 + \alpha L_n/\gamma_n)$, rather than L, is constant. This is equivalent to one of the "scaled" periodic systems discussed by Kyhl.[6]

The corresponding matrix is given by

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{2}Q & 1 \end{pmatrix}\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix}\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{2}Q & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{1}{2}Q^2\ell^2 & 2\ell(1 + \frac{1}{2}Q\ell) \\ -\frac{1}{2}Q^2\ell(1 - \frac{1}{2}Q\ell) & 1 - \frac{1}{2}Q^2\ell^2 \end{pmatrix} \qquad (31)$$

Applying the results quoted in Section III.A, we have

$$\cos\theta = 1 - \tfrac{1}{2}Q^2\ell^2 \qquad (32)$$

$$\sin\theta = |Q|\ell\sqrt{1 - \tfrac{1}{4}Q^2\ell^2} \qquad (33)^*$$

It may be shown that Eq. (31) defines the maximum a_{12}, provided that Q is positive:

$$a^*_{12} = 2\ell(1 + \tfrac{1}{2}|Q|\ell) \qquad (34)$$

Hence

$$x^2_{max} = \frac{2U}{Q^2\ell(1 - \frac{1}{2}|Q|\ell)} \qquad (35)$$

and

$$U^* = \tfrac{1}{2}Q^2\ell(1 - \tfrac{1}{2}|Q|\ell)a^2 \qquad (36)$$

---

*The sign of $\sin\theta$ has been chosen to make the phase space area, ν [Eq. (21)], positive.

The phase-space admittance is

$$A = \frac{\pi U^*}{\sin\theta} = \frac{\pi}{2}|Q|a^2\sqrt{\frac{1-\frac{1}{2}|Q|\ell}{1+\frac{1}{2}|Q|\ell}} \tag{37}$$

If the parameters vary adiabatically, the variation in amplitude of any orbit is given, with the help of Eq. (27), by

$$x^2_{max} \propto \frac{1}{|Q|}\sqrt{\frac{1+\frac{1}{2}|Q|\ell}{1-\frac{1}{2}|Q|\ell}} \tag{38}$$

Note that if Q and L are held constant in a long machine, then

$$\ell = \frac{1}{\alpha}\log\left(1+\frac{\alpha L}{\gamma}\right)$$

$$\to 0 \qquad \text{as} \qquad \gamma \to \infty$$

so that the beam size shrinks appreciably according to Eq. (38) (e.g., by a factor of $3^{\frac{1}{4}} = 1.3$ if $|Q|\ell$ is initially unity). This effect may be pictured as "smoothing out the jags" in the orbits.

2. Case (2): Reference Planes at Center of Drift Space

In this case the basic period is as shown in Fig. 7.

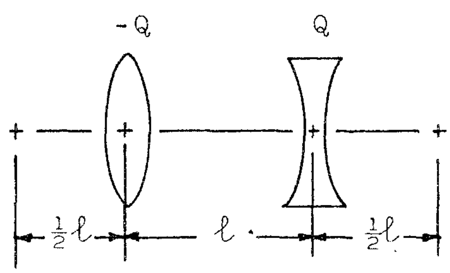


FIG. 7--Basic period of thin-lens alternating-gradient quadrupole system with reference planes the center of drift space.

The transfer matrix turns out to be

$$\begin{pmatrix} 1 - Q\ell - \frac{1}{2}Q^2\ell^2 & 2\ell\left(1 - \frac{1}{8}Q^2\ell^2\right) \\ -Q^2\ell & 1 + Q\ell - \frac{1}{2}Q^2\ell^2 \end{pmatrix} \tag{39}$$

The parameter θ and the admittance area are of course the same as in Case (1) above.

