SLAC-PUB-8988
August 2001

Microbunching Due to Coherent Synchrotron Radiation in a Bunch Compressor *

S. Heifets and G. Stupakov
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA

Abstract

The coherent synchrotron radiation of a bunch in a bunch compressor may lead to the microwave instability producing longitudinal modulation of the bunch. This modulation generates coherent radiation with the wave length small compared to the bunch length. It can also be a source of an undesirable emittance growth in the compressor. We derive and analyze the equation that describes linear evolution of the microwave modulation. Numerical solution of this equation for the LCLS bunch compressor reveals such an instability, in qualitative agreement with numerical simulations.

*Work supported by Department of Energy contract DE–AC03–76SF00515.

1 Introduction

A relativistic electron beam in a bunch compressor can radiate coherently if the wavelength of the radiation exceeds the length of the bunch. This radiation results in an undesirable growth of the beam emittance [1], which can however be cured (at least partially) by a special design of the compressor [2].

As was pointed out in our paper [3], the coherent synchrotron radiation of the beam can also be a source of modulation of the beam density at wavelengths small compared to the bunch length. It follows from the results of Ref. [3] that such modulation occurs at the wavelengths λ that are larger than the critical wavelength λ_0

$$\lambda \gtrsim \lambda_0 \equiv \pi R \left(\frac{\alpha \gamma \delta_0^2}{n_b r_e} \right)^{3/2} \quad , \tag{1}$$

with a typical growth rate

$$\operatorname{Im} \omega \sim \frac{c \alpha \delta_0}{\lambda_0} \quad , \tag{2}$$

where n_b is the number of particles in the beam per unit length, γ is the relativistic factor, α is the momentum compaction number, r_e is the classical electron radius, and δ_0 is the rms relative energy spread in the beam. Of course, this instability occurs only if the synchrotron radiation at this wavelength is not suppressed by the shielding effect of the conducting walls of the vacuum chamber. Also, for a bunched beam, the wavelength λ_0 should be much smaller then the length of the bunch.

The results of Ref. [3] refer to the microbunching instability in a ring, however a similar effect can also occur in a bunch compressor where coherent synchrotron radiation often plays a role. Indeed, the effect of microbunching caused by CSR has been observed in computer simulations of the bunch compressor [4] designed for the Linac Coherent Light Source (LCLS) at SLAC [5]. Analytical estimates of the CSR effects in bunch compressors has been recently published in Ref. [6].

In this paper we develop a linear theory of the microbunching in a bunch compressor generated by the CSR wakefield [7, 8]. We assume that the shielding effect of the conducting walls on CSR is not important which is usually true when the bunch length is short enough. We will also neglect the transient effects in the CSR wake [9] occurring in short magnets.

2 Basic equations

Consider a Gaussian bunch at the entrance to a bunch compressor with the rms bunch length σ_l and the rms relative energy spread δ_0, $\delta_0^2 = \langle(\Delta E)^2\rangle/E^2$, where E is the beam energy, and ΔE is the energy deviation. The distribution function of the bunch is

$$\rho_{\mathrm{in}}(z,\delta) = \frac{N}{2\pi\sigma_l\delta_0}\exp\left(-\frac{z^2}{2\sigma_l^2} - \frac{\delta^2}{2\delta_0^2}\right) \quad , \tag{3}$$

where N is the number of particles in the bunch and z is the longitudinal coordinate within the bunch with positive z corresponding to the head of the bunch. The rf voltage V_{rf} of the acceleration section introduces the $\delta - z$ correlation in a bunch changing particle energy $\delta \to \delta - uz$, where $u = 2\pi V_{\mathrm{rf}}/(\lambda_{\mathrm{rf}}E)$, and λ_{rf} is the rf wave length. The distribution function $\rho(s,z,\delta)$ after acceleration (at the entrance to the dispersion section, $s = 0$) takes form

$$\rho(0,z,\delta) = \frac{N}{2\pi\sigma_l\delta_0}\exp\left(-\frac{z^2}{2\sigma_l^2} - \frac{(\delta+uz)^2}{2\delta_0^2}\right) \quad . \tag{4}$$

When bunch enters the dispersion section the coordinate z is transformed $z \to z + R_{56}\delta$. For concreteness, we assume that the parameter R_{56} in the compressor is a positive function of s, then compression occurs if u is also positive.

