SLAC-PUB-8803
March 2001

Derivation of microwave instability dispersion relation with account of synchrotron damping and quantum fluctuations *

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Abstract

A dispersion relation for a microwave instability of a coasting beam is derived from the Vlasov-Fokker-Plank equation which takes into account the effects of synchrotron damping and quantum fluctuations. This derivation generalizes the standard analysis of the beam stability in which the diffusion and damping are usually neglected. Our results are also applicable for a bunched beam when the wavelength of the instability is much smaller than the bunch length.

*Work supported by Department of Energy contract DE–AC03–76SF00515.

1 Introduction

Consider a coasting beam with the line density n_b and the relativistic factor γ. Let us use (p, z) variables, where $p = \delta/\delta_0$ is the relative energy variation $\delta = \Delta E/E$ of a particle in units of the rms relative energy spread $\delta_0 = \langle \delta^2 \rangle^{1/2}$, z is the longitudinal coordinate measured relative to the reference particle with the nominal energy, and $s = ct$.

The beam is described by the longitudinal distribution function $\rho(p, z, s)$ normalized so that $\int \rho(p, z, s) dz dp = N$, where N is the number of particles in the beam. This function satisfies the Vlasov-Fokker-Planck equation

$$\begin{aligned} &\frac{\partial \rho}{\partial s} - \eta \delta_0 p \frac{\partial \rho}{\partial z} - \frac{r_0}{\gamma \delta_0} \frac{\partial \rho}{\partial p} \int_{-\infty}^{\infty} dz' dp' W(z' - z) \rho(p', z', s) \\ = \; &\frac{\gamma_d}{c} \frac{\partial}{\partial p} \left(\frac{\partial \rho}{\partial p} + p\rho \right), \end{aligned} \tag{1}$$

where η is the momentum compaction factor, r_0 is the classical electron radius, γ_d is the single particle radiation damping, and the function $W(z' - z)$ is the longitudinal wake function per unit length of the path. The right-hand-side of Eq. (1) describes diffusion and damping caused by the synchrotron radiation (SR).

We represent the distribution function ρ as a sum of the equilibrium distribution function ρ_0 and a perturbation ρ_1

$$\rho = \rho_0(p) + \rho_1(s, p, z), \tag{2}$$

with $\rho_1 \ll \rho_0$. Note that the equilibrium beam density n_b is equal to $n_b = \int \rho_0(p) dp$, and the density perturbation n_1 is given by $n_1 = \int \rho_1(p) dp$. Linearizing Eq. (1) and assuming that

$$\rho_1 = \hat{\rho}_1(s, p) e^{ikz}, \tag{3}$$

where k is the wavenumber, we find

$$\frac{\partial \hat{\rho}_1}{\partial s} - ik\eta\delta_0 p \hat{\rho}_1 - \frac{r_0}{\gamma \delta_0} \frac{\partial \rho_0}{\partial p} Z(k) \int dp \, \hat{\rho}_1(s, p) = \frac{\gamma_d}{c} \frac{\partial}{\partial p} \left(\frac{\partial \hat{\rho}_1}{\partial p} + p \hat{\rho}_1 \right), \tag{4}$$

where $Z(k)$ is the impedance of the ring,

$$Z(k) = \int_0^{\infty} d\zeta W(\zeta) e^{ik\zeta}. \tag{5}$$

For stability analysis it is often assumed that diffusion and damping are negligible provided the growth time of the instability is much shorter than the single particle SR damping time $\tau_d = \gamma_d^{-1}$. Then the the RHS can be omitted, and the remaining Vlasov equation has a solution

$$\hat{\rho}_1 = \hat{n}_1 \frac{icr_0 Z(k)}{\gamma\delta_0(\omega + ck\eta\delta_0 p)} \frac{d\rho_0}{dp} e^{-i\omega s/c}, \tag{6}$$

where ω satisfies the dispersion relation

$$1 = \frac{ir_0 c Z(k)}{\gamma\delta_0} \int \frac{dp\,(d\rho_0/dp)}{\omega + ck\eta\delta_0 p}, \tag{7}$$

and $\hat{n}_1$ in Eq. (6) is the amplitude of the density perturbation, $\hat{n}_1 = \int dp \hat{\rho}_1$.

Assumption that the RHS of Eq. (4) is negligible is not quite satisfactory. It is certainly not valid at the threshold of the instability where the growth rate vanishes. For the beam density n_b close to the threshold of the instability, the effect of the SR is usually taken into account by defining the threshold value n_{th} as the beam density at which the growth rate calculated without SR effects is equal to the damping rate γ_d, $\operatorname{Im}\omega \simeq \gamma_d$. This approach is equivalent to replacing the RHS of the linearized Fokker-Plank equation (4) by $-\gamma_d \hat{\rho}_1/c$. Such an approximation, qualitatively reasonable and common in kinetics and plasma physics, does not give the exact value of frequency and the growth rate near the threshold.