3. Circularly Symmetric System

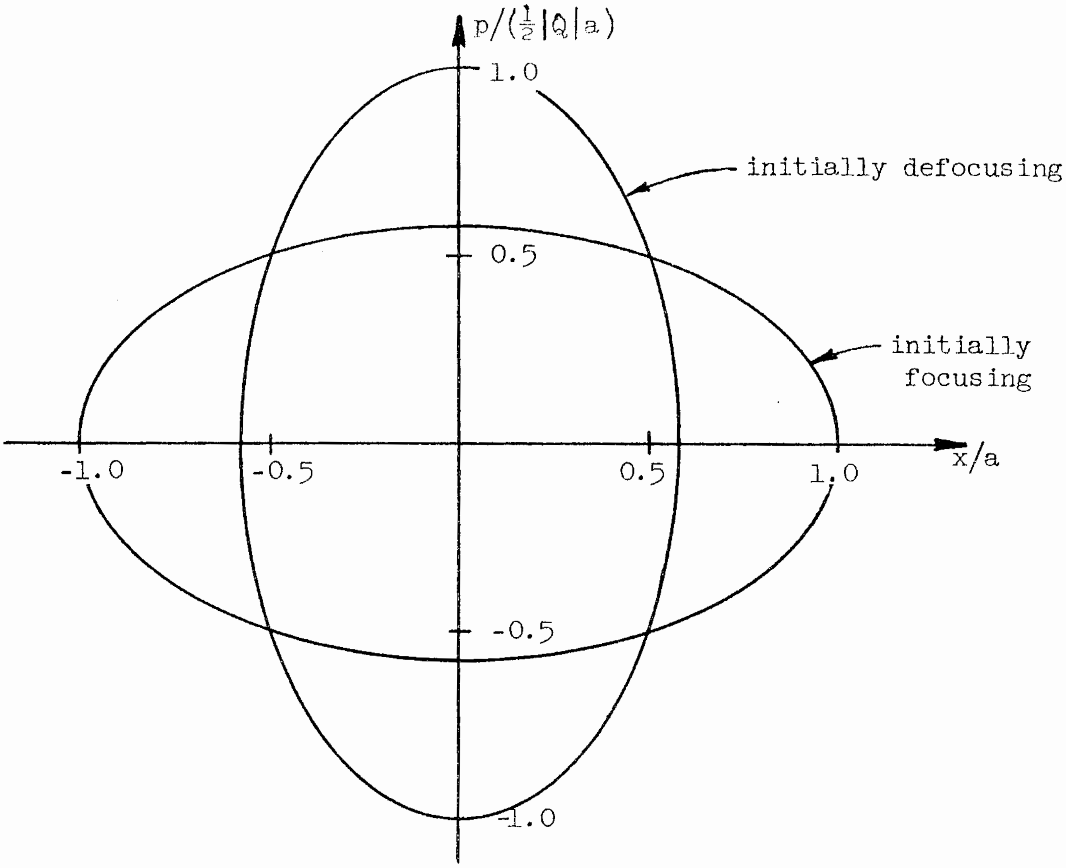
The characteristic admittance ellipses, for the two examples given above, are shown in Fig. 8 for the case of $Q\ell = \pm 1$. Note that both systems are quite unsymmetric with respect to the initially focusing and initially defocusing (x and y) planes. However, the admittance of Case (2) can be made circularly symmetric (i.e., equivalent in the x and y planes) by preceding it by a "matching" quadrupole, Q_M, given by

$$Q_M = \frac{Q}{2(1 - \frac{1}{8}Q^2\ell^2)} \tag{40}$$

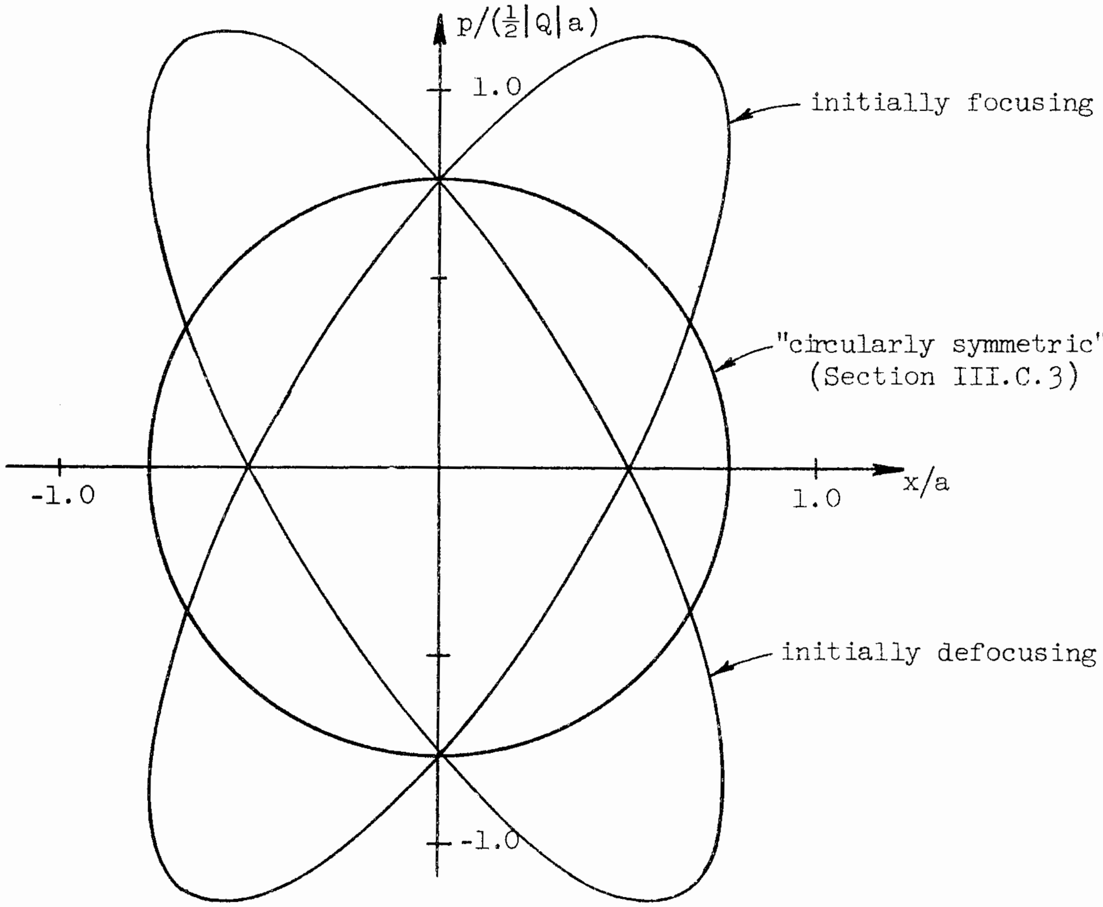
It turns out that the emittance also becomes circularly symmetric if the section is followed by a quadrupole of value $-Q_M$. Thus a special case of a circularly symmetric focusing section is as shown in Fig. 9.
The transfer matrix for this section is

