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Notes from the Underground:
À Propos of Givental's Conjecture

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Abstract

These brief notes record our puzzles and findings surrounding Givental's recent conjecture which expresses higher genus Gromov-Witten invariants in terms of the genus-0 data. We limit our considerations to the case of a projective line, whose Gromov-Witten invariants are well-known and easy to compute. We make some simple checks supporting his conjecture.

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1 Brief Summary

These incomplete notes[1] are brief sketches of our troubles and findings, which would not have seen the light of day under more favorable circumstances for the authors, surrounding a work of Givental.

Let $\mathcal{F}_g$ be the generating function in the small phase space for genus-g Gromov-Witten (GW) invariants of a manifold X with a semi-simple Frobenius structure. Then, Givental's conjecture, whose equivariant counter-part he has proved [G2], is

$$e^{\sum_{g\geq 2}\lambda^{g-1}\mathcal{F}_g(t)} = \left[e^{\frac{\lambda}{2}\sum_{k,l\geq 0}\sum_{i,j} V^{ij}_{kl}\sqrt{\Delta_i}\sqrt{\Delta_j}\partial_{q^i_k}\partial_{q^j_l}} \prod_j \tau(\lambda\Delta_j;\{q^j_n\}) \right]\Bigg|_{q^j_n = T^j_n}, \tag{1.1}$$

where $i,j = 1,\ldots \dim H^*(X,\mathbb{Q})$, τ is the KdV tau-function governing the intersection theory on the Deligne-Mumford space $\overline{\mathcal{M}}_{g,n}$, and V^{ij}_{kl}, Δ_j, and T^j_n are functions of the small phase space coordinates $t \in H^*(X,\mathbb{Q})$ and are defined by solutions to the flat-section equations associated with the genus-0 Frobenius structure of X [G2]. This remarkable conjecture organizes the higher genus GW-invariants in terms of the genus-0 data and the τ-function for a point. The motivation for our work lies in verifying the conjecture for $X = \mathbb{P}^1$, which is the simplest example with a semi-simple Frobenius structure and whose GW-invariants can be easily computed.

We have obtained two particular solutions to the flat-section equations (3.1), an analytic one encoding the two-point descendant GW-invariants of $\mathbb{P}^1$ and a recursive one corresponding to Givental's fundamental solution. According to Givental, both of these two solutions are supposed to yield the same data V^{ij}_{kl}, Δ_j, and T^j_n. Unfortunately, we were not able to produce the desired information using our analytic solutions, but the recursive solutions do lead to sensible quantities which we need. Combined with an expansion scheme which allows us to verify the conjecture at each order in λ, we thus use our recursive solutions to check the conjecture (1.1) for $\mathbb{P}^1$ up to order λ^2. Already at this order, we need to expand the differential operators in (1.1) up to λ^6 and need to consider up to genus-3 free energy in the τ-functions, and the computations quickly become cumbersome with increasing order. We have managed to re-express the conjecture for this case into a form which resembles the Hirota-bilinear relations, but at this point, we have no insights into a general proof. It is nevertheless curious how the numbers work out, and we hope that our results would provide a humble support for Givental's master equation.

Many confusions still remain – for instance, the discrepancy between our analytic and recursive solutions. As mentioned above, Givental's conjecture for $\mathbb{P}^1$ can be re-written in a form which resembles the Hirota-bilinear relations for the KdV hierarchies (see (4.9)). It would thus be interesting to speculate a possible relation between his conjecture and the conjectural Toda hierarchy for $\mathbb{P}^1$.

[1]The reader may immediately discern the injustice we have brought onto Dostoevsky by shamelessly borrowing his ingenious title. Our work admittedly lacks his psychological insights and profundity, but it does contain many confusions and an acute sense of despair, which are often characteristic of his writings and which thus compel us to adapt his mind-set in its original form. May his soul rest in peace, despite our abuse, which is rendered, nevertheless, with our greatest respect for his genius.

We have organized our notes as follows: in §2, we review the canonical coordinates for $\mathbb{P}^1$, to be followed by our solutions to the flat-section equations in §3. We conclude by presenting our checks in §4.

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2 Canonical Coordinates for $\mathbb{P}^1$.