In the case when effect of the wake fields is negligible, the equations of longitudinal motion are $d\delta/ds = 0$, $dz/ds = R'_{56}\delta$, where the prime denotes the derivative with respect to s, and the distribution function in the compressor is

$$\rho(s,z,\delta) = \frac{N}{2\pi\sigma_l\delta_0}\exp\left(-\frac{(z-\delta R_{56})^2}{2\sigma_l^2} - \frac{[\delta+u(z-\delta R_{56})]^2}{2\delta_0^2}\right) \quad . \tag{5}$$

In the presence of the wake field $W(z)$ the equations of motion are modified

$$\frac{d\delta}{ds} = -\frac{r_e}{\gamma}\int dz' W(z-z')n(s,z') \quad , \quad \frac{dz}{ds} = \delta R'_{56} \quad , \tag{6}$$

where $n(s,z) = \int \rho(s,z,\delta)d\delta$. Equations of motion Eq. (6) corresponds to the Vlasov equation for the beam density $\rho(s,z,\delta)$,

$$\frac{\partial\rho}{\partial s} + \delta R'_{56}\frac{\partial\rho}{\partial z} - \frac{r_e}{\gamma}\frac{\partial\rho}{\partial\delta}\int_{-\infty}^{\infty} dz' W(z-z')n(s,z') = 0 \quad . \tag{7}$$

For the wake we will assume a steady-state CSR wake which is valid for long magnets and neglects the transient effects at the entrance to and the exit from the magnet [9]. This wake can be written as [8]:

$$W(z) = \frac{2}{(3R^2)^{1/3}} \frac{\partial}{\partial z} \frac{1}{z^{1/3}} \quad , \quad z \geq 0 \quad , \tag{8}$$

where R is the curvature radius of a particle trajectory. The wake $W(z) = 0$ for $z < 0$. The derivative $\partial/\partial z$ in Eq. (8) indicates that substituting the wake in Eq. (7) one has to integrate by parts over z, which recasts the integral into one that converges at $z = 0$.

3 Linearized equations for perturbation

We are interested here in the bunch modulation with the wave length much smaller than the rms bunch length σ_l. In this case, we can neglect the spatial variation of the beam density and instead of Eq. (3) consider a coasting beam with the initial linear density $n_{\rm in}$ and a Gaussian distribution in δ, $\rho_{\rm in}(\delta) = (n_{\rm in}/\sqrt{2\pi}\delta_0)\exp(-\delta^2/2\delta_0^2)$. Eq. (5) then is reduced to

$$\rho_0(s, z, \delta) = \frac{n_{\rm in}}{\sqrt{2\pi}\delta_0} \exp\left[-\frac{[\delta + u(z - \delta R_{56}(s))]^2}{2\delta_0^2}\right] \quad . \tag{9}$$

The beam density at the location s is given by the following equation

$$n_0(s, z) = \int d\delta \rho_0(s, z, \delta) = \frac{n_{\rm in}}{1 - uR_{56}(s)} \quad , \tag{10}$$

where $(1 - uR_{56}(s))^{-1}$ is the compression factor. Note that in our model of the coasting beam, the beam density n_0 tends to infinity when $R_{56}(s)$ approaches $1/u$. In what follows, we assume that $0 < R_{56}(s) < 1/u$.

It is easy to see, that the distribution function given by Eq. (9) satisfies Eq. (7) because the beam density is independent of z and the integral term in Eq. (5) vanishes.

To study the bunch stability consider a small perturbation ρ_1 of the distribution function

$$\rho(s, z, \delta) = \rho_0(s, z, \delta) + \rho_1(s, z, \delta) \quad . \tag{11}$$

The linearized Vlasov equation for ρ_1 reads,

$$\frac{\partial \rho_1}{\partial s} + \delta R'_{56} \frac{\partial \rho_1}{\partial z} - \frac{r_e}{\gamma} \frac{\partial \rho_0}{\partial \delta} \int dz' d\delta' W(z - z') \rho_1(s, z', \delta') = 0 \quad . \tag{12}$$

Eq. (12) can be simplified by using variables ζ, p instead of z and δ:

$$\zeta = z - R_{56}(s)\delta, \quad p = \delta + u(z - R_{56}(s)\delta) \quad , \tag{13}$$

with the inverse transform

$$\delta = p - u\zeta, \quad z = (1 - uR_{56}(s))\zeta + R_{56}(s)p \quad , \tag{14}$$

and assuming that ρ_1 is a function of s, ζ and p.

