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Derived Categories and Zero-Brane Stability

Paul S. Aspinwall

Center for Geometry and Theoretical Physics,
Box 90318,
Duke University,
Durham, NC 27708-0318

Albion Lawrence

Theoretical Physics Group, MS 81,
Stanford Linear Accelerator Center,
2575 Sand Hill Road,
Menlo Park, CA 94025;
and
Stanford University Physics Department,
Stanford, CA 94305

Abstract

We define a particular class of topological field theories associated to open strings and prove the resulting D-branes and open strings form the bounded derived category of coherent sheaves. This derivation is a variant of some ideas proposed recently by Douglas. We then argue that any 0-brane on any Calabi–Yau threefold must become unstable along some path in the Kähler moduli space. As a byproduct of this analysis we see how the derived category can be invariant under a birational transformation.

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1 Introduction

The idea that a D-brane is simply some subspace of the target space where open strings are allowed to end is clearly too simple. Even at zero string coupling, we are faced with carefully analyzing the non-linear σ-model of maps from the string worldsheet into the target space. It is well-known that the non-linear σ-model modifies the usual rules of classical geometry in the target space. We expect that the notion of a D-brane should get quite complicated once such notions of stringy geometry are taken into account.

One computational handle on this issue is via string-scale target spaces that have an exact CFT description. There, D-branes can be explicitly constructed as "boundary states" after the fashion of [1, 2]. Such states have been studied at the Gepner point of a variety of Calabi-Yau threefold compactifications. Existing techniques have yielded only a handful of boundary states at these points. Furthermore their geometric or non-geometric nature is obscure. Their charges in a large radius basis are computable following the techniques in [3]; but the moduli space or any geometric interpretation of these objects as submanifolds or otherwise is difficult to find.

In fact, we can do better by studying general aspects of the open string worldsheet. Suppose our target space is a Calabi–Yau variety X, and we study D-branes which preserve an $N = 2$ worldsheet supersymmetry, *i.e.* the "B-type" branes of [4]. It was first proposed by Kontsevich [5] and then more recently by Douglas [6] that (at least in the context of zero string coupling) B-type D-branes are actually objects in the bounded derived category of coherent sheaves $\mathbf{D}(X)$. Further work related to forging a link between D-branes and $\mathbf{D}(X)$ has also appeared in [7–9] for example. In this paper we will analyze this connection in detail and assess to what extent this is true. Using this description we will begin to see how classical geometric notions of a D-brane can break down when stringy geometry becomes important.

The key idea will be to define a specific set of topological field theories with target space X. The sum of all such topological field theories is then equivalent to $\mathbf{D}(X)$. The objects in $\mathbf{D}(X)$ can be called "topological branes" following [6]. Building these topological field theories is a three-step process:

1. Begin with Witten's "B-model" of [10, 11].

2. Add the notion of integral "grading" following the ideas of Douglas [6].

3. Consider specific kinds of "marginal" deformations of this theory to form a more general class.

Defining this set of topological quantum field theories and showing it forms $\mathbf{D}(X)$ constitutes section 2 of this paper.This section borrows heavily from ideas by Kontsevich [5, 12] and especially Douglas [6] but some of the details of our construction are a little different from previous work and we try to spell out each step fairly explicitly given the complexity of the subject. Note that we address all of the words "bounded", "derived category" and "coherent" in "bounded derived category of coherent sheaves"!

Such topological field theories can often be associated to twisting the $N = 2$ CFT associated to a configuration of branes. When the volume of X is large, a single D-brane wrapped around a holomorphic cycle is expected to preserve target space supersymmetry and therefore B-type D-branes should be topological branes in the above sense, *i.e.* objects in $\mathbf{D}(X)$. On the other hand, the converse statement that all objects in $\mathbf{D}(X)$ correspond to physical D-branes is far from true.

The great advantage of using the derived category picture of D-branes as opposed to, say, K-theory is that the former picture contains considerably more data specifying the D-brane. For example, 0-branes at different points in X are different objects in $\mathbf{D}(X)$. We will use this extra knowledge to analyze the stability conditions on 0-branes in a Calabi–Yau threefold. In section 3 we will see that for any Calabi–Yau threefold embedded as a complete intersection in a toric variety, one may make any given 0-brane unstable by a process involving shrinking the overall size of the Calabi–Yau.

In section 4 we briefly discuss three issues. First, we discuss the relation of the category of topological branes to the stable branes at a given point in the Kähler moduli space, and make some suggestions about the origins of monodromy. We will then discuss briefly the appearance in the open string linear sigma models of [9,13,14] of structures similar to those found in this paper and in [6]. Finally, we discuss a very easy proof showing how $\mathbf{D}(X)$ is often invariant under a birational transformation of X.

2 The Derived Category and Topological D-Branes

The purpose of this section is to give a complete proof that the sum of a certain class of topological field theories on an algebraic variety X is equivalent (in a sense to be made precise) to the bounded derived category of coherent sheaves. Many of the ideas of this section are copied from Douglas [6]. Some of the details of the model are a little different from Douglas and the logical order of the proof is changed. We hope that this section clarifies some of the aspects of the technically difficult subject of associating D-branes to objects in the derived category.

2.1 Witten's topological field theory

We begin with a finite-dimensional complex vector bundle E with connection A over a complex Calabi–Yau manifold X. Imagine open strings propagating in X, with Chan-Paton factors living in E. Ref. [11] introduced an associated topological "B-model" for maps Φ of a worldsheet Σ, with boundaries C_k, into X. The action for the bulk of the worldsheet is:

$$\int_\Sigma \Big(g_{IJ}\partial_z\Phi^I\partial_{\bar z}\Phi^J + i\eta^{\bar\imath}(D_z\rho^i_{\bar z} + D_{\bar z}\rho^i_z)g_{i\bar\imath} + i\theta_i(D_{\bar z}\rho^i_z - D_z\rho^i_{\bar z}) + R_{i\bar\imath j\bar\jmath}\rho^i_z\rho^j_{\bar z}\eta^{\bar\imath}\theta_k g^{k\bar\jmath}\Big) d^2z\ , \qquad (1)$$

and is identical to the action for the topological "B-model" defined on closed string worldsheets [10]. Here θ and η are fermions transforming as 0-forms on Σ; and ρ is a fermion transforming as a 1-form. The indices I, J represent real coordinates on X and the corresponding lower-case indices represent (anti)holomorphic coordinates on X.

For each boundary C_k of Σ, there is an additional term in the action:

$$S_{C_k} = \oint_{C_k} \left(\Phi^*(A) - i\eta^{\bar{i}} F_{\bar{i}j} \rho^j \right) . \tag{2}$$

This appears in the path integral with a trace over the structure group of E:

$$\mathrm{Tr}_k \, \mathrm{P} \exp \left(-S_{C_k} \right) . \tag{3}$$

This topological quantum field theory on Σ has a BRST symmetry whose transformation laws are given by

$$\begin{aligned} \delta\Phi^i &= 0 \\ \delta\Phi^{\bar{i}} &= i\alpha\eta^{\bar{i}} \\ \delta\eta^{\bar{i}} &= 0 \\ \delta\theta_i &= 0 \\ \delta\rho^i &= -\alpha \, d\Phi^i , \end{aligned} \tag{4}$$

where α is a globally-defined fermionic parameter. The BRST operator Q is then defined by $\delta\Lambda = -i\alpha\{Q, \Lambda\}$ for any field Λ. The invariance of the action under Q requires that A be a *holomorphic* connection. That is, the (2,0) and (0,2) part of the curvature $F_{i\bar{j}}$ must vanish. Thus the open string "B-model" exists for "B-type" boundary conditions [3] as defined in [4, 15].

The operators in this topological field theory are then given by BRST cohomology. For open string states the operators correspond to the bundle-valued *Dolbeault* cohomology:

$$\phi \in H^{0,p}(X, \mathrm{End}\, E) . \tag{5}$$

The ghost number of a given operator is given by p. The operator product algebra is then generated by this cohomology with the product being given by the usual wedge product together with the natural composition for the group $\mathrm{End}\, E$.