In this note, we derive the dispersion relation for a coasting beam taking into account the correct right-hand site of the linearized Fokker-Plank equation. The coasting beam result should also be valid for a bunched beam for harmonics with large harmonic wavenumber k, such that $k\sigma_z \gg 1$ where σ_z is the rms bunch length.

2 Dispersion relation with damping

First we introduce dimensionless variables ξ, Λ and Γ

$$\xi = sk\eta\delta_0, \qquad \Lambda = \frac{r_0 Z(k)}{\gamma k\eta\delta_0^2}, \qquad \Gamma = \frac{\gamma_d}{ck\eta\delta_0}, \tag{8}$$

and write the linearized Fokker-Plank equation in the following form,

$$\frac{\partial\hat{\rho}_1}{\partial\xi} - ip\hat{\rho}_1 - \Lambda\frac{\partial\rho_0}{\partial p}\int_{-\infty}^{\infty} dp\,\hat{\rho}_1(\xi, p) = \Gamma\frac{\partial}{\partial p}\left(\frac{\partial\hat{\rho}_1}{\partial p} + p\hat{\rho}_1\right). \tag{9}$$

To solve Eq. (9) we use the Fourier transform over p,

$$\hat{\rho}_1(\xi,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq f(\xi,q) e^{ipq}, \qquad f(\xi,q) = \int_{-\infty}^{\infty} dp \hat{\rho}_1(\xi,p) e^{-ipq}. \tag{10}$$

This gives the first-order differential equation for $f(\xi,q)$,

$$\frac{\partial f}{\partial \xi} + (1+\Gamma q)\frac{\partial f}{\partial q} = -\Gamma q^2 + \hat{n}_1(\xi) g_0(q), \tag{11}$$

where

$$g_0(q) = \Lambda \int_{-\infty}^{\infty} \frac{\partial \rho_0}{\partial p} e^{-ipq} dp, \tag{12}$$

and

$$\hat{n}_1(\xi) = f(\xi, q=0) = \int_{-\infty}^{\infty} dp \hat{\rho}_1(\xi,p). \tag{13}$$

Introducing a new variable ζ,

$$\zeta(\xi,q) = \frac{1}{\Gamma}\left[(1+\Gamma q)e^{-\Gamma\xi} - 1\right], \tag{14}$$

we will use ζ, ξ as independent variables instead of q and ξ. This defines a new function $F(\xi,\zeta)$ which is the function f expressed in terms of variables ζ, ξ: $F(\xi,\zeta) = f(\xi, q(\xi,\zeta))$, where the dependence $q(\xi,\zeta)$ can be found by inverting Eq. (14)

$$q(\xi,\zeta) = \frac{1}{\Gamma}\left[(1+\Gamma\zeta)e^{\Gamma\xi} - 1\right]. \tag{15}$$

The equation for $F(\zeta,\xi)$ can be now obtained from Eq. (11),

$$\frac{\partial F}{\partial \xi} = -\Gamma q^2(\xi,\zeta) F + \hat{n}_1(\xi) g_0(q(\xi,\zeta)). \tag{16}$$

It can be easily integrated:

$$F(\xi,\zeta) = F_0(\zeta) e^{-H(\xi,\zeta)} + e^{-H(\xi,\zeta)} \int_0^{\xi} d\xi' e^{H(\xi',\zeta)} \hat{n}_1(\xi') g_0(q(\xi',\zeta)), \tag{17}$$

where $F_0(\zeta)$ is the initial value of $F(\xi,\zeta)$ at $\xi = 0$, and

$$\begin{aligned} H(\xi,\zeta) &= \Gamma \int_0^{\xi} d\xi' q^2(\xi',\zeta) \\ &= \frac{1}{2\,\Gamma^2}\Big[3 + 2\Gamma\zeta - \Gamma^2\zeta^2 - 4e^{\Gamma\xi}\left(1+\Gamma\zeta\right) \\ &+ e^{2\Gamma\xi}(1+\Gamma\zeta)^2 + 2\Gamma\xi\Big]. \end{aligned} \tag{18}$$

Note that $q(\xi = 0, \zeta) = \zeta$, and $F_0(\zeta)$ is given by

$$F_0(\zeta) = f(0, \zeta) = \int_{-\infty}^{\infty} dp \hat{\rho}_1(0, p) e^{-ip\zeta} dp. \tag{19}$$

