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Self-consistent analysis of three dimensional uniformly charged ellipsoid with zero emittance

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Abstract

A self-consistent treatment of a three-dimensional ellipsoid with negligible emittance in time-dependent external field is performed. Envelope equations describing the evolution of an ellipsoid boundary are discussed. For a complete model it is required that the initial particle momenta be a linear function of the coordinates. Numerical example and verification of the problem by a 3-dimensional particle-in-cell simulations are given.

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Envelope equations for three-dimensional (3D) uniformly charged ellipsoid are widely used in accelerator theory. Meanwhile, self-consistent analysis which could result in such equations, is unknown. There is no distribution function in 6-dimensional phase space, which leads in solution of self-consistent problem as uniformly charged 3D ellipsoid in real space [1]. In this paper we demonstrate existence of self-consistent solution for a 3D time-dependent ellipsoid with zero phase space volume. Time - independent solutions for an azimutally-symmetric ellipsoid (2-dimensional spheroid) were treated in Refs. [2], [3] and time -dependent solution for the same ellipsoid was found in Ref. [4].

Consider evolution of an uniformly charged ellipsoid propagating in z-direction of Cartesian system of coordinates (x, y, z) with average velocity of $v_s = \beta_s c$. Space charge density of ellipsoid is given by

$$\rho = \begin{cases} \dfrac{3}{4\pi}\dfrac{Q}{R_x R_y R_z}, & \dfrac{x^2}{R_x^2} + \dfrac{y^2}{R_y^2} + \dfrac{(z - v_s t)^2}{R_z^2} \leq 1 \\ 0, & \dfrac{x^2}{R_x^2} + \dfrac{y^2}{R_y^2} + \dfrac{(z - v_s t)^2}{R_z^2} > 1 \end{cases} , \qquad (1)$$

where Q is a charge of ellipsoid and R_x, R_y, R_z are semi-axes of ellipsoid. Let us introduce new canonical-conjugate variables (ζ, p_z), where $\zeta = z - v_s t$ is a deviation from center of ellipsoid and $p_z = p - p_s$ is a deviation from longitudinal momentum of center particle. We consider regime, when spread of particle energies is much smaller, than the average energy of the beam, $\Delta\gamma/\gamma << 1$, where $\gamma = (1 - v_s^2/c^2)^{-1/2}$. Under that assumption, Hamiltonian of particle motion in new variables is given by [5]

$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + \frac{p_z^2}{2\, m\, \gamma^3} + q\, U_{ext} + q\, \frac{U_b}{\gamma^2} , \qquad (2)$$

where p_x, p_y are transverse momentum, U_{ext} is a potential of external field:

$$U_{ext}(x, y, \zeta, t) = G_x(t)\frac{x^2}{2} + G_y(t)\frac{y^2}{2} + G_z(t)\frac{\zeta^2}{2}\ , \qquad (3)$$

$G_x(t)$, $G_y(t)$, $G_z(t)$ are time-dependent gradients of the external field and U_b is a space charge potential of the beam.

Let us introduce system of coordinates (x', y', ζ'), which moves with the same velocity as that of the center of ellipsoid. Coordinates of both frames are connected via Lorentz transformation:

$$x' = x\,, \quad y' = y\,, \quad \zeta' = \gamma\zeta\ . \qquad (4)$$

In the moving system of coordinates vector potential of the beam is zero, $\vec{A}_b = 0$, and beam field is described by a scalar potential [6]:

$$U_b'(x',y', \zeta') = -\frac{\rho'}{4\,\varepsilon_o} R_x'R_y'R_z' \int_0^\infty \left(\frac{x'^2}{R_x'^2 + s} + \frac{y'^2}{R_y'^2 + s} + \frac{\zeta'^2}{R_z'^2 + s}\right) \frac{ds}{\sqrt{(R_x'^2+s)(R_y'^2+s)(R_z'^2+s)}}, \quad (5)$$

where $R_x' = R_x$, $R_y' = R_y$, $R_z' = \gamma R_z$ are semi-axes and ρ' is a space charge density of ellipsoid in moving system:

$$\rho' = \frac{3}{4\pi}\frac{Q}{R_x'R_y'R_z'} = \frac{\rho}{\gamma}\ . \qquad (6)$$