For closed string states some of the BRST cohomology corresponds to deformations of the complex structure of the X. Deformations of the *Kähler* structure are BRST-exact and decouple from the theory [3, 6, 10].[1] Therefore the open topological B-model is independent of the Kähler deformations of the theory.

Finally, the correlators in the topological theory receive contributions only from constant maps into the target space. Furthermore, there is a "ghost number" (fermion number) selection rule such that correlators on the disc must violate ghost number by exactly the dimension of X to be nonvanishing.

[1]The proof in [3] of this statement for open strings showed decoupling only up to a boundary term. This term is canceled by a boundary contact term added to the integrated vertex operator, as in [16].

2.2 Adding the grading

So far we have discussed the topological sigma model associated to N D-branes which wrap the entire Calabi–Yau manifold X, and some vector bundle on them. Here N is the rank of bundle. In other words, for a Calabi–Yau threefold we have just discussed N D6-branes.[2] We consider this system a "single" D-brane. We wish to generalize this to include strings stretching between a finite collection of such objects. Eventually we will consider deformations of the topological field theory which encompass much more general situations.

One of the most important observations by Douglas in [6] is the essential rôle of a *grading* for D-branes. Consider the (untwisted) $N = 2$ worldsheet theory for strings. In the case of closed strings there is a left-moving and a right-moving spectral flow operator for the internal CFT, both of which make up part of the operator generating spacetime supersymmetry transformations. When one passes to open strings, the left-moving and right-moving sectors mix on boundaries and these two spectral flow operators cease to be independent. If the D-brane at one end of the string preserves half of the spacetime supersymmetries, the linear combination of spectral flow operators which vanishes at the boundary corresponds to the unbroken spacetime supercharge.

To each D-brane we can associate the phase shift between the left-moving and right-moving spectral flow operator at the boundary ending on that D-brane. This phase is the "grade" of the D-brane. Since the spectral flow operator can be constructed from the bosonized $\mathrm{U}(1)_R$ current of the $N = 2$ worldsheet algebra, the grade lives in $\mathrm{U}(1)$. Two branes are mutually supersymmetric only if their grade coincides. At this point the topological operators will have integral $U(1)$ charge as demanded by spacetime supersymmetry.

The mirror of our B-type D-branes gives an intuitive picture of the relative grading. For A-type branes associated to special Lagrangian submanifolds, the relative grading of two branes is literally the angle between them. Whether or not spacetime supersymmetry is broken can be determined by this angle [17, 18]. Within this framework, the strings stretching between the two D-branes can be created by a "twist field" changing the moding of the worldsheet fields; the grade is simply the worldsheet $\mathrm{U}(1)$ charge of this twist field [17].

More precisely, the grade for an A-brane wrapped on a special Lagrangian submanifold Γ is the phase of the associated period:

$$\frac{1}{\pi}\,\mathrm{Im}\log\int_\Gamma \Omega\ . \tag{6}$$

The normalization is fixed so that the grading is initially defined mod 2. If the A-brane corresponds to a BPS particle in four dimensions, this grading is just the phase of the central charge of the brane [18].

If we vary the complex structure of the target space, this grading for A-branes will vary. In particular if we initially define the relative grading of the ends of a string to be in the range $[0, 2)$, we can easily move in moduli space so that the relative grading lies outside this

[2] Our notation is that a Dp-brane wraps p real dimensions of X. That is, if we compactify on X, a Dp-brane appears as a particle in the non-compact dimensions.

range. Because of this we allow the grading to be defined in $\mathbb{R}$ rather than U(1). Note that there is no absolute meaning to a shift of the grading by two; such a shift should therefore considered a gauge symmetry.

Shifting the grade by 1, but otherwise preserving the boundary conditions, changes the linear combination of $N = 2$ spacetime supercharges that the D-brane preserves. This adds a factor of (-1) to the central charge, indicating that the shifted brane is simply the anti-brane. We may therefore refine the gauge symmetry of the previous paragraph by defining it to be *a shift in the grading by one, combined with exchanging the rôle of branes and anti-branes.*

In the open string CFT, the grading for B-branes is determined by $B + iJ$ and is given essentially by the mirror of (6):

$$\frac{1}{\pi}\,\mathrm{Im}\log\int_X e^{B+iJ}\,ch(E)\sqrt{td(T_X)}+\dots\,, \tag{7}$$

where E is the bundle (or sheaf or complex of sheaves etc.) representing the D-brane.

This grading depends essentially on $B + iJ$; therefore it will not appear as physical data in the topological B-model. Nonetheless, we wish to modify Witten's B-model to include some notion of grading. We will do this by decomposing our collection of branes as:

$$E = \bigoplus_{n=-\infty}^{\infty} E^n\,. \tag{8}$$

Since E is finite dimensional, only a finite number of the bundles E^n are nonzero. The index $n \in \mathbb{Z}$ defines the grading associated to the bundle E^n. The end of a given open string must be associated to a definite grading and hence a single summand E^n.

It is important to note two essential differences between the gradings for the physical B-branes constructed in the untwisted CFT, and the grading we have introduced into our topological field theory:

1. The gradings in the topological field theory have been fixed to be integers. In this sense the topological field theory is *less* general than the physical B-branes.

2. The gradings are fixed and do not depend upon $B + iJ$. There may be no value of $B+iJ$ for the physical branes which yields a given choice of gradings in the topological field theory. In this sense the topological field theory is *more* general than the physical B-branes.

Note also that since the gradings do not depend on $B + iJ$, our modified topological field theory is still invariant under deformations of the Kähler structure.

There is still a strong connection between our topological field theory and physical B-branes. Suppose we consider a combination of branes and anti-branes which preserve spacetime supersymmetry.[3] By an overall shift of the grading (which is physically meaningless)

[3] We call an anti-brane relative to a brane an object whose grading differs by an odd integer from the brane.

we can make the gradings of all the branes an even integer and the gradings of all the anti-branes an odd integer. If we topologically twist this theory following [10, 11, 19] we obtain a topological field theory consistent with our assumptions above. Following [6] we will refer to the boundaries in our topological field theory as "topological D-branes" to distinguish them from physical D-branes. We will address the connection between physical topological branes further in section 4.1. We note for now, however, that any single D-brane by itself preserves supersymmetry and so falls into the class of interest.

When adding the notion of grading, very little is changed in Witten's B-model. Operators are now elements of:

$$\phi \in H^{0,p}(X, (E^m)^\vee \otimes E^n) \ . \tag{9}$$

Note that the open strings are *oriented*. The above string goes from E^m to E^n. The grading is associated to the $\mathrm{U}(1)_R$ symmetry of the $N = 2$ worldsheet theory. To maintain the relationship to the untwisted theory, we define our topological field theory so that grading is associated to the ghost number. For field ϕ in (9), this is given by $p + n - m$.

It is now useful to change language a little. Rather than thinking of E^n as a complex vector bundle over the complex manifold X, we consider $\mathscr{E}^n$ to be the associated *locally free sheaf* over the algebraic variety X. Using ideas of sheaf cohomology we can rewrite (9) as (see section III.6 of [20] for example)

$$\phi \in \mathrm{Ext}^p(\mathscr{E}^m, \mathscr{E}^n) \ . \tag{10}$$

The product of operators is given by the "Yoneda pairing"

$$\mathrm{Ext}^p(\mathscr{E}^l, \mathscr{E}^m) \otimes \mathrm{Ext}^q(\mathscr{E}^m, \mathscr{E}^n) \to \mathrm{Ext}^{p+q}(\mathscr{E}^l, \mathscr{E}^n) \ , \tag{11}$$

signifying two open strings joining along a common boundary $\mathscr{E}^m$.