Let us consider a perturbation that is sinusoidal in ζ with the wave vector k,

$$\rho_1(s, \zeta, p) = \hat{\rho}_1(s, p) e^{ik\zeta} \quad . \tag{15}$$

In original variables z, δ, this perturbation corresponds to the dependence

$$\rho_1(s, z, \delta) = \hat{\rho}_1\left(s, \delta(1 - uR_{56}(s)) + uz\right) e^{ik(z - \delta R_{56}(s))} \quad , \tag{16}$$

and corresponds to a sinusoidally modulated distribution function $\rho_1(0, z, \delta) = \hat{\rho}_1(s, \delta)\, e^{ikz}$ at the entrance to the compressor (before the application of the RF voltage). The perturbation of the bunch density $n_1(s, z) = \int d\delta \rho_1(s, z, \delta)$ can now be found by integrating Eq. (16) which gives an explicit dependence of n_1 versus z, $n_1 \propto e^{ikz/(1-uR_{56}(s))}$. Note that the wavelength of the modulation $2\pi(1 - uR_{56}(s))/k$ decreases during the bunch compression inversely to the compression factor.

It is convenient to introduce a new function $G(s)$ such that

$$n_1(s, z) = \frac{e^{ikz/(1-uR_{56}(s))}}{(1 - uR_{56}(s))^{8/3}} G(s) \quad . \tag{17}$$

Introducing also a new variable $\xi = R_{56}(s)/(1 - uR_{56}(s))$ which varies from 0 to ∞ when $R_{56}(s)$ varies from 0 to $1/u$, we will consider G as a function of ξ. As shown in Appendix A, the function $G(\xi)$ satisfies the following integral equation,

$$G(\xi) = G_0(\xi) + (1 + u\xi)^{-5/3} \int_0^{\xi} d\xi' M(\xi, \xi') G(\xi') \quad , \tag{18}$$

where G_0 is the initial value of G,

$$G_0(\xi) = \frac{1}{(1+u\xi)^{5/3}} \int dp \hat{\rho}_1(0,p) e^{-ikp\xi} \quad , \tag{19}$$

and

$$M(\xi,\xi') = -iA \frac{k\Lambda(\xi')}{R'_{56}(\xi')} \int dp \frac{\partial \rho_0(p)}{\partial p} e^{-ikp(\xi-\xi')} \quad , \tag{20}$$

with

$$\Lambda(s) = \frac{n_{\rm in} r_e}{\gamma k^{2/3} R(s)^{2/3}}, \tag{21}$$

and $A = 1.63i - 0.94$.

For a Gaussian distribution function given by Eq. (9), $\rho_0(p) = (1/\sqrt{2\pi}\delta_0) \times \exp(-p^2/2\delta_0^2)$, and the kernel M reduces to

$$M(\xi,\xi') = A\mu(\xi') k^2 \delta_0^2 (\xi - \xi') e^{-(k\delta_0(\xi-\xi'))^2/2} \quad , \tag{22}$$

where $\mu(\xi) = \Lambda(\xi)/\delta_0^2 R'_{56}(\xi)$. In numerical calculations we also assumed a Gaussian initial perturbation, $\hat{\rho}_1(0,p) = \rho_0(p)$, which gives $G_0(\xi) = (1 + u\xi)^{-5/3} \exp(-(k\xi\delta_0)^2/2)$.

Eqs. (18)-(20) constitute a full set of equations that define bunch modulation $n_1(s,z)$ for $s > 0$.

It is interesting to note that in the absence of the wakefield, when the kernel vanishes, the solution of Eq. (18) is $G = G_0$, and an initial density perturbation exponentially decays in time $n_1 \propto \exp(-(k\xi\delta_0)^2/2)$. The wake, however, not only prevent this perturbation from decaying, but can actually amplify it, as we will see below.

4 Comparison with storage ring

It is instructive to compare the results obtained above with the similar problem of the bunch stability in the storage ring [3]. The storage ring result corresponds to the limit $u \to 0$ and a linear dependance $R_{56}(s) = \text{const} - \alpha s$, where α is the momentum compaction of the ring. We also assume here a constant bending radius R. In this limit $\xi = \text{const} - \alpha s$, and Eq. (18) reduces to

$$G(\xi) = G_0(\xi) + \int_0^{\xi} d\xi' M(\xi - \xi') G(\xi') \quad , \tag{23}$$

where the kernel M given by Eq. (22) is now a function of the difference $\xi - \xi'$ as explicitly indicated in Eq. (23). Note that the parameter μ is constant in this limit.

Eq. (23) can be solved by using the Laplace transform of $G(\xi)$,

$$\tilde{G}(\Omega) = \int_0^\infty d\xi G(\xi) e^{-i\Omega\xi} \quad , \tag{24}$$

where we use the complex variable $i\Omega$ as the Laplace variable. Note that the physical frequency ω is related to Ω by $\omega = \Omega c\alpha$.