According to Lorentz transformation, scalar potential in laboratory system is $U_b = \gamma U_b'$:

$$U_b(x,y, \zeta) = -\frac{\rho}{4\,\varepsilon_o} R_x R_y \gamma R_z \int_o^\infty \left(\frac{x^2}{R_x^2 + s} + \frac{y^2}{R_y^2 + s} + \frac{\gamma^2\zeta^2}{\gamma^2 R_z^2 + s}\right) \frac{ds}{\sqrt{(R_x^2+s)(R_y^2+s)(\gamma^2 R_z^2+s)}}\ . \quad (7)$$

From Hamiltonian, Eq. (2), Lorentz force of the beam is connected with space charge potential by relationship:

$$\vec{F}^{(b)} = - \frac{1}{\gamma^2} \text{grad } U_b \, , \tag{8}$$

and components of Lorentz force are:

$$F_x^{(b)} = - \frac{1}{\gamma^2} \frac{\partial U_b}{\partial x} = \frac{\rho \, M_x}{\gamma^2 \varepsilon_o} x, \qquad M_x(R_x, R_y, \gamma R_z) = \frac{1}{2} \int_o^\infty \frac{R_x R_y \, \gamma R_z \, ds}{(R_x^2+s) \sqrt{(R_x^2+s)(R_y^2+s)(\gamma^2 R_z^2+s)}}, \tag{9}$$

$$F_y^{(b)} = - \frac{1}{\gamma^2} \frac{\partial U_b}{\partial y} = \frac{\rho \, M_y}{\gamma^2 \varepsilon_o} y, \qquad M_y(R_x, R_y, \gamma R_z) = \frac{1}{2} \int_o^\infty \frac{R_x R_y \, \gamma R_z \, ds}{(R_y^2+s) \sqrt{(R_x^2+s)(R_y^2+s)(\gamma^2 R_z^2+s)}} \, , \tag{10}$$

$$F_z^{(b)} = - \frac{1}{\gamma^2} \frac{\partial U_b}{\partial \zeta} = \frac{\rho \, M_z}{\varepsilon_o} \zeta, \quad M_z(R_x, R_y, \gamma R_z) = \frac{1}{2} \int_o^\infty \frac{R_x R_y \, \gamma R_z \, ds}{(\gamma^2 R_z^2+s) \sqrt{(R_x^2+s)(R_y^2+s)(\gamma^2 R_z^2+s)}} \, . \tag{11}$$

Consider dynamics of an arbitrary element inside the ellipsoid with coordinates (x, x+ dx), (y, y + dy), (z, z + dz) which contains dN (x, y, z) particles. External focusing fields are linear functions of coordinates:

$$F_x^{(ext)} = - G_x(t) \, x, \qquad F_y^{(ext)} = - G_y(t) \, y, \qquad F_z^{(ext)} = - G_z(t) \, \zeta \, . \tag{12}$$

Assume that the ellipsoid remains uniformly populated, therefore equations of particle motion under the external field and space charge forces of the ellipsoid are linear:

$$\frac{dx}{dt} = \frac{p_x}{m\gamma}, \qquad \frac{dy}{dt} = \frac{p_y}{m\gamma}, \qquad \frac{d\zeta}{dt} = \frac{p_z}{m\gamma^3}, \tag{13}$$

$$\frac{dp_x}{dt} = - qG_x(t)x + q \, \frac{\rho(t) \, M_x(t)}{\gamma^2 \varepsilon_o} x, \tag{14}$$

$$\frac{dp_y}{dt} = - qG_y(t)y + q\,\frac{\rho(t)\,M_y(t)}{\gamma^2\varepsilon_o}\,y, \tag{15}$$

$$\frac{dp_z}{dt} = - qG_z(t)\zeta + q\,\frac{\rho(t)\,M_z(t)}{\varepsilon_o}\,\zeta\ . \tag{16}$$