2.3 A category

In this section we will describe the topological field theory of the previous sections as a *category*. At first sight this looks like introducing mathematical mumbo jumbo without any real need for it. We hope that by the end of this paper the reader will be convinced that category theory is indispensable for describing D-branes. We would also like to point out that this is not the same use of category theory that was used in the context of topological field theory in knot theory (as in [21] for example). The category language for D-branes has also been used in other works such as [22].

The objects in our category will be the finite collection of nontrivial sheaves $\mathscr{E}^n$ and the morphisms will be the operators ϕ from (10). In other words, the objects are D-branes and the morphisms are open strings. Composition of morphisms is then given by the Yoneda pairing (11).

There are only two conditions that these objects and morphisms need to satisfy in order for this to be a category:

- For every object there exists an identity morphism. Clearly this is given by $1_n \in \mathrm{Ext}^0(\mathscr{E}^n, \mathscr{E}^n)$.
- The morphisms are associative. This is equivalent to the associativity of the operator product algebra of the topological field theory.[4]

The content of a topological field theory is given completely by its operator algebra. The topological field theory we are discussing is therefore completely equivalent to this category.

Note that by construction we have described a category with a finite number of objects. Later on we will also consider the category $\mathbf{T}(X)$ of all possible topological D-branes on X . This latter category has an infinite number of objects. The category associated to a particular topological field theory is a full subcategory of $\mathbf{T}(X)$.

2.4 Deformations

So far we do not have anything that resembles the derived category of coherent sheaves. The topological field theory we have discussed so far is not general enough. In this section we will look for deformations of Witten's B-model.

A deformation of a topological field theory amounts to adding a Q-closed object to the action. More specifically, in a topological CFT, take a Q-closed "local" boundary operator φ with ghost charge h. An integrated boundary vertex operator, suitable for adding to the action, is [3, 10, 19]:

$$\delta_\varphi S = t \oint_{C_k} \{G, \varphi\} \ . \tag{12}$$

where G is the fermionic spin-one current in the twisted $N = 2$ superalgebra. The BRST variation of the integrand is a total derivative. This deformation has ghost number $(h - 1)$.

Adding $\delta_\varphi S$ to the topological worldsheet action implies adding φ as a boundary term to the BRST current Q_0. To see this we can vary the integrand according to (4) with α having arbitrary dependence on the boundary coordinate τ. The coefficient of $\partial_\tau \alpha$ in δS is the deformation δQ of the BRST charge. Since $\{G, Q\} = \partial_\tau$, after an integration by parts we find that

$$\delta Q = t\varphi \equiv d \ , \tag{13}$$

where φ is supported on the boundary C_k.

In order that the topological field theory retain its possible identity as a twisted version of an $N = 2$ SCFT we require that φ has ghost number one. This is equivalent to demanding

[4] The reader may recall that in string *field* theory the multiplication of states is not always associative [23]. This arises in considering open string states created by the open-closed string vertex. The multiplication of open string vertex operators is associative, however.

that it appears as a marginal operator in the untwisted theory. The candidate ghost number one operators are

$$\begin{aligned}
&\mathrm{Ext}^0(\mathscr{E}^n, \mathscr{E}^{n+1}) = \mathrm{Hom}(\mathscr{E}^n, \mathscr{E}^{n+1}) \\
&\mathrm{Ext}^1(\mathscr{E}^n, \mathscr{E}^n) \\
&\mathrm{Ext}^2(\mathscr{E}^n, \mathscr{E}^{n-1}) \\
&\qquad\vdots
\end{aligned} \tag{14}$$

and so on up to the dimension of X.

The group $\mathrm{Ext}^1(\mathscr{E}^n, \mathscr{E}^n)$ represents first-order deformations of the sheaf $\mathscr{E}^n$. These yield obvious deformations of Witten's B-model.

Of considerably more interest are the operators living in $\mathrm{Hom}(\mathscr{E}^n, \mathscr{E}^{n+1})$. These will be of central importance to us in this paper. The higher operators $\mathrm{Ext}^2(\mathscr{E}^n, \mathscr{E}^{n-1})$, etc. will produce yet more deformations of the topological field theory. While these deformations will be distinct from $\mathrm{Ext}^1(\mathscr{E}^n, \mathscr{E}^n)$ and $\mathrm{Hom}(\mathscr{E}^n, \mathscr{E}^{n+1})$, we will see in section 2.8 that, once we introduce the derived category, they essentially add nothing new.

Our more general topological field theory will therefore be described by a set of locally free sheaves $\mathscr{E}^n$ together with holomorphic maps (i.e., morphisms in the category of sheaves)

$$d_n : \mathscr{E}^n \to \mathscr{E}^{n+1} \ . \tag{15}$$

which correspond to marginal operators in the topological sigma model. As these operators map between different D-branes, they are "boundary condition-changing operators" in the sense of [24].

Upon turning on these deformations, we have argued that Q becomes $Q_0 + d(\sigma = 0) + d(\sigma = \pi)$, where Q_0 is the original BRST operator of section 2.1 and d on each boundary is a sum of the associated ghost number 1 operators φ. In order that the deformed theory remain topological, the deformations must be integrable; *i.e.* they must be invariant under the deformed BRST charge. This implies:

$$d_{n+1} d_n = 0 \ , \qquad \forall n \ , \tag{16}$$

which is also necessary for the nilpotency of $Q_0 + \delta Q$. In other words, the topological field theory is now specified by a *complex* of locally free sheaves:

$$\ldots \longrightarrow \mathscr{E}^{-1} \xrightarrow{d_{-1}} \mathscr{E}^0 \xrightarrow{d_0} \mathscr{E}^1 \xrightarrow{d_1} \ldots \tag{17}$$

Since the original vector bundle E in section 2.1 was finite-dimensional, this complex is *bounded.*[5] We will denote this complex $\mathscr{E}^\bullet$.

[5]All complexes and derived categories in this paper are bounded from now on whether we explicitly state this or not.

2.5 The new Q-cohomology

Now that we have deformed our topological field theory and thus the Q-operator, we need to recompute the Q-cohomology to find the operator algebra of the new topological field theory.

First note that we are no longer free to associate the end of the string with a particular $\mathscr{E}^n$. We must include both boundary operators defined for a given boundary condition, and boundary condition-changing operators. In other words, switching on the d-maps generically tangles the terms in the complex together into one D-brane.

In this section we would like to consider a string stretching between two possibly *distinct* D-branes. We may achieve this as follows.

Assume that each $\mathscr{E}^n$ is actually a direct sum of two sheaves for all n. Without trying to confuse the reader too much we will call this sum $\mathscr{E}^n \oplus \mathscr{F}^n$. Now we will restrict the maps in this complex to being block-diagonal. In other words we have maps

$$
\begin{aligned}
d_n^E &: \mathscr{E}^n \to \mathscr{E}^{n+1} \\
d_n^F &: \mathscr{F}^n \to \mathscr{F}^{n+1}\ ,
\end{aligned}
\tag{18}
$$

with no d maps mixing the $\mathscr{E}$'s and $\mathscr{F}$'s.

This gives us the notion of two D-branes — one associated to $\mathscr{E}^\bullet$ and one associated to $\mathscr{F}^\bullet$. We can then assume that the boundary conditions for the start of the string are given by $\mathscr{E}^\bullet$ and the end of the string are given by $\mathscr{F}^\bullet$.