Using the identity

$$\int_0^\infty t e^{-t^2/2 - i\Omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{p e^{-p^2/2} dp}{p + \Omega} \quad , \tag{25}$$

valid for $\mathrm{Im}\,\Omega < 0$, Eq. (23) can be written in the form

$$\tilde{G}(\Omega) = \tilde{G}_0(\Omega) + \tilde{G}(\Omega) \frac{A\mu}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{p e^{-p^2/2} dp}{\Omega/k\delta_0 + p} \quad , \tag{26}$$

where $\tilde{G}_0(\Omega)$ is the Laplace image of $G_0(\xi)$. The bunch stability is defined by the dispersion equation

$$1 = -\frac{A|\mu|}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{p e^{-p^2/2} dp}{\Omega/k\delta_0 + p} \quad , \tag{27}$$

which is obtained from Eq. (26) by dropping the G_0 term on the right hand side. The parameter $|\mu|$ is equal to $\Lambda/\alpha\delta_0^2$. This equation agrees with the dispersion relation obtained for the storage ring in Ref. [3] (see Eq. (7) of Ref. [3]).

The analysis of the dispersion relation (27) carried out in Ref. [3] shows that the system is stable for $|\mu| < 0.62$. Above the threshold, $|\mu| > 0.62$, the imaginary part of Ω becomes positive and an initial perturbation exponentially grows with s. For given beam parameters, the maximum growth rate $\mathrm{Im}\,\omega$ occurs at a wavelength that correspond to the value of $\mu = 1.3$ and is approximately given by Eq. (2). In the next section we will apply these results for qualitative analysis of the situation with $u \neq 0$.

5 Qualitative analysis and numerical solution

In the bunch compressor, when $u > 0$, Eq. (16) cannot be cast into a dispersion relation similar to Eq. (27). However, an analysis based on the results of the previous section provides valuable insight into the behavior of the solution in this case.

Note that the parameter μ in the kernel of Eq. (18) is multiplied by the additional factor $(1+u\xi)^{-5/3}$. Defining an effective $\mu_{\rm eff}$ as $\mu_{\rm eff} = \mu(1+u\xi)^{-5/3}$ we see that $\mu_{\rm eff}$ decreases with compression, and the beam becomes more stable. Assuming that, for a given wavelength, $\mu \gg 1$ at the beginning of the compressor, the perturbation will be unstable for small values of ξ. If, however, the bunch compressor is long enough, so that at some point $\mu(1+u\xi)^{-5/3}$ becomes smaller than the critical value of 0.6, one can expect the suppression of the instability in this part of the compressor with the initial exponential growth changing into an exponential decay.

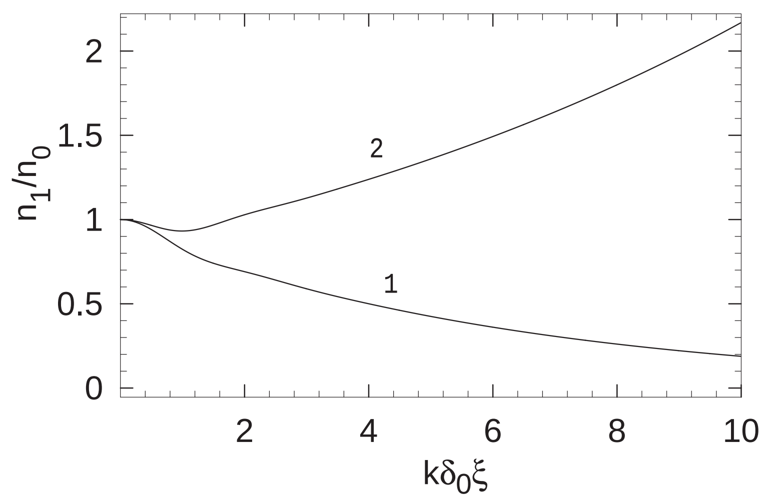

Figure 1: Relative density perturbation n_1/n_0 as function of $k\delta_0\xi$ in the case $u = 0$. Curve *1* – $\mu = 0.5$, curve *2* – $\mu = 0.7$.