The general solution of the set of linear differential equations of the first order is the linear combination of the initial conditions:

$$\begin{vmatrix} x(t) \\ p_x(t) \end{vmatrix} = \begin{vmatrix} a_{11}(t) & a_{12}(t) \\ a_{11}(t) & a_{12}(t) \end{vmatrix}\ \begin{vmatrix} x_o \\ p_{xo} \end{vmatrix}\ , \tag{17}$$

where $a_{ij}(t)$, i,j =1, 2 are coefficients of the solution matrix. Similar solutions are valid for the y, p_y and ζ, p_z variables. Let us introduce an additional requirement that the initial particle momenta are linear functions of the coordinates:

$$p_{xo} = \alpha_x\cdot x_o\ , \qquad p_{yo} = \alpha_y\cdot y_o\,, \qquad p_{zo} = \alpha_z\cdot\zeta_o\ . \tag{18}$$

In this case the solution, x(t), is a linear function of the initial particle position:

$$x(t) = a_{11}(t)\,x_o + a_{12}(t)\,\alpha_x x_o = c_x(t)\cdot x_o\ , \tag{19}$$

and similarly,

$$y(t) = c_y(t)\cdot y_o, \tag{20}$$

$$\zeta(t) = c_z(t)\cdot\zeta_o\ . \tag{21}$$

At a fixed moment of time the volume of a selected element, $dV(t) = dx(t)\,dy(t)\,d\zeta(t)$, is connected with the initial volume, $dV_o = dx_o\,dy_o\,d\zeta_o$, by the linear relationship

$$dx(t)\,dy(t)\,d\zeta(t) = c_x(t)\,c_y(t)\,c_z(t)\,dx_o\,dy_o\,d\zeta_o, \quad \text{or} \quad dV(t) = c(t)\,dV_o\ . \tag{22}$$

The number of particles inside the selected element is conserved, dN = const, because no one particle can penetrate the boundary of an element due to the linear transformation of particle positions, Eqs.(13) - (16). Therefore, the particle density, $\rho(t) = \frac{dN}{dV(t)}$, is connected with the initial density, $\rho_o = \frac{dN}{dV_o}$, by linear equation

$$\rho(t) = \rho_o \frac{dV_o}{dV(t)}, \qquad \text{or,} \qquad \rho(x,y,\zeta,t) = \frac{\rho(x_o,y_o,\zeta_o,0)}{c(t)} \quad . \tag{23}$$

Eq. (23) indicates that the initially uniformly populated ellipsoid, $\rho(x_o,y_o,\zeta_o,0) = \text{const}$, remains uniformly populated while propagating in linear field. The space charge density of the ellipsoid, $\rho(x,y,\zeta,t)$, depends only on time according to Eq. (23) and is not a function of coordinates x, y, ζ. Such an ellipsoid delivers linear space charge forces according to Eqs. (9) - (11). Therefore, the original suggestion about particle motion in a linear field is proved to be correct.

Due to the absence of momentum spread in the beam, particles at the surface of an ellipsoid remain there during the evolution of the ellipsoid, and envelope equations can be written as equations for maximum extended particles with coordinates $x = R_x$, $y = R_y$, $\zeta = R_z$:

$$\frac{d^2R_x}{dt^2} + \frac{q\,G_x(t)}{m\gamma} R_x - \frac{3}{4\pi}\frac{q}{m\gamma^3}\frac{Q}{\varepsilon_o}\frac{M_x}{R_y\,R_z} = 0 \ , \tag{24}$$

$$\frac{d^2R_y}{dt^2} + \frac{q\,G_y(t)}{m\gamma} R_y - \frac{3}{4\pi}\frac{q}{m\gamma^3}\frac{Q}{\varepsilon_o}\frac{M_y}{R_x\,R_z} = 0 \ , \tag{25}$$

$$\frac{d^2R_z}{dt^2} + \frac{q\,G_z(t)}{m\gamma^3} R_z - \frac{3}{4\pi}\frac{q}{m\gamma^3}\frac{Q}{\varepsilon_o}\frac{M_z}{R_x\,R_y} = 0 \ . \tag{26}$$