In order to compute how Q acts on such a string we will need to make use of the idea of collapsing a double complex into a single complex. Let us first consider the *sheaf*[6] given by $\mathscr{H}om(\mathscr{E}^m, \mathscr{F}^n)$. The maps d^E and d^F induce a double complex:

$$
\begin{array}{ccccc}
 & \big\downarrow{\scriptstyle d^E_1} & & \big\downarrow{\scriptstyle d^E_1} & \\
\xrightarrow{d^F_{-1}} & \mathscr{H}om(\mathscr{E}^1, \mathscr{F}^0) & \xrightarrow{d^F_0} & \mathscr{H}om(\mathscr{E}^1, \mathscr{F}^1) & \xrightarrow{d^F_1} \\
 & \big\downarrow{\scriptstyle d^E_0} & & \big\downarrow{\scriptstyle d^E_0} & \\
\xrightarrow{d^F_{-1}} & \mathscr{H}om(\mathscr{E}^0, \mathscr{F}^0) & \xrightarrow{d^F_0} & \mathscr{H}om(\mathscr{E}^0, \mathscr{F}^1) & \xrightarrow{d^F_1} \\
 & \big\downarrow{\scriptstyle d^E_{-1}} & & \big\downarrow{\scriptstyle d^E_{-1}} &
\end{array}
\tag{19}
$$

We may now form the single complex

$$
\cdots \longrightarrow \mathscr{H}om^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \xrightarrow{\bar{d}_0} \mathscr{H}om^1(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \xrightarrow{\bar{d}_1} \cdots
\tag{20}
$$

[6] Do not confuse this with the vector space $\mathrm{Hom}(\mathscr{E}^m, \mathscr{F}^n)$. $\mathscr{H}om(\mathscr{E}^m, \mathscr{F}^n)$ is the sheaf which associates an open set U with local holomorphic maps from sections of the bundle E^m over U to sections of the the bundle F^n over U.

by defining:

$$\mathscr{H}om^q(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = \bigoplus_n \mathscr{H}om(\mathscr{E}^n, \mathscr{F}^{n+q}) \; ; \tag{21}$$

and $\bar{d} = d^E + d^F$, where d^E and d^F anti-commute.

The cohomology of the complex (20) may be computed by using a *spectral sequence* (see [25] for example). In this case we have a spectral sequence which is a mixture of the usual homological and cohomological spectral sequence. One way to write it is that

$$(E_0)^n_m = \mathscr{H}om(\mathscr{E}^m, \mathscr{F}^n) \Rightarrow H^{n-m}(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) \; . \tag{22}$$

Now the problem at hand is to compute the cohomology of $Q = Q_0 + \bar{d}$. Note that Q_0 and $\bar{d}$ anti-commute. This suggests another double complex:

$$\begin{array}{ccccccc}
 & & \uparrow{\scriptstyle Q_0} & & \uparrow{\scriptstyle Q_0} & & \\
\xrightarrow{\bar{d}} & & \Omega^1(\mathscr{H}om^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & \xrightarrow{\bar{d}} & \Omega^1(\mathscr{H}om^1(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & \xrightarrow{\bar{d}} & \\
 & & \uparrow{\scriptstyle Q_0} & & \uparrow{\scriptstyle Q_0} & & \\
\xrightarrow{\bar{d}} & & \Omega^0(\mathscr{H}om^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & \xrightarrow{\bar{d}} & \Omega^0(\mathscr{H}om^1(\mathscr{E}^\bullet, \mathscr{F}^\bullet)) & \xrightarrow{\bar{d}} & \\
 & & \uparrow{\scriptstyle Q_0} & & \uparrow{\scriptstyle Q_0} & &
\end{array} \tag{23}$$

Here we use the notation Ω^p for the complex of "things" for which Q_0 is a boundary map. What is this exactly? We saw in section 2.1 that Q_0 basically acts as the boundary operator on the twisted Dolbeault complex. That is, if we go back to differential geometry, $\Omega^p(\mathscr{H}om^q(\mathscr{E}^\bullet, \mathscr{F}^\bullet))$ may be thought of as "$(0,p)$-forms with values in the bundle associated to $\mathscr{H}om^q(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$." We want to use the corresponding sheaf cohomology for the vertical maps. That is, we should do something along the lines of having injective resolutions of $\mathscr{H}om^q(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ in the vertical direction.

Anyway, consider a spectral sequence for the complex (23). Take cohomology in the horizontal direction and then the vertical direction to obtain:

$$E_2^{p,q} = H^p(X, H^q(\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet))) \Rightarrow H_Q^{p+q} \; , \tag{24}$$

where H_Q is the desired Q-cohomology. Now (24) is a kind of "local to global" spectral sequence (see section 4.2 of [26]). This implies that the cohomology group H_Q^P is actually given by the group $\mathrm{Hom}^P(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$. We may describe this group as follows.

Any locally free sheaf $\mathscr{F}^n$ has an injective resolution:

$$0 \to \mathscr{F}^n \to I^0(\mathscr{F}^n) \to I^1(\mathscr{F}^n) \to \dots \; , \tag{25}$$

where $I^s(\mathscr{F}^n)$ is an "injective object" in the category of quasi-coherent sheaves. Note that such injective objects are very peculiar things and look nothing like locally free sheaves. We may then similarly replace the entire complex $\mathscr{F}^\bullet$ by a complex of injective objects. That is, replace $\mathscr{F}^n$ in the complex by $\bigoplus_s I^s(\mathscr{F}^{n-s})$. The maps between $\mathscr{F}^n$ then have natural lifts to the maps between $I^s(\mathscr{F}^{n-s})$. We will denote this complex of injectives $\mathscr{F}^\bullet_{\text{inj}}$.

Now define a complex $\text{hom}^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ by

$$\text{hom}^P(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = \bigoplus_n \text{Hom}(\mathscr{E}^n, \mathscr{F}^{n+P}_{\text{inj}}) \ , \tag{26}$$

and the obvious maps $\text{hom}^P(\mathscr{E}^\bullet, \mathscr{F}^\bullet_{\text{inj}}) \to \text{hom}^{P+1}(\mathscr{E}^\bullet, \mathscr{F}^\bullet_{\text{inj}})$ induced by $d^E + d^F$. One may show that the groups $\text{Hom}^P(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ are then given by the cohomology of this chain complex.

We have therefore defined the group $\text{Hom}^P(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ in which the operators of the topological field theory live. Clearly P is the ghost number.

The operator product is a simple generalization of the Yoneda pairing

$$\text{Hom}^P(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \otimes \text{Hom}^Q(\mathscr{F}^\bullet, \mathscr{G}^\bullet) \to \text{Hom}^{P+Q}(\mathscr{E}^\bullet, \mathscr{G}^\bullet) \ . \tag{27}$$

2.6 Enter the derived category

Let $\mathbf{T}(X)$ be the category of all possible topological field theories of the type considered in section 2.5 with target variety X. The objects are naïvely all possible bounded complexes of locally free sheaves and the morphisms are the open string operators given by $\text{Hom}^P(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$. We now need to consider more carefully when two different complexes really should be considered "different" objects in $\mathbf{T}(X)$.

The physical content of a topological field theory is completely described by the operator product algebra. Let us consider two D-branes $\mathscr{E}_1^\bullet$ and $\mathscr{E}_2^\bullet$. Add $\mathscr{E}_1^\bullet$ to an arbitrary collection of D-branes and find the operator algebra of the resulting topological field theory. Now add $\mathscr{E}_2^\bullet$ to the same arbitrary collection of D-branes and find the operator algebra of this second theory. If these two algebras are isomorphic for any collection of additional D-branes then $\mathscr{E}_1^\bullet$ and $\mathscr{E}_2^\bullet$ physically represent the same D-brane and must be considered the same object in $\mathbf{T}(X)$.

After modding out by these identifications, we will show in this section that $\mathbf{T}(X)$ is very closely related to $\mathbf{D}(X)$, the derived category of coherent sheaves in X. To do this we will use the definition/theorem of the derived category $\mathbf{D}(\mathbf{A})$ of a given category $\mathbf{A}$ from chapter 4, 1.3 of [27]:

Theorem 1 *Let* $\mathbf{A}$ *be an abelian category, and let* $\mathbf{Kom}(\mathbf{A})$ *be the category of complexes over* $\mathbf{A}$. *There exists a category* $\mathbf{D}(\mathbf{A})$ *and a functor* $\mathsf{Q} : \mathbf{Kom}(\mathbf{A}) \to \mathbf{D}(\mathbf{A})$ *with the following properties:*

1. $\mathsf{Q}(f)$ *is an isomorphism for any quasi-isomorphism* f.