To verify the prediction of this qualitative analysis, we wrote a computer program that solves Eq. (18)) numerically. In the numerical algorithm, the interval of interest $0 < \xi < \xi_{\rm max}$ is divided into $n_{\rm max}$ subintervals and the unknown function G is defined at the mesh points of the grid. The contribution of each subinterval into the integral of Eq. (18) is evaluated using linear interpolation of G. As a result, the integral equation (18) is converted into a system of linear equations which can be solved numerically. Typical values of $n_{\max}$ used in the calculation were between 1000 and 3000. It was verified that the obtained result did not depend on the size of the mesh.

The numerical algorithm has been tested for the case $u = 0$ where we found a good agreement with the dispersion relation described in the previous section. For illustration, numerical solutions are shown in Fig. 1 for the case $u = 0$ for two values of parameter μ — below and above the critical value $\mu_{\text{crit}} = 0.62$. It is seen, that the solution grows or decays depending on whether the value of μ is below or above the threshold. A more detailed study demonstrates also an agreement between the growth rates found numerically and the ones computed from the dispersion relation.

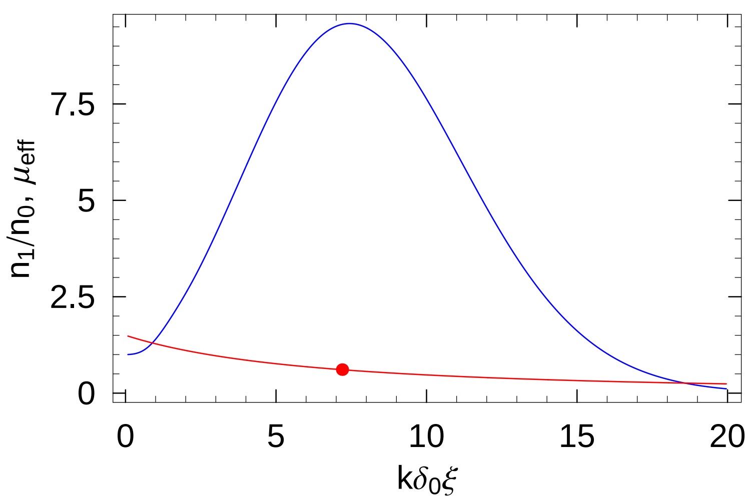

Figure 2: Relative density perturbation n_1/n_0 (upper curve) and μ_{eff} (lower curve) as functions of $k\delta_0\xi$ in the case $u = 0.1k\delta_0$ and $\mu = 10$. The dot shows the point where μ_{eff} reaches the critical value of 0.62.

A numerical solution with $u = 0.1k\delta_0$ and $\mu = 10$ is shown in Fig. 2 for a linear dependence of $R_{56}(s)$. We also plot the value of $\mu_{\text{eff}}(s)$ which according to the qualitative analysis above is correlated with the growth and decay of the density perturbation. As one can see, indeed in the region where $\mu_{\rm eff}(s)$ is smaller than the critical value 0.62, the solution is unstable; it stabilizes and begins to decay when $\mu_{\rm eff}$ drops below approximately the critical value.

6 LCLS bunch compressor

The second LCLS compressor BC2 consists of eight dipole magnets of length 0.8 m. It is located at the point in the linac where the beam energy is equal to 4.5 GeV, and compresses the rms bunch length σ_l from 195 microns down to 22 microns, (see [5]). Other relevant parameters of the bunch compressor are: uncorrelated rms relative energy spread at the entrance to the compressor $\delta_0 = 1.6 \cdot 10^{-5}$, number of particles in the bunch $N = 6.5 \cdot 10^{-9}$, the energy chirp parameter u in Eq. (4) is 41 m^{-1}. The bending radius in the first 4 magnets is 16.2 meters, and the last four — 37.2 meters. The calculated R_{56} as a function of s is shown in Fig. 3.

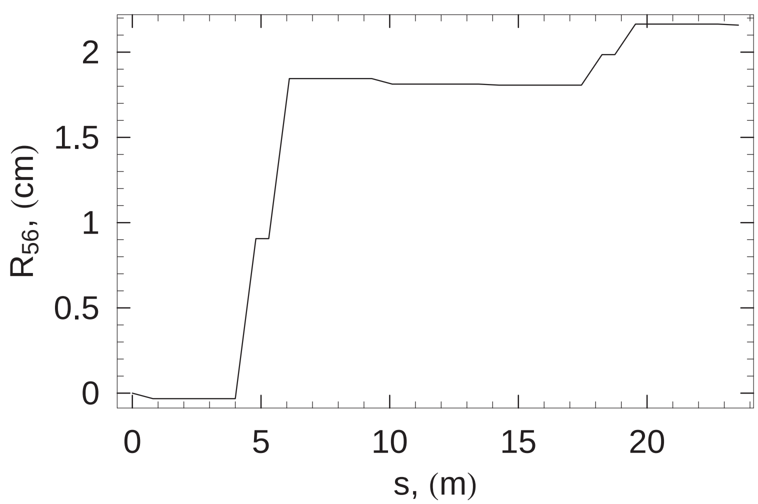

Figure 3: Plot of R_{56} for the LCLS bunch compressor.