We here review the canonical coordinates $\{u_\pm\}$ for $\mathbb{P}^1$ [D, DZ, G1]. Recall that a Frobenius structure on $H^*(\mathbb{P}^1, \mathbb{Q})$ carries a flat pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ defined by the Poincaré intersection pairing. The canonical coordinates are defined by the property that they form the basis of idempotents of the quantum cup-product, denoted in the present note by $\circ$. The flat metric $\langle\cdot,\cdot\rangle$ is diagonal in the canonical coordinates, and following Givental's notation, we define $\Delta_\pm := 1/\langle\partial_{u_\pm}, \partial_{u_\pm}\rangle$.

Let $\{t^\alpha\}, \alpha \in \{0,1\}$ be the flat coordinates of the metric and let $\partial_\alpha := \partial/\partial t^\alpha$. The quantum cohomology of $\mathbb{P}^1$ is

$$\partial_0 \circ \partial_\alpha = \partial_\alpha \qquad \text{and} \qquad \partial_1 \circ \partial_1 = e^{t^1}\,\partial_0.$$

The eigenvalues and eigenvectors of $\partial_1\circ$ are

$$\pm\, e^{t^1/2} \qquad \text{and} \qquad (\pm e^{t^1/4}\,\partial_0 + e^{-t^1/4}\,\partial_1),$$

respectively. So, we have

$$(\pm e^{t^1/4}\,\partial_0 + e^{-t^1/4}\,\partial_1)\circ(\pm e^{t^1/4}\,\partial_0 + e^{-t^1/4}\,\partial_1) = \pm 2\,e^{t^1/4}\,(\pm e^{t^1/4}\,\partial_0 + e^{-t^1/4}\,\partial_1),$$

which implies that

$$\frac{\partial}{\partial u_\pm} = \frac{\partial_0 \pm e^{-t^1/2}\,\partial_1}{2},$$

such that

$$\partial_{u_\pm}\circ\partial_{u_\pm} = \partial_{u_\pm} \qquad \text{and} \qquad \partial_{u_\pm}\circ\partial_{u_\mp} = 0\,.$$

We can solve for $u_\pm$ up to constants as

$$u_\pm = t^0 \pm 2\,e^{t^1/2}\,. \tag{2.1}$$

To compute $\Delta_\pm$, note that

$$\frac{1}{\Delta_\pm} := \langle\partial_{u_\pm}, \partial_{u_\pm}\rangle = \pm\frac{1}{2e^{t^1/2}}\,.$$

The two bases are related by

$$\partial_0 = \partial_{u_+} + \partial_{u_-} \qquad \text{and} \qquad \partial_1 = e^{t^1/2}\,(\partial_{u_+} - \partial_{u_-})\,.$$

Define an orthonormal basis by $f_i = \Delta_i^{1/2} \frac{\partial}{\partial u_i}$. Then the transition matrix Ψ from $\{\frac{\partial}{\partial t_\alpha}\}$ to $\{f_i\}$ is given by

$$\Psi_\alpha^{\ i} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-t^1/4} & -i\, e^{-t^1/4} \\ e^{t^1/4} & i\, e^{t^1/4} \end{pmatrix} = \begin{pmatrix} \Delta_+^{-1/2} & \Delta_-^{-1/2} \\ \frac{1}{2}\Delta_+^{1/2} & \frac{1}{2}\Delta_-^{1/2} \end{pmatrix}, \tag{2.2}$$

such that

$$\frac{\partial}{\partial t_\alpha} = \sum_i \Psi_\alpha^{\ i}\, f_i.$$

We will also need the inverse of (2.2):

$$(\Psi^{-1})_i^{\ \alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{t^1/4} & e^{-t^1/4} \\ i\, e^{t^1/4} & -i\, e^{-t^1/4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\Delta_+^{1/2} & \Delta_+^{-1/2} \\ \frac{1}{2}\Delta_-^{1/2} & \Delta_-^{-1/2} \end{pmatrix}. \tag{2.3}$$

3 Solutions to the Flat-Section Equations

The relevant data V_{kl}^{ij}, Δ_j and T_n^j are extracted from the solutions to the flat-section equations of the genus-0 Frobenius structure for $\mathbb{P}^1$. We here find two particular solutions. The analytic solution correctly encodes the two-point descendant GW-invariants, while the recursive solution is used in the next section to verify Givental's conjecture.