$$\begin{pmatrix} 1 - \frac{1}{2}Q^2\ell^2 & 2\ell\left(1 - \frac{1}{8}Q^2\ell^2\right) \\ -\frac{Q^2\ell}{2}\frac{\left(1 - \frac{1}{4}Q^2\ell^2\right)}{\left(1 - \frac{1}{8}Q^2\ell^2\right)} & 1 - \frac{1}{2}Q^2\ell^2 \end{pmatrix} \tag{41}$$

The admittance area is the same as in Cases (1) and (2).



A. Reference planes at center of quadrupoles (Section III.C.1).



B. Reference planes at center of drift space (Section III.C.2).

FIG. 8--Admittance ellipses for equally-spaced quadrupoles for $Q\ell = \pm 1$.

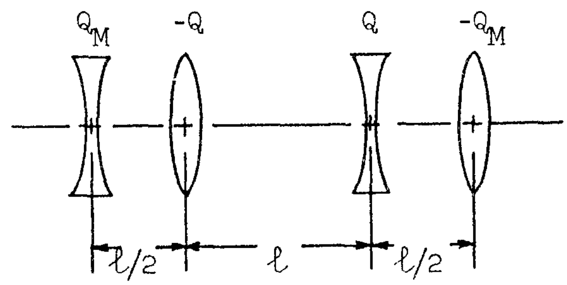


FIG. 9--Circularly symmetric focusing section.

If these sections are cascaded, the "matching" quadrupole disappears, except at the beginning and end, so that N cascaded sections contain only 2N + 2 (not 4N) quadrupoles.

It will sometimes be convenient to deal with systems which are circularly symmetric, and the present example is probably the simplest system which has this property. In practice the approximation

$$Q_M \approx \tfrac{1}{2}Q \qquad (40')$$

usually will be adequate.

IV. CHOICE OF PARAMETERS

The properties of the periodic system (quadrupoles with approximately equal spacing), discussed in Section III.C, are described by the two parameters Q and ℓ. It will be convenient first to choose the product, $Q\ell$.

A. CHOICE OF $Q\ell$

The requirement that the parameter θ be real imposes the restriction*

$$- 1 \leqslant \cos\,\theta \leqslant 1$$

Thus, by Eq. (32),

$$|Q|\ell < 2 \qquad (42)$$

From the definition of ℓ in Eq. (30)

$$\ell = \frac{1}{\alpha} \log \left(1 + \frac{\alpha L}{\gamma}\right)$$

it will be seen that the condition of Eq. (42) defines a low-energy cutoff for a system of given α, Q, and L.

In order to choose $Q\ell$ more definitely, we might want to minimize either the number of quadrupoles or the amount of quadrupole power. For a given phase-space admittance, Eq. (37) gives

$$\text{No. of quadrupoles} \propto \frac{1}{\ell} \propto \frac{1}{|Q|\ell} \left(\frac{1 + \frac{1}{2}|Q|\ell}{1 - \frac{1}{2}|Q|\ell}\right)^{\frac{1}{2}}$$

For quadrupoles of a given design,

$$\text{Quadrupole power} \propto \frac{Q^2}{\ell} \propto \frac{1}{|Q|\ell} \left(\frac{1 + \frac{1}{2}|Q|\ell}{1 - \frac{1}{2}|Q|\ell}\right)^{3/2}$$

---

*If θ is imaginary, the orbits will diverge; the admittance approaches zero as $|Q|\ell \to 2$ and becomes imaginary for $|Q|\ell > 2$.