Performed analysis allows us to consider practical examples of bunched beam dynamics. One of the problem is expansion of a bunched beam in a drift space under self space charge forces. Drift of ellipsoidal bunch in a free space is described by Eqs. (24) - (26) with $G_x = G_y$

$= G_z = 0$. In Figs. 1, 2 numerical results of the drift of an proton ellipsoidal bunch with charge Q = 60 nK, initial semi-axes $R_x = 1.5$ cm, $R_y = 2.5$ cm, $R_z = 4.5$ cm and energy $\gamma = 2$ are presented. Numerical calculations were performed using the three-dimensional particle-in-cell code BEAMPATH [7] utilizing $2 \cdot 10^4$ particles on the grid $1/2\ N_x \times N_y \times N_z = 64 \times 128 \times 512$. The difference in analytical and numerical values of the ellipsoid sizes are within 2% of each other.

Another practical example is dynamics of bunched beam in rf accelerating field. In most of accelerators, transverse focusing is provided by combination of alternative-gradient quadrupole lenses, therefore, transverse gradients are functions of z:

$$G_x = G_x(z), \qquad G_y = G_y(z)\ . \tag{27}$$

Taking into account that $z = \zeta + v_s t$, transverse gradients, Eqs. (27), appear to be functions of both ζ ant t. Therefore, analysis of Section 3 is not valid in this case, because gradients have to be the functions of time only, see Eqs. (13) - (16). In smooth approximation to particle dynamics, alternative-gradient focusing is substituted by an effective continuous focusing and potential of external field is given by [5]:

$$U_{ext} = \frac{E}{k_z}\left[\ I_o\left(\frac{k_z r}{\gamma}\right)\sin(\varphi_s - k_z\,\zeta) - \sin\varphi_s + k_z\,\zeta\cos\varphi_s\right] + G_t\,\frac{r^2}{2}, \tag{28}$$

where E is an amplitude of accelerating field, φ_S is a synchronous phase, $k_z = 2\pi/(\beta_s\lambda)$ is a wave number, G_t is a constant gradient of focusing field, and $r = \sqrt{x^2 + y^2}$ is a particle radius. The potential of the external field, U_{ext}, is a nonlinear function of the coordinates z, r. In the vicinity of a synchronous particle, $k_z\zeta << 1$, $k_z r << 1$, the following expansions are valid:

$$\sin(\varphi_s - k_z\zeta) \approx \sin\varphi_s - (k_z\zeta)\cos\varphi_s - \frac{1}{2}(k_z\zeta)^2\sin\varphi_s\ , \tag{29}$$

$$I_o\left(\frac{k_z r}{\gamma}\right) \approx 1 + \frac{1}{4}\left(\frac{k_z r}{\gamma}\right)^2\ . \tag{30}$$

Under these restrictions, the potential, Eq. (28), becomes:

$$U_{ext} = G_z \frac{\zeta^2}{2} + G_t \frac{r^2}{2} [1 - \frac{G_z}{2\,\gamma^2 G_t} \frac{\sin(\varphi_s - k_z\zeta)}{\sin\varphi_s}] \approx G_z \frac{\zeta^2}{2} + G_t(1 - \frac{G_z}{2\,\gamma^2 G_t})(\frac{x^2 + y^2}{2}) , \qquad (31)$$

where G_z is a longitudinal gradient of external field

$$G_z = 2\pi \frac{E\,|\sin\varphi_s|}{\beta\lambda} , \qquad (32)$$

and transverse gradients of external field are:

$$G_x = G_y = G_t(1 - \frac{G_z}{2\,\gamma^2 G_t}) . \qquad (33)$$

The envelope equations (24) - (26) describe in this case the evolution of an ellipsoidal bunch, which sizes are much smaller than separatrix size, in an external field with constant gradients, Eqs. (32), (33). Special solutions $R_x'' = R_y'' = R_z'' = 0$ give the conditions for a stationary (time-independent) bunch, which is in equilibrium with the external field:

$$G_x = \frac{3}{4\pi\,\gamma^2} \frac{Q}{\varepsilon_o} \frac{M_x}{R_x\,R_y\,R_z}, \quad G_y = \frac{3}{4\pi\,\gamma^2} \frac{Q}{\varepsilon_o} \frac{M_y}{R_x\,R_y\,R_z}, \quad G_z = \frac{3}{4\pi} \frac{Q}{\varepsilon_o} \frac{M_z}{R_x\,R_y\,R_z} . \qquad (34)$$

In Fig. 3 the results of proton bunched beam dynamics with Q = 16 nK, $\gamma = 2$ in a channel with G_t = 22.48 kV/cm^2, G_z= 13.56 kV/cm^2 and λ = 10 cm are presented. The values of $R_x = R_y$ = 0.5 cm, R_z = 1 cm correspond to a stationary bunch. The initial conditions for an ellipsoidal bunch were selected to be R_x = 0.4 cm, R_y = 0.6 cm, R_z = 0.8 cm. Results of simulation indicate good agreement between analytical and numerical models. Deviation from the stationary solution results in oscillations around equilibrium, while the ellipsoid remains uniformly populated.

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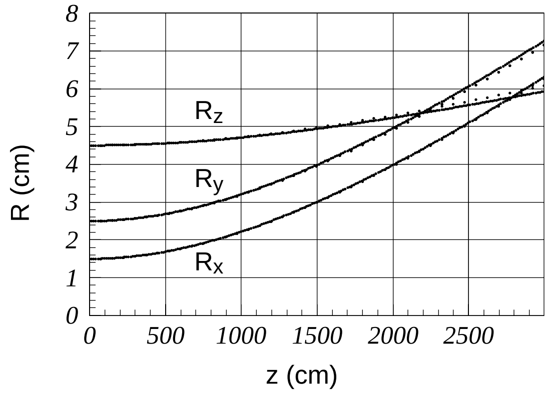

Fig. 1. Envelopes of a uniformly populated ellipsoid in a drift space: solid lines - particle-in-cell simulation, dotted lines - analytical solution of Eqs. (24) - (26).