3.1 Analytic Solution

The genus-0 free energy for $\mathbb{P}^1$ is

$$\mathcal{F}_0 = \frac{1}{2}(t^0)^2 t^1 + e^{t^1}.$$

Flat sections S_α of $TH^*(\mathbb{P}^1, \mathbb{Q})$ satisfy the equations

$$z\, \partial_\alpha\, S_\beta = \mathcal{F}_{\alpha\beta\mu}\, g^{\mu\nu}\, S_\nu\ , \tag{3.1}$$

where $z \neq 0$ is an arbitrary parameter and $\mathcal{F}_{\alpha\beta\mu} := \partial^3 \mathcal{F}/\partial t^\alpha \partial t^\beta \partial t^\mu$. The only non-vanishing components of $\mathcal{F}_{\alpha\beta\mu}$ are

$$\mathcal{F}_{001} = 1 \qquad \text{and} \qquad \mathcal{F}_{111} = e^{t^1}\ .$$

Hence, we find that the general solutions to the flat-section equations (3.1) are

$$S_0 = e^{t^0/z}\left[c_1\, I_0(2\, e^{t^1/2}/z) - c_2\, K_0(2\, e^{t^1/2}/z)\right] \tag{3.2}$$

and

$$S_1 = e^{t^0/z}\, e^{t^1/2}\left[c_1\, I_1(2 e^{t^1/2}/z) + c_2\, K_1(2 e^{t^1/2}/z)\right]\ , \tag{3.3}$$

where $I_n(x)$ and $K_n(x)$ are modified Bessel functions, and c_i are integration constants which may depend on z.

We would now like to find two particular solutions corresponding to the following Givental's expression:

$$S_{\alpha\beta}(z) = g_{\alpha\beta} + \sum_{n\geq 0, (n,d)\neq(0,0)} \frac{1}{n!}\, \langle \phi_\alpha \cdot \frac{\phi_\beta}{z-\psi} \cdot (t^0\phi_0 + t^1\phi_1)^n \rangle_d\,, \tag{3.4}$$

where $S_{\alpha\beta}$ denotes the α-th component of the β-th solution. Here, $\{\phi_\alpha\}$ is a homogeneous basis of $H^*(\mathbb{P}^1,\mathbb{Q})$, $g_{\alpha\beta}$ is the intersection paring $\int_{\mathbb{P}^1}\phi_\alpha\cup\phi_\beta$ and $\psi\in H^2(\overline{M}_{0,n+2}(\mathbb{P}^1,d),\mathbb{Q})$ is the first Chern class of the universal cotangent line bundle over the moduli space $\overline{M}_{0,n+2}(\mathbb{P}^1,d)$. In order to find the particular solutions, we compare our general solution (3.2) with the 0-th components of $S_{0\beta}$ in (3.4) *at the origin of the phase space.* The two-point functions appearing in (3.4) have been computed at the origin in [S] and have the following forms:

$$S_{00}|_{t^\alpha=0} = -\sum_{m=1}^{\infty}\frac{1}{z^{2m+1}}\frac{2\,d_m}{(m!)^2}\ , \quad \text{where } d_m=\sum_{k=1}^{m}1/k\,, \tag{3.5}$$

and

$$S_{01}|_{t^\alpha=0} = 1+\sum_{m=1}^{\infty}\frac{1}{z^{2m}}\frac{1}{(m!)^2}. \tag{3.6}$$

Using the standard expansion of the modified Bessel function K_0, we can evaluate (3.2) at the origin of the phase space to be

$$c_1\,I_0\left(\frac{2}{z}\right)-c_2\,K_0\left(\frac{2}{z}\right) \;=\; c_1\,I_0\left(\frac{2}{z}\right)-c_2\left[-(-\log(z)+\gamma_E)\,I_0\left(\frac{2}{z}\right)+\sum_{m=1}\frac{c_m}{z^{2m}(m!)^2}\right], \tag{3.7}$$

where γ_E is Euler's constant. Now matching (3.7) with (3.5) gives

$$c_1=-c_2\log(1/z)-c_2\gamma_E \qquad \text{and} \qquad c_2=\frac{2}{z}\,,$$

while noticing that (3.6) is precisely the expansion of $I_0(2/z)$ and demanding that our general solution coincides with (3.6) at the origin yields