2. *Any functor* $\mathsf{F} : \mathbf{Kom}(\mathbf{A}) \to \mathbf{C}$ *transforming quasi-isomorphisms into isomorphisms can be uniquely factorized through* $\mathbf{D}(\mathbf{A})$, *i.e., there exists a unique functor* $\mathsf{G} : \mathbf{D}(\mathbf{A}) \to \mathbf{C}$ *with* $\mathsf{F} = \mathsf{G} \circ \mathsf{Q}$.

Recall that a "quasi-isomorphism" is a chain map which induces an isomorphism on the cohomology groups of the complex.

Let $\mathbf{K}_{\mathrm{LF}}(X)$ be the category of complexes of locally free sheaves on X. The morphisms in this category are chain maps. We wish to construct a functor $\mathsf{F} : \mathbf{K}_{\mathrm{LF}}(X) \to \mathbf{T}(X)$. The definition of this functor is pretty obvious. A chain in $\mathbf{K}_{\mathrm{LF}}(X)$ maps to the corresponding D-brane in $\mathbf{T}(X)$. A chain map in $\mathbf{K}_{\mathrm{LF}}(X)$ maps to an element of $\mathrm{Hom}^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ in $\mathbf{T}(X)$. Note that homotopic chain maps are identified by F.

Now we wish to argue that a quasi-isomorphism in $\mathbf{K}_{\mathrm{LF}}(X)$ maps to an isomorphism in $\mathbf{T}(X)$. Note that the cohomology of a complex of locally-free sheaves is a set of *coherent sheaves* in general but this still allows us to define the notion of a quasi-isomorphism in $\mathbf{K}_{\mathrm{LF}}(X)$.

Consider a quasi-isomorphism between two complexes $f : \mathscr{E}_1^\bullet \to \mathscr{E}_2^\bullet$. Clearly this induces a map

$$f^* : \mathrm{Hom}^P(\mathscr{E}_2^\bullet, \mathscr{F}^\bullet) \to \mathrm{Hom}^P(\mathscr{E}_1^\bullet, \mathscr{F}^\bullet) \ , \tag{28}$$

from the diagram (for e.g., $P = 1$)

$$\begin{array}{ccccccccc} \longrightarrow & \mathscr{E}_1^0 & \longrightarrow & \mathscr{E}_1^1 & \longrightarrow & \mathscr{E}_1^2 & \longrightarrow & & \\ & \downarrow f & & \downarrow f & & \downarrow f & & & \\ \longrightarrow & \mathscr{E}_2^0 & \longrightarrow & \mathscr{E}_2^1 & \longrightarrow & \mathscr{E}_2^2 & \longrightarrow & & \\ \searrow & & \searrow & & \searrow & & \searrow & & \\ \longrightarrow & \mathscr{F}^0 & \longrightarrow & \mathscr{F}^1 & \longrightarrow & \mathscr{F}^2 & \longrightarrow & & \end{array} \tag{29}$$

We would like to show that f^* is actually an isomorphism.

Consider first the case where $\mathscr{E}^\bullet$ is acyclic, i.e., a complex with trivial cohomology and let $\mathscr{F}^\bullet$ be any complex. The spectral sequence (22) tells us that the cohomology of $\mathscr{H}om^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ is trivial. The spectral sequence (24) then tells us that the Q-cohomology of the topological field theory is trivial. Similarly if $\mathscr{F}^\bullet$ is acyclic then $\mathrm{Hom}^P(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = 0$ for any $\mathscr{E}^\bullet$.

Now return to the quasi-isomorphism $f : \mathscr{E}_1^\bullet \to \mathscr{E}_2^\bullet$. We may construct the "Cone" of the map f which is also a complex of sheaves. We refer to chapter 2, 2.2 of [27] for a definition of Cone. The cone of a map yields an acyclic complex if and only if the map is a quasi-isomorphism.

The map f will induce a chain map $f^\sharp$ as follows:

$$\begin{array}{ccccc} \longrightarrow & \mathrm{hom}^0(\mathscr{E}_1^\bullet, \mathscr{F}^\bullet) & \longrightarrow & \mathrm{hom}^1(\mathscr{E}_1^\bullet, \mathscr{F}^\bullet) & \longrightarrow \\ & \uparrow f^\sharp & & \uparrow f^\sharp & \\ \longrightarrow & \mathrm{hom}^0(\mathscr{E}_2^\bullet, \mathscr{F}^\bullet) & \longrightarrow & \mathrm{hom}^1(\mathscr{E}_2^\bullet, \mathscr{F}^\bullet) & \longrightarrow \end{array} \tag{30}$$

If f is a quasi-isomorphism then its cone is acyclic. Thus the groups $\mathrm{Hom}^P(\mathrm{Cone}(f), \mathscr{F}^\bullet)$ associated to the cone are zero. This in turn implies that the cone of $f^\sharp$ is acyclic which in turn shows that $f^\sharp$ is a quasi-isomorphism. Thus $\mathrm{Hom}^P(\mathscr{E}_1^\bullet, \mathscr{F}^\bullet) \cong \mathrm{Hom}^P(\mathscr{E}_2^\bullet, \mathscr{F}^\bullet)$ and (28) provides the canonical isomorphism.

Similarly one may show that for a quasi-isomorphism $g : \mathscr{F}_1^\bullet \to \mathscr{F}_2^\bullet$, the induced map $g_* : \mathrm{Hom}^P(\mathscr{E}^\bullet, \mathscr{F}_1^\bullet) \to \mathrm{Hom}^P(\mathscr{E}^\bullet, \mathscr{F}_2^\bullet)$ is an isomorphism.

We have shown that a quasi-isomorphism in $\mathbf{K}_{\mathrm{LF}}(X)$ maps to an isomorphism in $\mathbf{T}(X)$. However, in order to use theorem 1 we need to consider complexes over an *abelian* category. The category of locally-free sheaves is not abelian, since it does not contain its own cokernels. We may consider the larger category of *coherent* sheaves instead. Let $\mathbf{Kom}(X)$ denote the category of bounded complexes of coherent sheaves on X.

In order to define a functor from $\mathbf{Kom}(X)$ to $\mathbf{T}(X)$ we need to define the image of complexes of coherent sheaves which are not locally free. This is actually very easy given our above analysis of quasi-isomorphisms.

Given any coherent sheaf we can find a locally free resolution and hence a complex of locally free sheaves which is quasi-isomorphic to a complex containing only our original coherent sheaf. Furthermore given any map between coherent sheaves we may find locally free resolutions which allow this map to be lifted to a map between the complexes of locally free sheaves. Therefore any configuration of branes involving coherent sheaves may be rewritten in terms of locally free sheaves.

All this shows that $\mathbf{T}(X)$ already contains the image of complexes of all coherent sheaves. In a way we have justified various conjectures in the past (e.g. [28, 29]) that *coherent* sheaves can be relevant for describing D-branes.

We have now constructed a functor from $\mathbf{Kom}(X)$ to $\mathbf{T}(X)$. Therefore by theorem 1 we have constructed a functor

$$\mathsf{G} : \mathbf{D}(X) \to \mathbf{T}(X)\ . \tag{31}$$

2.7 G as an equivalence of categories

The statement that the category of all D-branes in our topological field theories is the same as the derived category of coherent sheaves on X amounts to saying that the functor G in (31) is an equivalence of categories.

As it stands this is not quite true. However, we may make some fairly innocuous changes to make it true. Let us define the category $\mathbf{T}_0(X)$ which is a subcategory of $\mathbf{T}(X)$. $\mathbf{T}_0(X)$ contains exactly the same objects as $\mathbf{T}(X)$ but we only consider morphisms $\mathrm{Hom}^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$. That is, we throw out all open strings of nonzero ghost number. It is easy to argue that $\mathbf{T}_0(X)$ is equivalent to $\mathbf{D}(X)$ as we now show.