Making inverse Fourier transform of F we find $\hat{\rho}_1(\xi, p)$

$$\begin{aligned}
\hat{\rho}_1(\xi, p) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dq F(\zeta(\xi, q), \xi) e^{iqp} \\
&= \frac{e^{\Gamma\xi}}{2\pi} \int_{-\infty}^{\infty} d\zeta F(\zeta, \xi) e^{iq(\zeta,\xi)p} \\
&= \frac{e^{\Gamma\xi}}{2\pi} \int_{-\infty}^{\infty} d\zeta e^{iq(\zeta,\xi)p - H(\xi,\zeta)} \\
&\times \left[F_0(\zeta) + \int_0^{\xi} d\xi' e^{H(\xi',\zeta)} \hat{n}_1(\xi') g_0(q(\xi', \zeta)) \right].
\end{aligned} \tag{20}$$

Now we integrate this equation over p to obtain an integral equation for $\hat{n}_1(\xi)$

$$\begin{aligned}
\hat{n}_1(\xi) &= \int_{-\infty}^{\infty} dp \frac{e^{\Gamma\xi}}{2\pi} \int_{-\infty}^{\infty} d\zeta e^{iqp - H(\xi,\zeta)} \left[F_0(\zeta) + \int_0^{\xi} d\xi' e^{H(\xi',\zeta)} \hat{n}_1(\xi') g_0(q(\zeta, \xi')) \right] \\
&= e^{\Gamma\xi} \int_{-\infty}^{\infty} d\zeta \delta(q(\zeta, \xi)) e^{-H(\xi,\zeta)} \left[F_0(\zeta) + \int_0^{\xi} d\xi' e^{H(\xi',\zeta)} \hat{n}_1(\xi') g_0(q(\zeta, \xi')) \right].
\end{aligned} \tag{21}$$

With the δ-function in the integrand, the integration over ζ will be carried out with the help of Eq. (15) which gives the following relations

$$q(\zeta, \xi')|_{q(\zeta,\xi)=0} = \frac{1}{\Gamma} \left(e^{-\Gamma(\xi-\xi')} - 1 \right) \equiv r(\xi - \xi'), \tag{22}$$

and

$$\begin{aligned}
&[H(\zeta, \xi) - H(\zeta, \xi')]|_{q(\zeta,\xi)=0} = \\
&\frac{1}{2\Gamma^2} \left(3 - 4e^{-\Gamma(\xi-\xi')} + e^{-2\Gamma(\xi-\xi')} - 2\Gamma(\xi - \xi') \right) \equiv h(\xi - \xi').
\end{aligned} \tag{23}$$

The solution of Eq. (21) then takes the form of the integral equation

$$\hat{n}_1(\xi) = m(\xi) + \Lambda \int_{-\infty}^{\infty} dp \frac{\partial \rho_0}{\partial p} \int_0^{\xi} G(p, \xi - \xi') \hat{n}_1(\xi') d\xi', \tag{24}$$

with

$$G(p,\tau) = e^{-ipr(\tau)-h(\tau)}, \tag{25}$$

where the term $m(\xi)$ is due to the initial value F_0. In stability analysis, the initial value on the perturbation of distribution function does not play a role, and we omit this term in what follows.

Laplace transform of Eq. (24) gives for $\tilde{n}_1(\Omega) = \int_0^\infty d\xi e^{i\Omega\xi}\hat{n}_1(\xi)$ (we use the variable $-i\Omega$ in Laplace transform, where Ω has a meaning of complex frequency)

$$\left[1 - \Lambda \int dp \frac{\partial \rho_0(p)}{\partial p} \int_0^\infty dx e^{i\Omega x - h(x) - ipr(x)}\right] \tilde{n}_1(\Omega) = 0. \tag{26}$$

A nontrivial solution $\tilde{n}_1(\Omega)$ exists only if the following dispersion relation is satisfied

$$1 = \Lambda \int dp \frac{\partial \rho_0(p)}{\partial p} \int_0^\infty dx e^{i\Omega x - h(x) - ipr(x)}. \tag{27}$$

In the limit when synchrotron radiation is negligibly small, $\Gamma \to 0$, we have $h(x) \approx -\Gamma x^3/3 \to 0$ and $r(x) \to x$ and Eq. (27) reduces to Eq. (7).