$$c_1=1 \qquad \text{and} \qquad c_2=0\ .$$

To recapitulate, we have found

$$\begin{aligned}
S_{00} &= -\frac{2e^{t^0/z}}{z}\left[(\gamma_E-\log(z))\;I_0\left(\frac{2\,e^{t^1/2}}{z}\right)+\;K_0\left(\frac{2\,e^{t^1/2}}{z}\right)\right],\\
S_{10} &= \frac{2e^{t^0/z}\,e^{t^1/2}}{z}\left[K_1\left(\frac{2e^{t^1/2}}{z}\right)-(\gamma_E-\log(z))\,I_1\left(\frac{2e^{t^1/2}}{z}\right)\right],\\
S_{01} &= e^{t^0/z}\,I_0\left(\frac{2\,e^{t^1/2}}{z}\right),\\
S_{11} &= e^{t^0/z}\,e^{t^1/2}\,I_1\left(\frac{2e^{t^1/2}}{z}\right).
\end{aligned} \tag{3.8}$$

We have checked that these solutions correctly reproduce the corresponding descendant Gromov-Witten invariants obtained in [S].

If the inverse transition matrix in (2.3) is used to relate the matrix elements $S_\alpha^{\ i}$ to $S_{\alpha\beta}$ as $S_\alpha^{\ i}=S_{\alpha\beta}((\psi^{-1})^t)^\beta_{\ j}\,\delta^{ji}$, then we should have

$$S_\alpha^{\ \pm}=\sqrt{\pm 2}\,e^{t^1/4}\left(\frac{1}{2}S_{\alpha 0}\pm\frac{e^{-t^1/2}}{2}S_{\alpha 1}\right). \tag{3.9}$$

3.2 Recursive Solution

In [G1, G2], Givental has shown that near a semi-simple point, the flat-section equations (3.1) have a fundamental solution given by

$$S_\alpha^{\ i} = \Psi_\alpha^{\ j}(R_0 + zR_1 + z^2R_2 + \cdots + z^nR_n + \cdots)_{jk}\,[\exp(U/z)]^{ki}\ ,$$

where $R_n = (R_n)_{jk}$, $R_0 = \delta_{jk}$ and U is the diagonal matrix of canonical coordinates. The matrix R_1 satisfies the relations

$$\Psi^{-1}\frac{\partial\Psi}{\partial t^1} = [\frac{\partial U}{\partial t^1}, R_1] \tag{3.10}$$

and

$$\left[\frac{\partial R_1}{\partial t^1} + \Psi^{-1}\left(\frac{\partial\Psi}{\partial t^1}\right)R_1\right]_{\pm\pm} = 0\,, \tag{3.11}$$

which we use to find its expression. From the transition matrix given in (2.2) we see that

$$\Psi^{-1}\frac{\partial\Psi}{\partial t^1} = \frac{1}{4}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

while taking the $(+-)$ component of the relation (3.10) gives

$$\frac{i}{4} = \frac{\partial U_{++}}{\partial t^1}(R_1)_{+-} - (R_1)_{+-}\frac{\partial U_{--}}{\partial t^1} = 2e^{t^1/2}(R_1)_{+-}\ ,$$

where in the last step we have used the definition (2.1) of canonical coordinates. We therefore have

$$(R_1)_{+-} = \frac{i}{8}e^{-t^1/2}\,,$$

and similarly considering the $(-+)$ component of (3.10) gives

$$(R_1)_{-+} = \frac{i}{8}e^{-t^1/2}.$$

The diagonal components of R_1 can be obtained from (3.11), which implies that

$$\frac{\partial(R_1)_{++}}{\partial t^1} = (R_1)_{+-}\frac{\partial U_{--}}{\partial t^1}(R_1)_{-+} - \frac{\partial U_{++}}{\partial t^1}(R_1)_{+-}(R_1)_{-+} = \frac{\exp(-t^1/2)}{32} = -\frac{\partial(R_1)_{--}}{\partial t^1}.$$

Hence, $(R_1)_{++} = -\exp(-t^1/2)/16 = -(R_1)_{--}$ and the matrix R_1 can be written as

$$(R_1)_{jk} = \frac{1}{16}e^{-t^1/2}\begin{pmatrix} -1 & 2i \\ 2i & 1 \end{pmatrix}. \tag{3.12}$$