To show that $\mathsf{G} : \mathbf{D}(X) \to \mathbf{T}_0(X)$ is an equivalence of categories we need to show that G is "full, faithful, and dense" (see for example section 14 of [30]).

The "dense" property asserts that every object in $\mathbf{T}_0(X)$ is isomorphic to some object in the image of G. This is clear — every D-brane can be represented by a complex of coherent sheaves.

Now fix a pair of objects $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$ in $\mathbf{D}(X)$. The morphisms between $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$ come from chain maps up to various equivalence relations upon building $\mathbf{D}(X)$. These morphisms are given exactly by $\mathrm{Hom}^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ — the ghost number zero open strings in $\mathbf{T}_0(X)$. Thus G is an isomorphism for morphisms between any pair $\mathscr{E}^\bullet$, $\mathscr{F}^\bullet$. This shows that G is "full" and "faithful".

Having shown that $\mathsf{G} : \mathbf{D}(X) \to \mathbf{T}_0(X)$ is an equivalence of categories, it is fairly trivial to see that $\mathbf{T}_0(X)$ contains essentially the same information as $\mathbf{T}(X)$. It is easy to show that

$$\mathrm{Hom}^P(\mathscr{E}^\bullet, \mathscr{F}^\bullet) = \mathrm{Hom}^0(\mathscr{E}^\bullet, \mathscr{F}^\bullet[P]) \ , \tag{32}$$

where $[P]$ means shift the complex P places to the left. This means that the open strings with nonzero ghost number are also found in $\mathbf{T}_0(X)$.

In this sense *the category of topological field theories given by* $\mathbf{T}(X)$ *is essentially the same thing as the bounded derived category of coherent sheaves* $\mathbf{D}(X)$.

2.8 "Tachyon" condensation

Finally in our review and analysis of the rôle of the derived category we would like to discuss the "tachyon condensation" of [31].[7] This is nothing more than giving a vacuum expectation value to a string vertex operator in the worldsheet theory. In the physical theory the tachyon corresponds to a relevant boundary operator. One can understand the endpoint of condensation as the endpoint of the RG flow induced by a relevant perturbation [33, 34]. On the other hand, we can move to a line of marginal stability where the vertex operator becomes marginal and (if it is exactly marginal) study the deformation entirely in the CFT.

Consider the open strings between complexes given by $\mathscr{E}^\bullet$ and $\mathscr{F}^\bullet$. As in section 2.4 we will consider an operator of ghost number one $\phi \in \mathrm{Hom}^1(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$. What happens to the topological field theory if we use ϕ as a deformation?

Using exactly the same argument as in section 2.4 we find that the resulting topological field theory has a single D-brane consisting of the complex

$$\longrightarrow \mathscr{E}^0 \oplus \mathscr{F}^0 \xrightarrow{\begin{pmatrix} d_E & 0 \\ \phi & d_F \end{pmatrix}} \mathscr{E}^1 \oplus \mathscr{F}^1 \xrightarrow{\begin{pmatrix} d_E & 0 \\ \phi & d_F \end{pmatrix}} \mathscr{E}^2 \oplus \mathscr{F}^2 \longrightarrow \ . \tag{33}$$

This is nothing other than $\mathrm{Cone}(\phi : \mathscr{E}^\bullet[-1] \to \mathscr{F}^\bullet)$.

At a general value of $(B + iJ)$ the operator ϕ may or may not represent a tachyon in the untwisted theory. The mass of the spacetime field associated to ϕ depends upon $B + iJ$; so the topological theory is independent of the mass. If ϕ happens to be be tachyonic then $\mathrm{Cone}(\phi : \mathscr{E}^\bullet[-1] \to \mathscr{F}^\bullet)$ represents a "bound state" of $\mathscr{E}^\bullet[-1]$ and $\mathscr{F}^\bullet$. This is the basic idea behind the "Π-stability" of [6, 18]. Note that there are other formulations of stability as in [35] for example.

[7] Tachyon condensation and brane-anti-brane annihilation in the topological A-model has been discussed in [32].

Let us consider as an example the case of a Calabi–Yau threefold X and a particular D-brane/anti-D-brane pair. The D-brane $\mathscr{F}^\bullet$ is the simple, one-term complex

$$\longrightarrow 0 \longrightarrow \underline{\mathscr{O}_X} \longrightarrow 0 \longrightarrow \ , \tag{34}$$

where $\mathscr{O}_X$ is the structure sheaf of X. This complex represents the trivial line bundle, or basic D6-brane, over X. In our notation the underline represents the location of the zeroth position in the complex.

Let the anti-D-brane $\mathscr{E}^\bullet$ be given by

$$\longrightarrow 0 \longrightarrow \mathscr{O}_X(-D) \longrightarrow \underline{0} \longrightarrow \ , \tag{35}$$

where D is a particular 4-cycle $D \subset X$. That is, we have an anti-D6-brane with a D4-brane charge given by D. (This is an anti-brane rather than a brane because it appears in an odd position in the complex.)

We now have a map $f : \mathscr{O}_X(-D) \to \mathscr{O}_X$ in $\mathrm{Hom}^1(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ giving a cone:

$$\longrightarrow 0 \longrightarrow \mathscr{O}_X(-D) \xrightarrow{f} \underline{\mathscr{O}_X} \longrightarrow \ , \tag{36}$$

which is quasi-isomorphic to

$$\longrightarrow 0 \longrightarrow \underline{\mathscr{O}_D} \longrightarrow 0 \longrightarrow \ , \tag{37}$$

where $\mathscr{O}_D$ is the structure sheaf of D extended by zero over X.

This is the topological field theory statement of a D6-brane/anti-D6-brane pair forming a D4-brane state. Again let us emphasize that the choice of grading above for this pair would imply unbroken supersymmetry and hence marginal stability. Typically, if X is large, this will not be the case and f will be a tachyon which in turn makes the D4-brane a bound state. At a given value of $(B+iJ)$ the *stable* object will be described by a particular tachyon vev.

Finally, let us discuss the higher Ext's of section 2.4, which we chose to ignore at that point. For example, consider $\mathrm{Ext}^2(\mathscr{E}^n, \mathscr{E}^{n-1})$. Writing the sheaves $\mathscr{E}^n$ and $\mathscr{E}^{n-1}$ as one-term complexes we may write $\mathrm{Ext}^2(\mathscr{E}^n, \mathscr{E}^{n-1}) = \mathrm{Hom}^1(\underline{\mathscr{E}}^n, \underline{\mathscr{E}}^{n-1}[1])$. Thus, even though $\mathrm{Ext}^2(\mathscr{E}^n, \mathscr{E}^{n-1})$ gives distinct deformations for a particular topological field theory, once we pass to the big category $\mathbf{T}(X)$, there are enough objects (e.g. $\underline{\mathscr{E}}^{n-1}[1]$) for us to replace all higher Ext's by the deformations of the form we considered in (33). That is, at least to first order, these higher Ext's do not deform the theories outside the class given by $\mathbf{T}(X)$.

3 Zero-Brane Stability

In this section we will discuss 0-brane stability based on some observations about monodromy. The basic idea is as follows. Let us begin with a 0-brane which, we assume, is a stable object on large Calabi–Yau threefold. Now follow this object around a loop in the moduli space of complexified Kähler form on X. If the resulting monodromy results in something which is manifestly unstable then we must have crossed a line of marginal stability for the 0-brane during our travels, resulting in 0-brane decay.

This situation is similar to that in $N = 2, d = 4$ $SU(2)$ Yang-Mills theory [36]. There the charge spectrum of stable BPS states at weak coupling is not invariant under the monodromy group of the theory. If we follow the theory around a loop which generates one of the offending monodromy transformations, the BPS spectrum jumps.