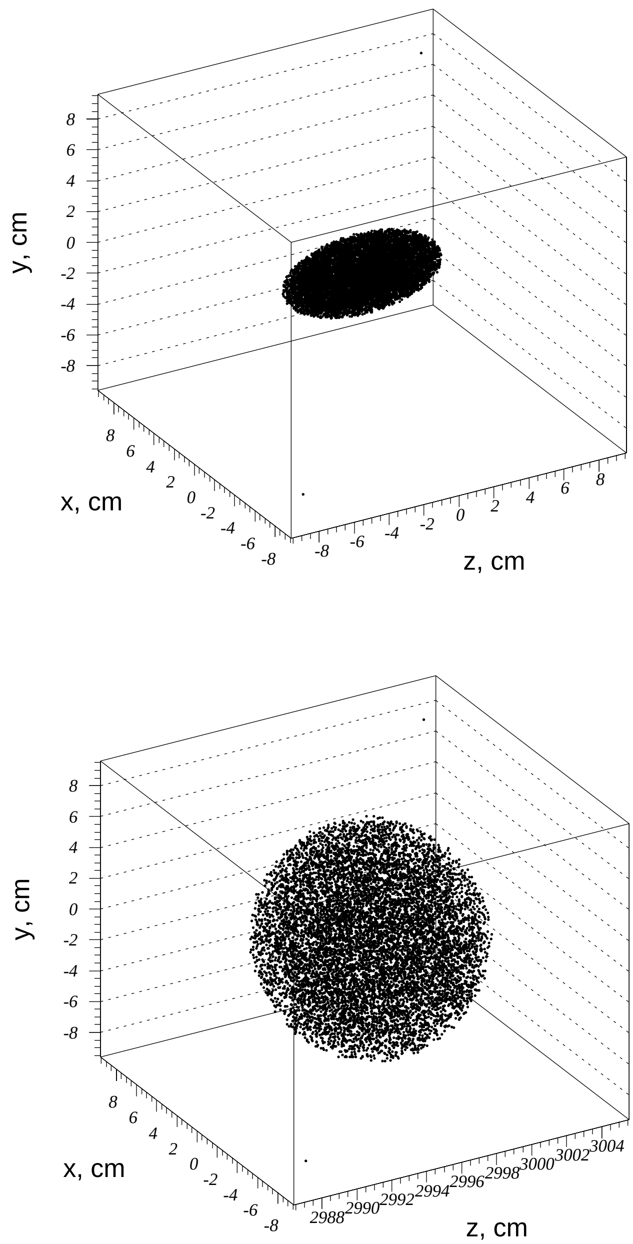

Fig. 2. Uniformly populated ellipsoid in drift space: (a) t = 0, (b) t = $1.15 \cdot 10^{-7}$ sec.

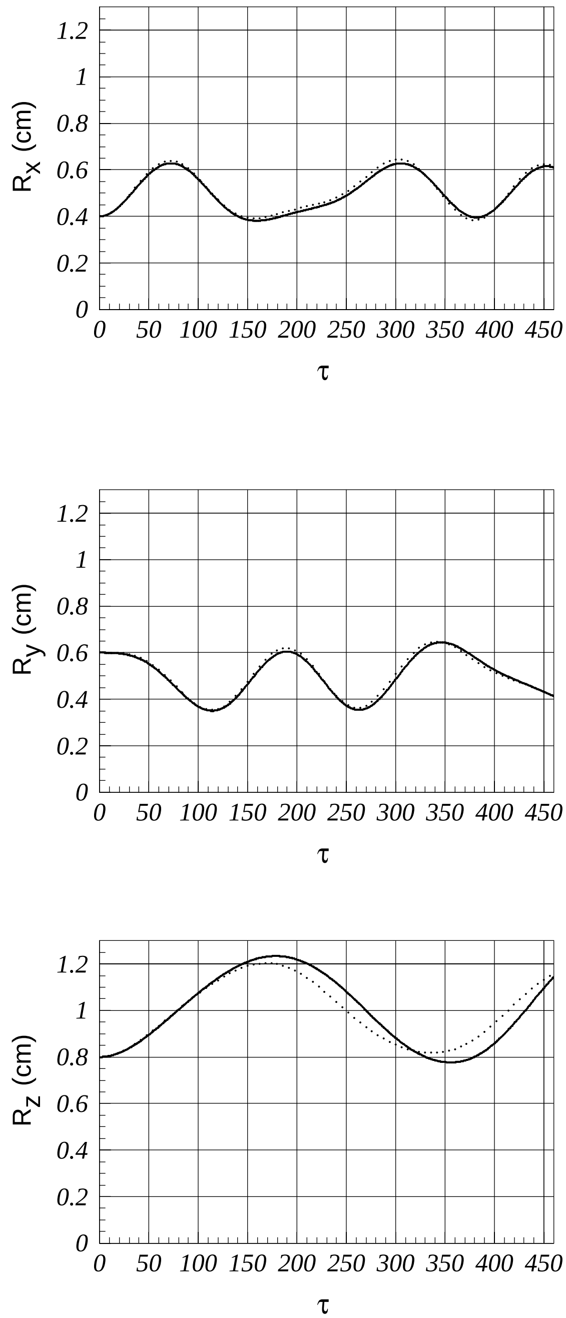

Fig. 3. Envelopes of an ellipsoid in an accelerating-focusing channel, $\tau = tc/\lambda$; solid lines - particle-in-cell simulation, dotted lines - analytical solution.