In general, the matrices R_n satisfy the recursion relations [G1]

$$\left(d + \Psi^{-1}d\Psi\right)R_n = [dU, R_{n+1}]\ , \tag{3.13}$$

which, for our case, imply the following set of equations:

$$\frac{\partial R_n}{\partial t^0} = 0 \, , \tag{3.14}$$

$$\frac{\partial (R_n)_{++}}{\partial t^1} = -\frac{i}{4}(R_n)_{-+} \, , \tag{3.15}$$

$$(R_{n+1})_{-+} = -\frac{1}{2}e^{-t^1/2}\left[\frac{\partial (R_n)_{-+}}{\partial t^1} - \frac{i}{4}(R_n)_{++}\right] \, , \tag{3.16}$$

$$\frac{\partial (R_n)_{--}}{\partial t^1} = \frac{i}{4}(R_n)_{+-} \, , \tag{3.17}$$

$$(R_{n+1})_{+-} = \frac{1}{2}e^{-t^1/2}\left[\frac{\partial (R_n)_{+-}}{\partial t^1} + \frac{i}{4}(R_n)_{--}\right] \, . \tag{3.18}$$

LEMMA 3.1 *For $n \geq 1$, the matrices R_n in the fundamental solution are given by*

$$(R_n)_{ij} = \frac{(-1)^n}{(2n-1)}\frac{\alpha_n}{2^n}\, e^{-nt^1/2}\begin{pmatrix} -1 & (-1)^{n+1}\, 2n\, i \\ 2n\, i & (-1)^{n+1} \end{pmatrix}, \tag{3.19}$$

where

$$\alpha_n = (-1)^n \frac{1}{8^n n!}\prod_{\ell=1}^{n}(2\ell-1)^2 \; , \; \alpha_0 = 1.$$

These solutions satisfy the unitarity condition

$$\boldsymbol{R}(z)\boldsymbol{R}^t(-z) := (1+zR_1+z^2R_2+\cdots+z^nR_n+\cdots)(1-zR_1^t+z^2R_2^t+\cdots+(-1)^n z^n R_n^t+\cdots) = 1$$

and the homogeneity condition and, thus, are unique.

Proof: For $n = 1$, $\alpha_1 = -1/8$ and (3.19) is equal to the correct solution (3.12). The proof now follows by an induction on n. Assume that (3.19) holds true up to and including $n = m$. Using the fact that

$$\alpha_{m+1} = -\frac{(2m+1)^2}{8(m+1)}\alpha_m,$$

we can show that R_{m+1} in (3.19) satisfies the relations (3.15)–(3.18) as well as (3.14).

To check unitarity, consider the z^k-term $P_k := \sum_{\ell=0}^{k}(-1)^\ell\, R_{k-\ell}\, R_\ell^t$ in $\boldsymbol{R}(z)\boldsymbol{R}^t(-z) = \sum_{k=0} P_k z^k$. As shown by Givental, the equations satisfied by the matrices R_n imply that the off-diagonal entries of P_k vanish. As a result, combined with the anti-symmetry of P_k for odd k, we see that P_k vanishes for k odd. Hence, we only need to show that for our solution, P_k vanishes for all positive even k as well. To this end, we note that Givental has also deduced from the equation $dP_k + [\Psi^{-1}d\Psi, P_k] = [dU, P_{k+1}]$ that the diagonal entries of P_k are constant. The expansion of P_{2k} is

$$P_{2k} = R_{2k} + R_{2k}^t + \cdots \, ,$$

where the remaining terms are products of R_ℓ, for $\ell < 2k$. Now, we proceed inductively. We first note that R_1 and R_2 given in (3.19) satisfy the condition $P_2 = 0$, and assume that

R_ℓ's in (3.19) for $\ell < 2k$ satisfy $P_\ell = 0$. Then, since the off-diagonal entries of P_n vanish for all n, the expansion of P_{2k} is of the form

$$P_{2k} = A\, e^{-2k\, t^1/2} + B,$$

where A is a constant diagonal matrix resulting from substituting our solution (3.19) and B is a possible diagonal matrix of integration constants for R_{2k}. But, since the diagonal entries of P_n are constant for all n, we know that $A = 0$. We finally choose the integration constants to be zero so that $B = 0$, yielding $P_{2k} = 0$. Hence, the matrices in our solution (3.19) satisfy the unitarity condition and are manifestly homogeneous. It then follows by the proposition in [G2] that our solutions R_n are unique. ■