Minimization of the first function gives, for <u>minimum number of quadrupoles</u>,

$$|Q|\ell = \sqrt{5} - 1 = 1.236$$

The second function gives, for <u>minimum quadrupole power</u>,

$$|Q|\ell = \sqrt{13} - 3 = 0.605$$

Thus if the design were to be based entirely on economy, the choice of $|Q|\ell$ would probably fall within the range

$$0.6 \lessapprox |Q|\ell \lessapprox 1.236$$

and a reasonable choice would be

$$|Q|\ell \approx 1$$

which may be called the "$\pi/3$ mode" because it gives

$$\cos\,\theta = \frac{1}{2}$$

In practice it is likely that some maximum spacing such as $L = 40$ ft will be chosen and the above consideration will be important mainly at the low-energy end.

B. CHOICE OF QUADRUPOLE STRENGTH

Referring to Eq. (37), and assuming that $|Q|\ell$ has been chosen at least approximately, we have

$$|Q| \geqslant \frac{2}{\pi}\,\frac{A}{a^2}\sqrt{\frac{1 + \frac{1}{2}|Q|\ell}{1 - \frac{1}{2}|Q|\ell}} \tag{43}$$

The radical in the expression varies from unity in the close-spacing limit ($\ell \to 0$), to 2.06 in the case of minimum number of quadrupoles ($|Q|\ell = \sqrt{5} - 1$), with a typical value of $\sqrt{3}$ in the $\pi/3$-mode case. Equation (43) specifies $|Q|$ in terms of the admittance once $|Q|\ell$ has been chosen.

C. QUADRUPOLE SPACING.

It has been proposed that provision be made for small-aperture (≈ 1-inch-diameter) quadrupoles at fixed spacing along the machine. This spacing at present* has been set tentatively at 40 ft or four accelerator sections, on the basis of considerations other than focusing and outside the scope of the present paper.

In regions of the machine where low energies may be present, such as near injectors and position converters, it may be necessary to decrease the spacing in order to avoid the low-energy cutoff (Section IV.A). This will call for a number of special quadrupoles of large aperture (large enough to clear the accelerator cooling jacket of ≈ 6 inches diameter).

To minimize the number of special quadrupoles, the most obvious procedure is to keep the parameters $|Q|$ and ℓ constant (i.e., strictly periodic system) so that the actual spacing L will increase exponentially in accordance with Eq. (30), until the 40-foot (fixed) spacing is reached.

The coordinate, z_n, of the n-th quadrupole, is given for the system of Section III.C.1 (reference plane at center of quadrupole) by

$$z_n = z_o + \frac{\gamma_o}{\alpha}\left(e^{n\alpha\ell} - 1\right), \; n = o, 1, \ldots. \tag{44}$$

where z_o, γ_o are the coordinate and energy at the initial reference plane; the spacing is

$$L_n = z_{n+1} - z_n = \frac{\gamma_o}{\alpha}\left(e^{\alpha\ell} - 1\right)e^{n\alpha\ell} \tag{45}$$

For the systems of Sections III.C.2 and III.C.3 (reference plane at center of drift space), the corresponding expressions are

$$z_n = z_o + \frac{\gamma_o}{\alpha}\left[e^{(n-\frac{1}{2})\alpha\ell} - 1\right], \; n = 1, 2, \ldots \tag{46}$$

*August, 1962.

and

$$L_n = \frac{\gamma_o}{\alpha}\left(e^{\alpha\ell} - 1\right)e^{(n-\frac{1}{2})\alpha\ell} \tag{47}$$

Assuming that a transition from tapered (constant ℓ) spacing to fixed (constant L) spacing is made, quadrupole strength could be increased slowly as a function of length beyond the transition. The criterion in the case of minimum number of quadrupoles per unit length would be

$$|Q|\ell \lesssim 1.236$$

or

$$|Q| \lesssim \frac{1.236\gamma}{L}\ (mc^2/e/cm) \tag{48}$$

where L is in cm, or

$$|Q| \lesssim \left(\frac{40}{L}\right)\frac{V}{0.296}\ (\text{kilogauss}) \tag{48'}$$

where L is in feet and V is beam energy in Bev. The orbit amplitudes then would shrink as $(|Q|)^{-\frac{1}{2}}$, in accordance with Eq. (38). Even if $|Q|$ were not increased, Eq. (38) shows that the orbit amplitudes would decrease by a factor of $\approx \left(\frac{1 + \frac{1}{2}|Q|\ell}{1 - \frac{1}{2}|Q|\ell}\right)^{\frac{1}{4}} \approx 1.43$ (for $|Q|_o\ell_o = 1.236$) as $\ell \approx \frac{L}{\gamma} \to 0$.