For Gaussian distribution function, $\rho_0(p) = N(2\pi)^{-1/2}e^{p^2/2}$, one can integrate Eq. (27) over p to obtain the dispersion relation for the frequency Ω as a function of parameters Γ and $N\Lambda$:

$$\begin{aligned} 1 &= \frac{iN\Lambda}{\Gamma} \int_0^\infty dx \left(e^{-x\Gamma} - 1\right) \exp\left[\frac{1}{2\,\Gamma^2}\left(1 - e^{-x\Gamma}\right)^2 \right. \\ &+ \left. \frac{1}{2\,\Gamma^2}\left(3 + e^{-2x\Gamma} - 4e^{-x\Gamma} - 2x\Gamma\right) + ix\Omega\right]. \end{aligned} \tag{28}$$

This equation was solved numerically for different values of Γ and the stability diagrams corresponding to the threshold of the instability ($\mathrm{Im}\,\Omega = 0$) are plotted in the complex plane of the variable $N\Lambda$ in Fig. 1 (note that in our definition of Z the inductive impedance has a negative imaginary part). For small value of Γ the stability curve takes a familiar "onion-like" shape characteristic for the $\Gamma = 0$ limit. Increasing the parameter Γ expands the stability region, as expected, due to the stabilizing effect of the synchrotron radiation damping.

Fig. 2 shows the ratio of the two threshold values of the number of particles in the beam, N_{th} and N_{appr}, for the impedance corresponding to the synchrotron radiation in free space $Z = \mathrm{const}(1.63 - 0.94i)$, [1]. The first number, N_{th}, was calculated using Eq. (28). The second number, N_{appr}, was

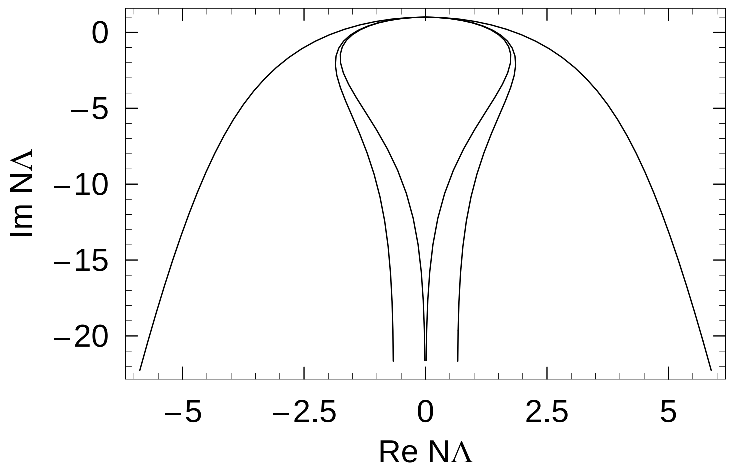

Figure 1: Stability diagrams ($\mathrm{Im}\,\Omega = 0$) for $\Gamma = 0.0001$, $\Gamma = 0.1$ and $\Gamma = 1$. The areas under the curves correspond to the stable beam.

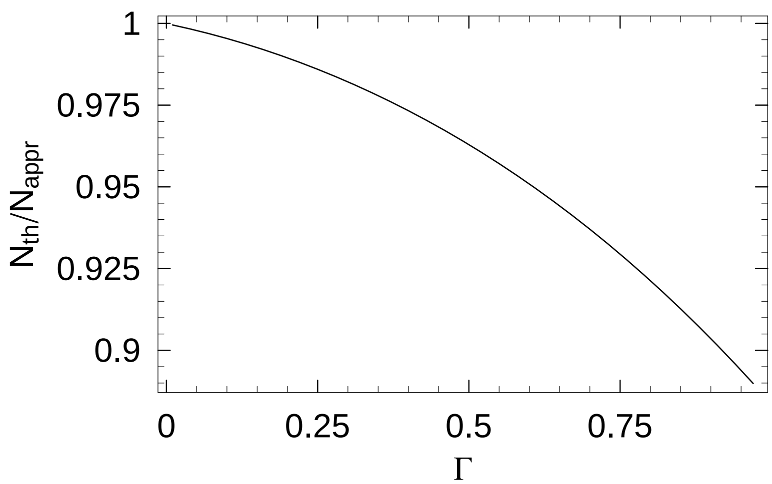

Figure 2: The ratio of exact and approximate threshold values of N as a function of Γ for the impedance of the "free-space" $Z = \mathrm{const}(1.63 - 0.94i)$

calculated with the help of a standard recipe that is often invoked for the estimation of the effect of the synchrotron radiation on the instability. Specifically, one first finds the frequency of the instability $\tilde{\Omega}(N)$ neglecting the synchrotron radiation, and *then* takes into account the radiation by subtracting Γ from the imaginary part of $\tilde{\Omega}$. For the threshold of the instability one uses the equation $\operatorname{Im}\tilde{\Omega}(N_{\mathrm{appr}}) = \Gamma$. As is seen from Fig. 2, both methods actually give a very close results for the value of the threshold.

References

[1] S. Heifets and G. Stupakov, "Beam instability and microbunching due to coherent synchrotron radiation," SLAC-PUB-8761.