Let $\boldsymbol{R} := (R_0 + zR_1 + z^2R_2 + \cdots + z^nR_n + \cdots)$. Then, we can use the matrices R_n from Lemma 3.1 to find

$$\begin{aligned} {S_0}^+ &= (\boldsymbol{R}_{++} - i\,\boldsymbol{R}_{-+})\,\frac{\exp(u_+/z)}{\sqrt{\Delta_+}} \\ &= \left[1 + \sum_{n=1}^{\infty} \frac{\alpha_n}{2^n} \exp\left(\frac{-nt^1}{2}\right)(-z)^n\right] \frac{\exp(u_+/z)}{\sqrt{\Delta_+}}\ , \end{aligned} \tag{3.20}$$

$$\begin{aligned} {S_0}^- &= (\boldsymbol{R}_{--} + i\,\boldsymbol{R}_{+-})\,\frac{\exp(u_-/z)}{\sqrt{\Delta_-}} \\ &= \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{\alpha_n}{2^n} \exp\left(\frac{-nt^1}{2}\right)(-z)^n\right] \frac{\exp(u_-/z)}{\sqrt{\Delta_-}}\ , \end{aligned} \tag{3.21}$$

$$\begin{aligned} {S_1}^+ &= (\boldsymbol{R}_{++} + i\,\boldsymbol{R}_{-+})\,\frac{\sqrt{\Delta_+}}{2}\exp(u_+/z) \\ &= \left[1 - \sum_{n=1}^{\infty} \frac{(2n+1)}{(2n-1)}\frac{\alpha_n}{2^n} \exp\left(\frac{-nt^1}{2}\right)(-z)^n\right] \frac{\sqrt{\Delta_+}}{2}\exp(u_+/z)\ , \end{aligned} \tag{3.22}$$

$$\begin{aligned} {S_1}^- &= (\boldsymbol{R}_{--} - i\,\boldsymbol{R}_{+-})\,\frac{\sqrt{\Delta_-}}{2}\exp(u_-/z) \\ &= \left[1 - \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)}{(2n-1)}\frac{\alpha_n}{2^n} \exp\left(\frac{-nt^1}{2}\right)(-z)^n\right] \frac{\sqrt{\Delta_-}}{2}\exp(u_-/z)\ . \end{aligned} \tag{3.23}$$

Using the above expressions for ${S_\alpha}^i(z)$, we can also find $V^{ij}(z,w)$, which is given by the expression

$$V^{ij}(z,w) := \frac{1}{z+w}\,[{S_\mu}^i(w)]^t\,[g^{\mu\nu}]\,[{S_\nu}^j(z)].$$

If we define

$$A_{p,q} := \frac{(4p\,q-1)}{(2p-1)(2q-1)}\,\frac{\alpha_p\,\alpha_q}{2^{p+q}}e^{\frac{-(p+q)t^1}{2}}$$

and

$$B_{p,q} := \frac{2(p-q)}{(2p-1)(2q-1)}\,\frac{\alpha_p\,\alpha_q}{2^{p+q}}e^{\frac{-(p+q)t^1}{2}},$$

then after some algebraic manipulations we obtain

$$V^{++}(z,w) = e^{u_+/w+u_+/z}\left\{\frac{1}{z+w}+\sum_{k,l=0}^{\infty}\left[\sum_{n=0}^{k}(-1)^n A_{l+n+1,k-n}\right](-1)^{k+l}\,w^k z^l\right\}, \quad (3.24)$$

$$V^{--}(z,w) = e^{u_-/w+u_-/z}\left\{\frac{1}{z+w}-\sum_{k,l=0}^{\infty}\left[(-1)^{k+l}\sum_{n=0}^{k}(-1)^n A_{l+n+1,k-n}\right](-1)^{k+l}\,w^k z^l\right\},$$

$$V^{+-}(z,w) = e^{u_+/w+u_-/z}\left\{\sum_{k,l=0}^{\infty}\left[i\,(-1)^l\sum_{n=0}^{k}B_{l+n+1,k-n}\right]\,(-1)^{k+l}\,w^k z^l\right\}, \quad (3.25)$$

$$V^{-+}(z,w) = e^{u_-/w+u_+/z}\left\{\sum_{k,l=0}^{\infty}\left[i\,(-1)^k\sum_{n=0}^{k}B_{l+n+1,k-n}\right]\,(-1)^{k+l}\,w^k z^l\right\}.$$