According to Eq. (48'), the maximum usable quadrupole strength at a 40-foot spacing would be ≈ 50,000 gauss at 15 Bev, but other considerations probably will impose a considerably lower practical limit on the strength.

V. NUMERICAL EXAMPLE: POSITRON BEAM

For purposes of illustration, the positron beam situation is assumed to be as illustrated by Fig. 10.

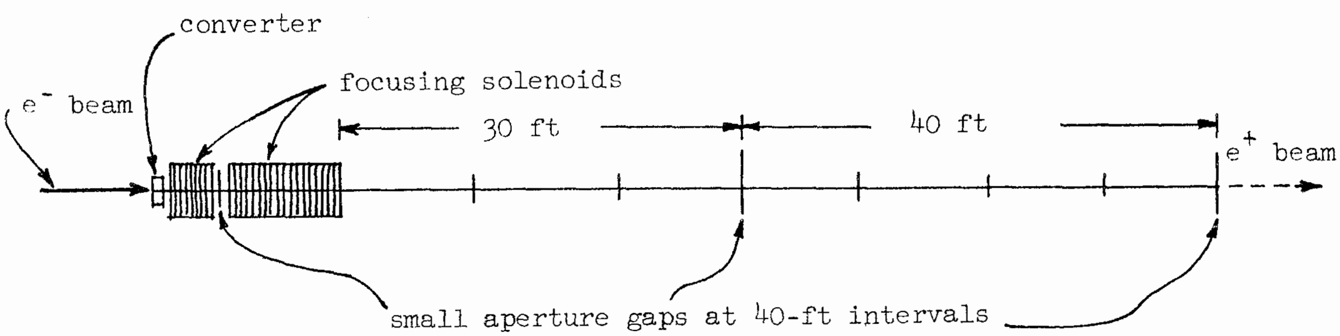


FIG. 10--Assumed layout for positron beam.

The first accelerator section beyond the converter is assumed to operate at $\alpha = 0.2\ mc^2/e/cm$ (3 Mev/ft); all succeeding sections are assumed to operate at $\alpha = 0.1$ (1.5 Mev/ft). Hence the energy at the beginning of the quadrupole system (assuming positrons produced at ≈ 7.5 Mev) and accelerated through the first section, will be

$$\alpha_o \approx 15 + 60 = 75\ mc^2$$

The design of the solenoid shown in Fig. 10 will not be considered here. Its function will be to transform the emittance of the converter in such a way that the admittance of the quadrupole system is filled with a uniform density of positrons. (It may turn out that the region labeled "solenoids" will consist partly of quadrupoles.)

Since a solenoid is essentially circularly symmetric, the quadrupole system of Section III.C.3 will be appropriate.

It will be reasonable to choose

$$|Q|\ell = \sqrt{5} - 1 = 1.236$$

since, as was shown in Section IV.A, this gives minimum number of quadrupoles for given admittance (hence maximum admittance for a given number of quadrupoles).

If we now specify the admittance, then $|Q|$ is determined by Eq. (43). A reasonable admittance is

$$A/\pi \approx 0.1 \text{ mc-cm}$$

corresponding to an angular divergence of $\approx \pm 10^{-5}$ radian in a 1-cm-diameter beam at 10 Bev. Taking the effective accelerator aperture as $a = 0.85$ cm (0.670-inch-diameter), Eq. (43) gives

$$|Q| \geqslant 0.57 \text{ mc}^2/\text{e}/\text{cm} = 970 \text{ gauss}$$

Hence

$$\ell = 1.236/|Q| \leqslant 2.17$$

The problem now is to specify a number of quadrupoles, N, such that

$$L_N \approx 40 \text{ ft}$$

and $z_N - z_o$ has one of the values 30 ft, 70 ft, 110 ft, 150 ft, ...; i.e., the initial 30 feet plus an integral multiple of the fixed spacing, 40 feet (see Fig. 10). To this end, Eqs. (46) and (47) can be solved for ℓ and N, giving

$$\ell = \frac{1}{\alpha} \log \left(1 + \frac{L_N}{z_N - z_o + \gamma_o/\alpha}\right)$$

and

$$N = \frac{1}{2} + \frac{1}{\alpha \ell} \log \left(1 + \frac{z_N - z_o}{\gamma_o / \alpha}\right)$$

Trial of the various permissible values of z_N gives:

$z_N - z_o$	ℓ	N
110 ft.	2.70	6.8
150	2.064	10.0
190	1.71	13.2.

The best choice of an <u>integral</u> value of N which gives $\ell < 2.17$ is

$$N = 10, \quad z_N - z_o = 150 \text{ ft}$$

The design value of ℓ, finally, is 2.064.