In this case we can make a finer statement. It may be that states with D0-brane charge continue to exist. But there are many objects in $\mathbf{D}(X)$ which carry D0-brane charge. Only some of these objects are 0-branes in the sense of being points in a large-volume Calabi-Yau. We will ask about the action of monodromy on these "point" objects.

3.1 What is a 0-brane?

Define $\underline{\mathscr{F}}$ as a complex containing all zeroes except $\mathscr{F}$ at the zeroth position. In the language of derived categories, a 0-brane on X will be an object in $\mathbf{D}(X)$ which can be represented by $\underline{\mathscr{O}_p}$, where $\mathscr{O}_p$ is the skyscraper sheaf of a point $p \in X$. Obviously p is the location of the 0-brane in X.

Locally, in affine coordinates, we may use a Koszul resolution in terms of free sheaves to represent the same object as:

$$0 \longrightarrow \mathscr{O} \xrightarrow{\begin{pmatrix} z \\ x \\ y \end{pmatrix}} \mathscr{O}^{\oplus 3} \xrightarrow{\begin{pmatrix} y & 0 & -z \\ -x & z & 0 \\ 0 & -y & x \end{pmatrix}} \mathscr{O}^{\oplus 3} \xrightarrow{(\, x \; y \; z \,)} \underline{\mathscr{O}} \longrightarrow 0 \; , \tag{38}$$

where p is at $(x, y, z) = (0, 0, 0)$.

Now given any object $\mathscr{F}^\bullet \in \mathbf{D}(X)$ we may compute the "D-brane charge" in $H^{\mathrm{even}}(X)$ in terms of a locally free resolution as

$$ch(\mathscr{F}^\bullet) = \sum_n (-1)^n \, ch(\mathscr{F}^n) \; . \tag{39}$$

The D-brane charge of a 0-brane on a Calabi–Yau threefold will be a 6-form in De Rham cohomology which is Poincaré dual to a point. The converse of this statement need not be true. There are many objects in $\mathbf{D}(X)$ which have the D-brane charge of a 0-brane but are not quasi-isomorphic to a complex containing just $\mathscr{O}_p$.

Thus there is more to being a 0-brane than simply having 0-brane charge. Let us discuss a particularly clear illustration from a *flop*, based on the work of Bridgeland [37] and also noted in [6].

Consider a Calabi–Yau threefold X and a flop X' of X. It was shown explicitly in [37] that there is an equivalence of categories $\mathbf{D}(X) \sim \mathbf{D}(X')$. Assuming X and X' are topologically inequivalent there is clearly no map equating the 0-branes of X with the 0-branes of X'. Actually the identification must be done as follows. Away from the exceptional locus of the flop we may identify 0-branes of X with 0-branes of X'. There are then objects in $\mathbf{D}(X)$

which correspond to 0-branes on the exceptional curve $C \subset X$ and there is a whole bunch of different objects in $\mathbf{D}(X)$ which correspond to 0-branes on the other exceptional curve $C' \subset X'$.

That is, there is a plethora of objects in $\mathbf{D}(X)$ which look like they might be 0-branes. One way of distinguishing them is via stability. The 0-branes corresponding to point on C' are presumably unstable when X is at large radius limit and *vice versa*.

3.2 Monodromy

Now consider the action of monodromy on objects in $\mathbf{D}(X)$ as we follow loops in the moduli space of the complexified Kähler form $B + iJ$. The topological field theory of section 2 is invariant under changes of the Kähler form and so the action of the monodromy is manifestly trivial!

In order to see monodromy we need to restore the dependence of the grading on $(B+iJ)$. Now if we identify a given physical D-brane as an object in $\mathbf{D}(X)$, the monodromy will act nontrivially to produce a different object in $\mathbf{D}(X)$. At this point we do not understand how to see the action of this monodromy directly. We will discuss some hints for how it arises in section 4.1.

Here we will instead assume a conjecture by Kontsevich, Morrison and Horja [12,38–40]. At least for Calabi–Yau threefolds embedded as complete intersections in toric varieties there is a distinguished divisor in the moduli space known as the "primary component of the discriminant" [39, 40].[8] It seems reasonable to assume that this component of the discriminant can be defined for general Calabi–Yau threefolds.

According to the conjecture by Kontsevich et al, a loop around the primary component of the discriminant transforms an element of $\mathbf{D}(X)$ as

$$T(\mathscr{F}^\bullet) = \mathrm{Cone}\Big(\hom(\underline{\mathscr{O}_X}, \mathscr{F}^\bullet) \otimes \underline{\mathscr{O}_X} \to \mathscr{F}^\bullet\Big) \ , \tag{40}$$

where the operator "$\otimes$" is defined in the obvious way in the derived category and is explained further, along with many other interesting facts about this Fourier–Mukai transform, in [41]. A generalization of this Fourier–Mukai transform is discussed in [42].

What happens to a 0-brane upon monodromy around the primary component? First note that if $\mathscr{F}^\bullet$ is of the form $\underline{\mathscr{F}}$ (i.e., concentrated at position zero) then the cohomology of the complex $\hom(\underline{\mathscr{O}_X}, \mathscr{F}^\bullet)$ is given by the sheaf cohomology of $\mathscr{F}$. The skyscraper sheaf has trivial cohomology given by $H^0 = \mathbb{C}$, all other cohomology vanishes. It follows that $\hom(\underline{\mathscr{O}_X}, \underline{\mathscr{O}_p}) \otimes \underline{\mathscr{O}_X}$ is given simply by $\underline{\mathscr{O}_X}$. Therefore

$$T(\underline{\mathscr{O}_p}) = \Big(\mathscr{O}_X \to \underline{\mathscr{O}_p}\Big) \ , \tag{41}$$

which is quasi-isomorphic to $\underline{\mathscr{I}_p}[1]$ — the ideal sheaf of a point p shifted left by one.

[8] Also known sometimes as the "principal" component.

3.3 Stability

Stability of the 0-brane at small volume now depends on whether $\mathscr{I}_p[1]$ can become a stable object in the large radius limit. By this we mean that either the $\overline{\text{anti}}$-D6-brane $\mathscr{O}_X$ and the D0-brane $\mathscr{O}_p$ are mutually supersymmetric, or they can form a supersymmetric bound state (as with the D0-D2 system). A signature of the latter would be an attractive force between the objects at short distance.

We can ask about stability at arbitrarily large volume; the force is then identical to that between the D0 and anti-D6 branes in flat space.

The static force is computed for the D0-D6 system in [43,44], and is repulsive. The answer is identical for the D0/anti-D6 system. The potential energy is equal to the one-loop vacuum energy for open strings stretched between the two branes. Reversing the charge of the D6-brane in this calculation simply changes the parity $(-1)^F$ of open string states preserved by the GSO projection. But due to fermion zero modes in the D0/anti-D6 system, $\mathrm{tr}(-1)^F q^{L_0}$ vanishes in both the Neveu-Schwarz and Ramond sectors [43]; so the one-loop amplitude is independent of the GSO projection. Therefore the D0- and anti-D6-branes also repel each other. Another argument is that the two D-branes break spacetime supersymmetry at large volume [45] (see also the earlier paper [46]).[9]

Using the same logic as in [36], we see that if one follows a loop around the primary component of the discriminant locus then we must cross a line of marginal stability for any 0-brane and hence the 0-branes will be unstable for some subset of the loop.

The location of the primary component of the discriminant was discussed in [40]. If one begins at large radius then a loop around the primary component will necessarily leave the "geometric phases" region of the moduli space. For example, in the simple case of the quintic threefold, the primary component lies in the phase boundary between the large-radius Calabi–Yau phase and the Landau–Ginzburg phase. In more complicated threefold examples, the primary component will appear in the wall of a phase boundary whenever the effective dimension of the target space decreases below three. For example, one will *not* see the primary component when blowing down a divisor to produce an orbifold singularity but one can see the primary component if one passes to a "hybrid" phase of a Landau–Ginzburg theory fibred over $\mathbb{P}^2$ say. One also passes to a non-geometric phase when performing the "exoflop" of [47].