We calculated the microbunching effect in the bunch compressor by numerically solving Eq. (18) . We assumed a Gaussian distribution of the beam density and used the relation for the linear particle density $n_{\rm in} = N/\sqrt{2\pi}\sigma_l$. We also assumed initial Gaussian distributions for $\rho_0(\delta)$ and $\hat{\rho}_1(0,\delta)$ with the rms value of the relative energy spread δ_0. At the entrance to the compressor, an initial density perturbation $n_1^{\rm init}$ with the wavelength λ has been specified and the ratio $|n_1(s,z)|/n(s)$ has been calculated throughout the compressor, where n_1 is given by Eq. (17) (note, that the absolute value $|n_1(s,z)|$ is a function of s only). The amplification factor P for the density perturbations is defined as

$$P(s) = \frac{|n_1(s,z)|}{n(s)} \frac{n_{\rm in}}{n_1^{\rm init}} \quad , \tag{28}$$

which characterizes the growth of the relative density perturbation of the beam (note that the linear beam density $n(s)$ grows by a factor of $(1 - uR_{56})^{-1} \approx 10$ at the end of the compressor).

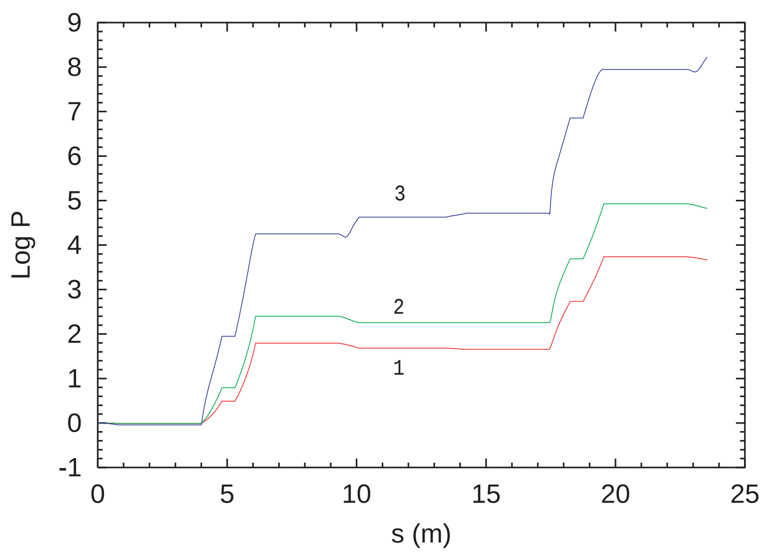

Figure 4: The logarithm of the amplification factor P for three values of λ: *1* – $\lambda = 10\,\mu$m, *2* – $\lambda = 5\,\mu$m, *3* – $\lambda = 1\,\mu$m.

The results of calculation are shown in Fig. 4. As one can see, according to the linear theory, the initial density perturbation with the initial wavelength in the range of 1 to 10 microns will grow several orders in magnitude over the length of the compressor. Of course, the linear theory is only valid if $n_1(s)/n(s) \ll 1$ and the large values of the amplification factor mean that the density perturbation will most likely grow into a nonlinear regime and saturate at the level where $n_1(s)/n(s) \sim 1$. One has also keep in mind that the wavelength of the perturbation λ is also compressed by the same factor

$(1-uR_{56})^{-1}$, so, for example, an initial 10 micron perturbation is transformed into approximately 1 micron wavelength at the end.

It is important to emphasize here that the wake Eq. (8) used in this paper may not be applicable for very short wavelength. Indeed, this wake was derived for a bunch that is infinitely thin in the transverse direction and assumes that all particle in the cross section of the bunch radiate coherently. However, the transverse coherence length $l_\perp \sim \lambda^{2/3}R^{1/3}$ decreases with the wavelength and at some point becomes smaller than the transverse dimension of the beam. For such wavelength, one has to use a wake that takes into account the transverse dimension of the beam.

Note also, that the formation length of the radiation involved is actually comparable with the length of the magnets, which means that the true wake in this case may somewhat differ from the steady-state expression Eq. (8).