The strength of the quadrupole is

$$|Q| = 1.236/\ell = 0.599 \text{ mc}^2/\text{e/cm} = 1020 \text{ gauss}$$

and the "matching" quadrupole, Q_M, by Eq. (40) is

$$|Q_M| = \frac{|Q|}{2(1 - \frac{1}{8} Q^2 \ell^2)} = 0.619 \; |Q| = 632 \text{ gauss}$$

The phase-space admittance is

$$A/\pi = 0.105 \text{ mc-cm}$$

The quadrupole positions are shown in Fig. 11 and tabulated in Fig. 12.

Note that the position of Q_{11} differs from the constant-ℓ design by only 0.1 foot out of 40 feet if constant spacing is instituted after Q_{10}. Hence the transition should go without serious phase-space mismatch.

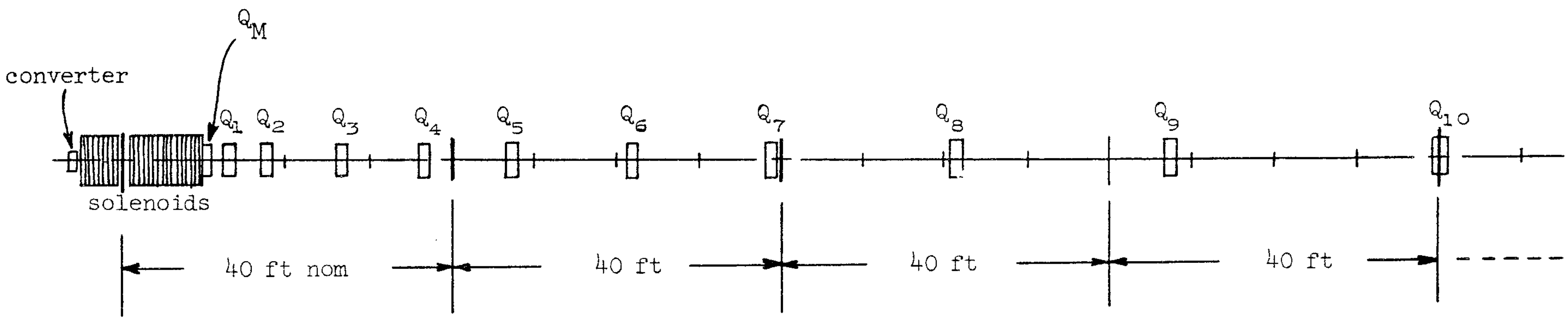


FIG. 11--Possible layout of special (large-aperture) quadrupoles following a positron converter. Ten special quadrupoles are required, counting the matching quadrupole Q_M.

n	Position (feet from Q_M)	Strength (gauss)
-	0	632
1	2.68	1020
2	8.9	1020
3	16.6	1020
4	26.1	1020
5	37.3	1020
6	51.9	1020
7	69.4	1020
8	91.1	1020
9	117.6	1020
10	150.0	1020
11	190.0 (190.1)*	1020

*190.1 ft would be the position of Q_{11} for ℓ = constant. It is assumed that the 40-foot spacing is maintained after Q_{10}.

FIG. 12--Layout of special quadrupoles for positron beam.

Energy bandwidth: In a positron converter of the type assumed here, the main positron yield probably will be in a range of ≈ 4 to 16 Mev, so that the important range of initial energies in the present system would be $\approx 68 \lessapprox \gamma_0 \lessapprox 90$. The criterion ℓ = constant in the tapered region can be true only at the design energy. For higher energies, the initial value of ℓ is smaller than necessary; the admittance area is determined by the parameters at large distance and therefore is essentially the same as for the design energy.

For energies lower than the design energy, the initial value of ℓ is too large so that the admittance is determined by the initial parameters. Assuming that adiabatic theory applies, we have [from Eq. (46)]

$$\ell_0 \approx \frac{2}{\alpha} \log \left(1 + \frac{z_1 - z_0}{\gamma_0/\alpha}\right)$$

Using the parameters of the above example and taking $\gamma_0 \approx 68$, we find

$$\ell_0 \approx 2.27$$

$$A/\pi \approx 0.094 \text{ mc-cm}$$

and the transmission, proportional to A^2, is down by a factor of $(0.094/0.105)^2 = 0.80$ from the design energy. Hence even at the low energy end of the positron band, the loss in transmission is not serious provided that the "solenoid" does an adequate job of phase-space matching.

The characteristic admittance ellipses [plot of Eq. (26) with $\nu = A$] for $\gamma_0 = 68$ and 75 are shown in Fig. 13.

Several concluding remarks may be made about the above example:

(1) It is based on Stage I (1.5 Mev/ft) but would be somewhat overdesigned for Stage II (3.0 Mev/ft) -- the admittance would be increased by the higher field by about 5%, the transmission by about 10%.