This shows that the paths associated with destabilizing the 0-brane pass outside the geometric phases. To determine the exact way in which the 0-brane becomes unstable requires further analysis which we do not do here.

There may well be objects which are stable in a non-geometric phase, such as a Landau–Ginzburg phase, and have the right charge to be a 0-brane but are not a 0-brane in the strict sense of section 3.1. This could explain the "zero brane" found at the Gepner point in [48]. Note that even if the three massless fields found in [48] are moduli, the moduli space may

[9]In [45] a stable state was found, but at large NS-NS B-field. We may avoid this issue by beginning and ending at $\mathrm{Re}(B + iJ) = 0$. It would be interesting to see if and how the result in [45] applies to stability issues on the Calabi-Yau threefold.

be different from the large-volume threefold.

Another explanation of the D0-brane in [48] is that the D0-brane really remains stable near the Gepner point. It would be interesting to understand this further.

We note in passing that the conjectured Matrix theory description of Calabi-Yau compactifications of M-theory is as a collection of $N \to \infty$ BPS objects with 0-brane charge at the primary component of the discriminant [49]. The instability of the "point-like" 0-brane at this component of the moduli space is highly relevant to the dynamics of this large-N theory.

3.4 Another time around[10]

It is interesting to ask what happens if we take our 0-brane around the primary component a *second* time. That is, what does (40) give for $T(\underline{\mathscr{I}_p}[1])$?

From the exact sequence $0 \to \mathscr{I}_p \to \mathscr{O}_X \to \mathscr{O}_p \to 0$ it is easy to compute that $H^3(X, \mathscr{I}_p) = \mathbb{C}$ with all other cohomology groups vanishing. This implies that

$$\hom(\underline{\mathscr{O}_X}, \underline{\mathscr{I}_p}[1]) \otimes \underline{\mathscr{O}_X} = \underline{\mathscr{O}_X}[-2] \ . \tag{42}$$

Now let $I^\bullet$ be an injective resolution of $\mathscr{I}_p$. It follows that $T(\underline{\mathscr{I}_p}[1]) = \mathrm{Cone}(\underline{\mathscr{O}_X}[-2] \to \underline{\mathscr{I}_p}[1])$ is given by

$$\begin{array}{ccccccccc} I^0 & \xrightarrow{i_0} & \underline{I^1} & \xrightarrow{i_1} & I^2 & \xrightarrow{i_2} & I^3 & \xrightarrow{i_3} & \\ & & & & \oplus & \nearrow_{f} & & & \\ & & & & \mathscr{O}_X & & & & \end{array} \tag{43}$$

where f is a map which generates the sheaf cohomology group $H^3(X, \mathscr{I}_p)$.

The complex (43) represents the weird and wonderful object in $\mathbf{D}(X)$ obtained by looping a 0-brane twice around the primary component of the discriminant. This complex represents one of the interesting objects in $\mathbf{D}(X)$ which cannot be reduced to its cohomology. To see this note first that the cohomology of (43) is the same as the cohomology of

$$0 \xrightarrow{0} \mathscr{I}_p \xrightarrow{0} \underline{0} \xrightarrow{0} \mathscr{O}_X \xrightarrow{0} \ . \tag{44}$$

One might therefore be tempted to say that (43) represents an anti-D6-brane/D0-brane pair at position -1 (giving $\underline{\mathscr{I}_p}[1]$) together with an an anti-D6-brane at position 1 (giving $\underline{\mathscr{O}_X}[-1]$). It turns out however that (43) is *not* quasi-isomorphic to (44). One may check this by computing $\mathrm{Hom}^P(\mathscr{O}_X, -)$ for each complex for example. Thus the full glory of the object (43) cannot be reduced to a statement of conventional branes.

Since our naïve picture of branes as subspaces, and hence coherent sheaves, is tied to large radius pictures, one might be tempted to speculate that exotic objects such as (43) will

[10] We are very grateful to S. Katz for conversations regarding this section.

not be stable at large radius limits. Certainly we expect this to be true in this case since our 0-brane already decayed after one trip around the primary component of the discriminant.

It would be interesting to prove the instability at large radius of these objects which defy a simple D-brane interpretation.

4 Further Discussion

4.1 Relation of topological to physical branes

We have emphasized that at a general point in the Kähler moduli space, a general object in $\mathbf{D}(X)$ will *not* correspond to a physical D-brane, and the topological field theory will not be a twist of a CFT at that point. This is in line with the ideology in [6] that when the D-branes are taken to fill our four-dimensional spacetime and so realize $N = 1$ compactifications, the objects in $\mathbf{D}(X)$ correspond to solutions to the F-term equations, while the stability conditions are essentially the D-term equations.

At the line of marginal stability, the tachyonic deformations of a collection of branes become marginal. If they are exactly marginal we may pick any deformation we like along the marginal directions and still have a CFT. We may twist this CFT to get a topological field theory with arbitrary deformations, as we have described.

As we move away from this line, the topological field theory is invariant. The flow of grading means the associated deformations will be tachyonic (dependig on which direction we move off the line of marginal stability). At a given value of $B + iJ$, only a subset of measure zero of the tachyon vevs will correspond to CFTs. This subset corresponds to the tachyon at the extrema of its effective potential, or to some stable solitonic configuration of the tachyon. The remaining topological field theories will describe "off-shell" configurations of string theory.

The change of grading indicates that the tachyon potential (and thus its extrema) will vary throughout the Kähler moduli space. Therefore the stable conformal field theories will change, giving some notion of a nontrivial "bundle" of CFT data over Kähler moduli space. This change should lead to monodromy action on $\mathbf{D}(X)$, at least on states in $\mathbf{D}(X)$ which exist as physical objects at our starting point in Kähler moduli space.

4.2 Linear sigma models for complexes

There are now constructions via linear sigma models (following [50]) of a class of D-branes described by monads [13,14,51] and by multi-step resolutions [9,14] in toric varieties.[11] Some of the structures in [6] and in the present work appear quite naturally and intuitively in the linear sigma model. The Chan-Paton factors are zero modes of boundary fields. The chain maps are mass terms pairing up the fields of neighboring complexes, and their appearance via boundary contributions to the worldsheet supercharge is clear.

[11] Previous work on open string linear sigma models includes [52,53].

Since one may pass between phases easily in the linear sigma model, it may be useful to study 0-brane decay in this framework. Hopefully using this framework, or that advocated in [54], one may be able to understand what the 0-brane decays to, and what if any object at small radius might have 0-brane charge.

4.3 $\mathbf{D}(X)$ as a birational invariant

Suppose X and X' are birationally equivalent projective threefolds with terminal singularities. It was shown in [37, 55] that $\mathbf{D}(X)$ is equivalent to $\mathbf{D}(X')$. We may use our topological field theory to easily prove[12] a generalization of this kind of statement.

Let X be a Calabi–Yau variety of dimension d. The phase picture of [47] shows how, for the case $d = 3$, a flop to X' may be seen as a change in $B + iJ$. The basic idea is that if you shrink a $\mathbb{P}^1$ down to zero size by changing the Kähler form, continuing this process through the wall of Kähler cone will result in a flop on this curve. Thus X and X' are related simply by a change in the Kähler structure. Now since $\mathbf{D}(X) \sim \mathbf{T}(X)$ is invariant under a change in $B + iJ$ it follows immediately that $\mathbf{D}(X) \sim \mathbf{D}(X')$!

Note also that this can be generalized to any d. If a birational transformation can be induced by a change in the Kähler structure then the derived category will be invariant. It would be interesting to know if all birational transformations can be induced this way for $d > 3$. This would complete a proof of the birational invariance of $\mathbf{D}(X)$.

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