7 Conclusion

In this paper, we developed a linear theory describing self-induced microbunching of a beam in a bunch compressor. The microbunching is a result of the microwave instability driven in a self-consistent way by the coherent synchrotron radiation of the short-wavelength modulation produced by the instability. An integral equation is derived that governs the development of initial sinusoidal density perturbation in the beam, assuming a steady-state CSR wake.

Numerical calculation for the LCLS bunch compressor shows that an initial density perturbation with a wavelength in the range 1–10 micron may be amplified by several orders of magnitude over the length of the compressor. In reality, however, such amplification will most likely be limited by nonlinear saturation of the beam modulation, which will set up both the typical length and the final amplitude of the microbunching. A more realistic theory has also to include a wake that takes into account the finite length of the magnets and transverse dynamics of the particles.

8 Acknowledgments

We thank P. Emma for numerous discussions of the issues of the LCLS bunch compressor, and M. Borland for sharing with us unpublished results of the

simulations demonstrating micro bunching effect due to the CSR wake.

This work was supported by Department of Energy contract DE–AC03–76SF00515.

References

[1] H. H. Braun, R. Corsini, L. Groening, F. Zhou, A. Kabel, T. Raubenheimer, R. Li, and T. Limberg, Phys. Rev. ST Accel. Beams **3**, 124402 (2000).

[2] P. Emma and R. Brinkmann, in *Proceedings of the 1997 Particle Accelerator Conference* (IEEE, Piscataway, NJ, 1997).

[3] S. Heifets and G. Stupakov, in *Proceedings of the 2001 Particle Accelerator Conference* (IEEE, Piscataway, NJ, 2001).

[4] M. Borland and P. Emma, private communication.

[5] *Linac Coherent Light Source (LCLS) Design Study Report: The LCSL Design Study Group*, Report SLAC-R-521, SLAC (1998).

[6] E. L. Saldin, E. A. Schneidmiller, and M. V. Yurkov, in *Proceedings of the FEL2001 Conference* (2001).

[7] S. K. J. B. Murphy and R. L. Gluckstern, in *Proc. IEEE Particle Accelerator Conference and International Conference on High-Energy Accelerators, Dallas, 1995* (IEEE, Piscataway, NJ, 1996), (IEEE Conference Record 95CH35843).

[8] Y. S. Derbenev, J. Rossbach, E. L. Saldin, and V. D. Shiltsev, *Microbunch Radiative Tail-Head Interaction*, DESY FEL Report TESLA-FEL 95-05, Deutsches Elektronen-Synchrotron, Hamburg, Germany (September 1995).

[9] E. L. Saldin, E. A. Schneidmiller, and M. V. Yurkov, Nuclear Instruments and Methods, Sec. A **398**, 373 (1997).

Appendix A

Introducing the new variables ζ and p given by Eqs. (13) we cast Eq. (12) to the following one

$$\begin{aligned}\left(\frac{\partial\rho_1}{\partial s}\right)_{\zeta,p} &= \frac{r_e}{\gamma}\frac{\partial\rho_0}{\partial\delta}\int_{-\infty}^{z} dz'd\delta' W(z-z')\hat{\rho}_1(s,z',\delta') \\ &= \frac{2r_e}{(3R(s))^{1/3}\gamma}\frac{\partial\rho_0}{\partial\delta}\int_{-\infty}^{z}\frac{dz'd\delta'}{(z-z')^{1/3}}\frac{\partial}{\partial z'}\hat{\rho}_1(s,z',\delta') \quad . \end{aligned} \quad \text{(A1)}$$

Here, in the left-hand-side (LHS) the partial derivative with respect to s is taken for constant values of ζ and p, and on the right-hand-side we used Eq. (8) for the wake and integrated by parts over z'.

Let us consider a perturbation of the distribution function given by Eq. (15). Using the invariance of the phase volume $dzd\delta = dpd\zeta$ and transforming to the variables p' and ζ' in the RHS of Eq. (A1) we obtain

$$\begin{aligned}\frac{\partial\hat{\rho}_1}{\partial s} &= -A\Lambda(s)(1-uR_{56}(s))^{2/3}\frac{\partial\rho_0(p)}{\partial p} \\ &\int_{-\infty}^{\infty} dp'\left[u\frac{\partial\hat{\rho}_1(s,p')}{dp'} + ik\hat{\rho}_1(s,p')\right]\exp\left(\frac{ikR_{56}(s)(p-p')}{1-uR_{56}(s)}\right) \quad . \end{aligned} \quad \text{(A2)}$$

Here

$$\Lambda(s) = \frac{n_{\mathrm{in}} r_e}{\gamma k^{2/3} R(s)^{2/3}} \quad , \quad \text{(A3)}$$

and we have used

$$\int_{-\infty}^{x}\frac{d\zeta}{(x-\zeta)^{1/3}}e^{ik\zeta} = -\frac{3^{1/3}A}{2k^{2/3}}e^{ikx} \quad , \quad \text{(A4)}$$

where $A = 3^{-1/3}\Gamma\left(\frac{2}{3}\right)\left(\sqrt{3}i-1\right) = 1.63i - 0.94$, with Γ the gamma-function.