(2) On the other hand, lower accelerating fields or loss of one or more klystrons in the critical region would decrease the transmission.

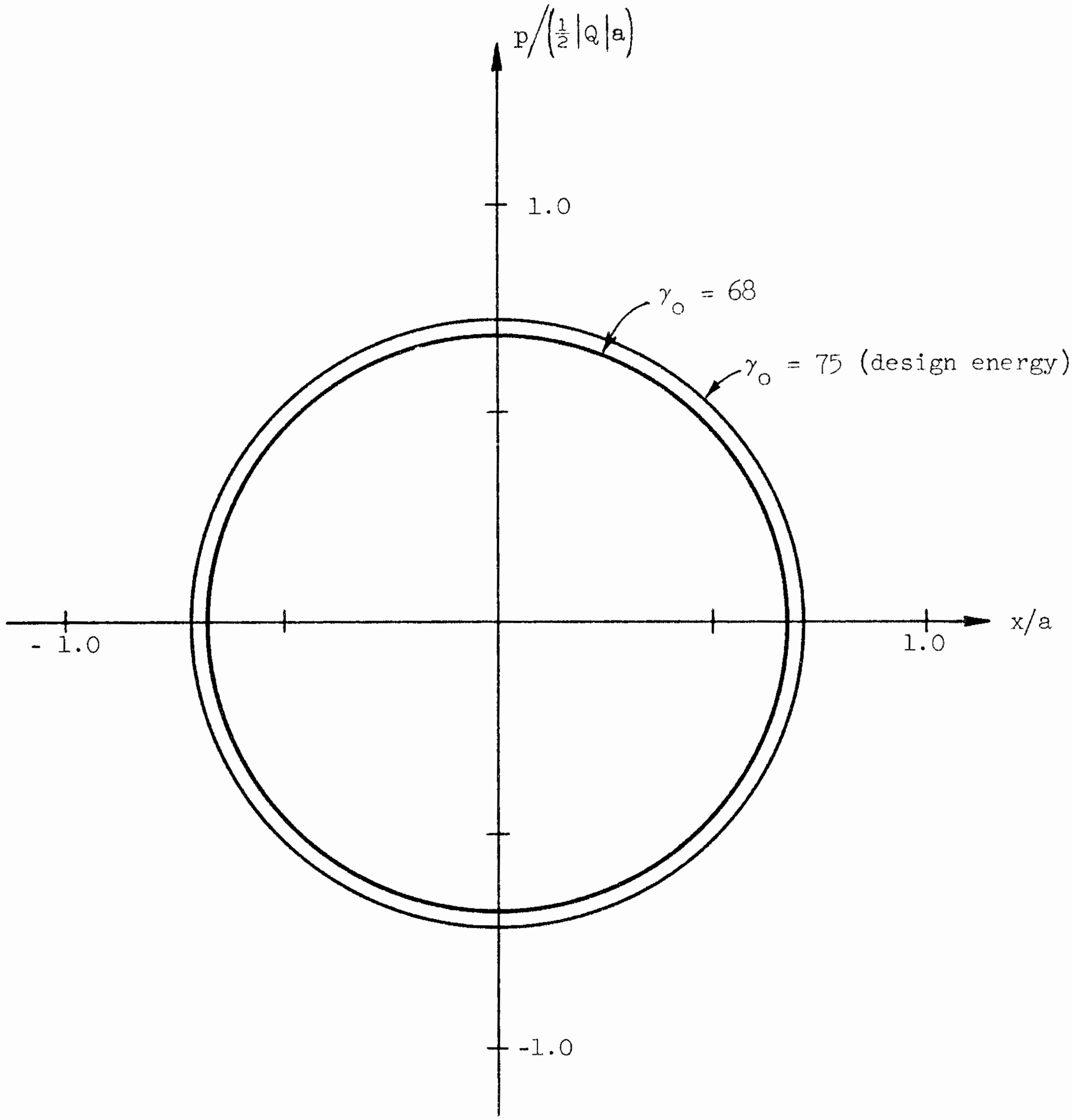


FIG. 13--Admittance of the system described in Section V for different initial energies. The curve for γ_o = 90, not shown, very nearly coincides with the one for γ_o = 75.

(3) In a design with larger admittance, the number of special quadrupoles would increase essentially linearly with A, as would the length of machine used by the tapered region; however, the transmission goes as A^2, so somewhat more expense might be justified. With $A/\pi \approx 0.4$ mc-cm, which is about the maximum admittance of the beam extraction system, we would have $Q \approx 4000$ gauss, number of special quadrupoles ≈ 42, length of machine with special quadrupoles ≈ 650 ft.

(4) Similar solutions, suitable for matching the electron beam from an injector into the constant-quadrupole-spacing, are readily found. Since the phase space is considerably smaller in this case, fewer special quadrupoles (on the order of 2 or 3) are required.

(5) Situations in which the beam is not circularly symmetric can be handled with a little more effort.

VI. PRELIMINARY DESIGN OF THE QUADRUPOLES

The quadrupole design may be worked out roughly on the basis of the criteria given by Brown[7]; coil cross-sectional area per pole $\approx \frac{|Q| b^2}{500\ d}$ (square inches) and $d \geqslant 4b$ where

b = radial aperture in inches

d = quadrupole length in inches

$|Q|$ = quadrupole strength in gauss

This presumes a current density of ≈ 500 amperes per square inch of total winding, for solid conductor without water cooling.

In addition, the average field across any section of the iron should not exceed 5000 to 6000 gauss, to avoid saturation effects.

A reasonable design value of $|Q|$ would appear to be ≈ 5000 gauss, corresponding to an admittance of $\approx 0.4\pi$ which has been assumed (Section I) to be about the maximum phase-space acceptance of the beam extraction system.

We are interested in two designs, a large-aperture ($2b \approx 6$ inches) "special" quadrupole and a small-aperture ($2b \approx 1.5$ inches) "standard" quadrupole.

A. SPECIAL QUADRUPOLE

Typical specifications are:

Maximum strength: 5000 gauss

Coil area: 8 to 10 sq. in. per pole

Length: 12 inches

Maximum flux density: in air: 2500 gauss; in iron: 5000 gauss

Maximum gradient: 500 gauss/inch

Weight: 300 to 400 lb.

Number required: 10 to 50

Maximum power: 500 watts per quadrupole

Maximum outside diameter: < 18 inches

B. STANDARD QUADRUPOLE

Typical specifications are:

Maximum strength: 5000 gauss

Coil area: 2 sq. in. per pole

Length: 3 inches

Maximum flux density { in air: 2500 gauss; in iron: 5000 gauss }

Gradient: < 2000 gauss/inch

Weight: 20 to 50 lb.

Number required: 240 (nominal)

Maximum power: < 100 watts per quadrupole

Maximum outside diameter: < 10 inches

Reasonable care should be taken to insure aberration-free design.

NOTE: It should be emphasized that these numbers are highly preliminary and are included merely to give a feel for the physical sizes of the magnets.

VII. SUMMARY AND CONCLUSIONS

The main conclusions of the present report may be summed up briefly:

(1) Transmission of the electron beam from the end of the injector to the end of the machine could be accomplished with something like one to three lenses (e.g., quadrupole doublets) of moderate strength.

(2) There are a number of reasons for wanting focusing somewhat stronger than the minimal requirement for electron transmission; for example, to reduce the beam size, to allow for possible non-conservation of transverse phase space, to take advantage of the beam-guiding properties of strong focusing, or to increase the yield in the positron beam. The number of focusing elements then becomes appreciable and the system takes on the properties of periodic focusing. Alternating gradient quadrupole focusing is the obvious expedient.

(3) The phase-space admittance, $\int dx dp_x$, for an alternating gradient quadrupole system is proportional to the quadrupole strength, $|Q|$. For admittances of the order of $(0.1 \text{ to } 0.4)\pi$ mc-cm corresponding to a beam diameter of one cm and angular divergence of $\pm(1 \text{ to } 4) \times 10^{-5}$ at 10 Bev, the quadrupole strength comes out to be about 1000 to 4000 gauss.

(4) Because of the low energy cutoff in a periodic system, the spacing of the quadrupoles must be closest where low beam energies are present. In regions following injectors and positron converters, the spacing generally will be less than the nominal 40 feet allowed for small-aperture quadrupoles; hence some "special" or large-aperture quadrupoles will be required.

(5) In a hypothetical positron-beam situation it is found, for example, that about 10 special quadrupoles are required to conduct the beam from the 38 Mev point up to the beginning of 40-foot spacing (the desired admittance is taken as $\approx 0.1\pi$ mc-cm). The number of special quadrupoles is essentially proportional to the design admittance.

(6) A typical "special" quadrupole for 5000 gauss strength is found to be about 16 to 18 inches maximum outside diameter; 12 inches in

length; weight 400 to 500 lbs; power requirement about 500 watts; number required, about 10 to 50.

(7) A typical "standard" quadrupole for 5000 gauss strength might be on the order of 6 to 8 inches maximum outside diameter; 3 inches long; weight < 50 lbs; power requirement < 100 watts; number required, 240 (nominal 40-foot spacing).

(8) All the above numbers are based only on focusing requirements. Other considerations may eventually change the size and number of quadrupoles somewhat.

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