Assuming that $\hat{\rho}_1(s,p)\to 0$ when $|p|\to\infty$ and integrating Eq. (A2) by parts yields

$$\frac{\partial\hat{\rho}_1(s,p)}{\partial s} = -\frac{ikA\Lambda(s)}{(1-uR_{56}(s))^{1/3}}\frac{\partial\rho_0(p)}{\partial p}g(s)\exp\left(\frac{ikR_{56}(s)p}{1-uR_{56}(s)}\right) \quad , \quad \text{(A5)}$$

where

$$g(s) = \int_{-\infty}^{\infty} dp\hat{\rho}_1(s,p)\exp\left(-\frac{ikR_{56}(s)p}{1-uR_{56}(s)}\right) \quad . \quad \text{(A6)}$$

Integrating Eq. (A5) over s from the entrance to the bunch compressor $s = 0$ to the current position s yields

$$\hat{\rho}_1(s,p) = \hat{\rho}_1(0,p) - ikA\frac{\partial\rho_0}{\partial p}\int_0^s \frac{ds'\Lambda(s')g(s')}{(1-uR_{56}(s'))^{1/3}}\exp\left(\frac{ikR_{56}(s')p}{1-uR_{56}(s')}\right) \quad , \tag{A7}$$

where $\hat{\rho}_1(0,p)$ is the initial value of $\hat{\rho}_1$ at $s = 0$. Combining Eq. (A6) and Eq. (A7), we obtain the integral equation for $g(s)$:

$$g(s) = g_0(s) + \int_0^s ds' K(s,s')g(s') \quad , \tag{A8}$$

where,

$$g_0(s) = \int_{-\infty}^{\infty} dp\hat{\rho}_1(0,p)\exp\left(-\frac{ikR_{56}(s)p}{1-uR_{56}(s)}\right) \quad , \tag{A9}$$

and the kernel

$$K(s,s') = \frac{-ikA\Lambda(s')}{(1-uR_{56}(s'))^{1/3}}\int_{-\infty}^{\infty} dp\frac{\partial\rho_0}{\partial p}\exp\left(\frac{ikpR_{56}(s')}{1-uR_{56}(s')} - \frac{ikpR_{56}(s)}{1-uR_{56}(s)}\right) \quad , \tag{A10}$$

The perturbation of the bunch density $n_1(s,z) = \int d\delta\rho_1(s,z,\delta)$ can now be found by integrating Eq. (16) and using Eq. (A6)

$$n_1(s,z) = \frac{g(s)}{1-uR_{56}(s)}\exp\left(\frac{ikz}{1-uR_{56}(s)}\right) \quad . \tag{A11}$$

Introducing a new variable $\xi = R_{56}(s)/(1-uR_{56}(s))$ and a new function $G = g(s)(1-uR_{56}(s))^{5/3}$ we will consider G as a function of ξ. Note that $1 - uR_{56}(s) = 1/(1+u\xi)$ and $g(s) = G(\xi)(1+u\xi)^{5/3}$. From Eq. (A8) it follows

$$G(\xi) = G_0(\xi) + (1+u\xi)^{-5/3}\int_0^\xi d\xi' M(\xi,\xi')G(\xi') \quad , \tag{A12}$$

where

$$G_0(\xi) = \frac{1}{(1+u\xi)^{5/3}}\int dp\hat{\rho}_1(0,p)e^{-ikp\xi} \quad , \tag{A13}$$

and

$$M(\xi,\xi') = -iA\frac{k\Lambda(\xi')}{R'_{56}(\xi')}\int dp\frac{\partial\rho_0(p)}{\partial p}e^{-ikp(\xi-\xi')} \quad . \tag{A14}$$

The perturbation of the bunch density in terms of the function $G(\xi)$ is given by the following equation

$$n_1(s,z) = (1+u\xi)^{8/3}e^{ikz(1+u\xi)}G(\xi) \quad . \tag{A15}$$