3.3 A Puzzle

Incidentally, we note that in the asymptotic limit $z \to 0$,

$$S_0{}^{+} = \Re\left[\sqrt{\frac{2\pi}{z}}e^{t^0/z}I_0\left(\frac{2e^{t^1/2}}{z}\right)\right]$$

and

$$S_0{}^{-} = -i\sqrt{\frac{2}{\pi z}}e^{t^0/z}K_0\left(\frac{2e^{t^1/2}}{z}\right)$$

reproduce the expansions in (3.20) and (3.21). This is in contrast to what was expected from the discussion leading to (3.9). Despaired of matching the two expressions, it seems to us that the analytic correlation functions obtained in §3.1 do not encode the right information that appear in Givental's conjecture. In the following section, we will use the recursive solutions from §3.2 to check Givental's conjectural formula at low genera.

4 Checks of the Conjecture at Low Genera

The T_n^i that appear in Givental's formula (1.1) are defined by the equations [G2]

$$S_0{}^{\pm} := \left[1-\sum_{n=0}^{\infty}T_n^{\pm}(-z)^{n-1}\right]\frac{\exp(u_\pm/z)}{\sqrt{\Delta_\pm}}.$$

From the computations of $S_0{}^{+}$ and $S_0{}^{-}$ in (3.20) and (3.21), respectively, one can extract T_n^i to be

$$T_n^{+} = \begin{cases} 0\,, & n=0,1\,,\\ -\dfrac{\alpha_{n-1}}{2^{n-1}}\exp\left[\dfrac{-(n-1)t^1}{2}\right], & n\geq 2\,, \end{cases} \quad (4.1)$$

$$T_n^{-} = \begin{cases} 0\,, & n=0,1\,,\\ -(-1)^{n-1}\dfrac{\alpha_{n-1}}{2^{n-1}}\exp\left[\dfrac{-(n-1)t^1}{2}\right], & n\geq 2\,. \end{cases} \quad (4.2)$$

Notice that

$$T_n^- = (-1)^{n-1}\, T_n^+ \,. \tag{4.3}$$

The functions V_{kl}^{ij} are defined[2] by the expansion [G2]

$$V^{ij}(z,w) = e^{u^i/w + u^j/z}\left[\frac{\delta^{ij}}{z+w} + \sum_{k,l=0}^{\infty}(-1)^{k+l}\, V_{kl}^{ij}\, w^k z^l\right],$$

and from (3.24) and (3.25) we see that

$$\begin{aligned} V_{kl}^{++} &= \sum_{n=0}^{k}(-1)^n A_{l+n+1,k-n} = \sum_{n=0}^{k}\frac{(-1)^n(4(l+n+1)(k-n)-1)}{(2l+2n+1)(2k-2n-1)}\, T_{l+n+2}^+ T_{k-n+1}^+\,, \\ V_{kl}^{+-} &= i(-1)^l\sum_{n=0}^{k} B_{l+n+1,k-n} = i(-1)^l\sum_{n=0}^{k}\frac{2(l+2n+1-k)}{(2l+2n+1)(2k-2n-1)}\, T_{l+n+2}^+ T_{k-n+1}^+. \end{aligned}$$

Now, the τ-function for the intersection theory on the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves is defined by

$$\tau(\lambda;\{q_k\}) = \exp\left(\sum_{g=0}^{\infty}\lambda^{g-1}\,\mathcal{F}_g^{\mathrm{pt}}(\{q_k\})\right)$$

and has the following nice scaling invariance: consider the scaling of the phase-space variables q_k given by

$$q_k \mapsto s^{k-1}\, q_k \tag{4.4}$$

for some constant s. Then, since a non-vanishing intersection number $\langle \tau_{k_1}\cdots\tau_{k_n}\rangle$ must satisfy

$$\sum_{i=1}^{n}(k_i-1) = \dim(\overline{\mathcal{M}}_{g,n}) - n = 3g-3\,,$$

we see that under the transformation (4.4), the genus-g generating function $\mathcal{F}_g^{\mathrm{pt}}$ must behave as

$$\mathcal{F}_g^{\mathrm{pt}}(\{s^{k-1}\, q_k\}) = (s^3)^{g-1}\,\mathcal{F}_g^{\mathrm{pt}}(\{q_k\}).$$

Hence, upon scaling the "string coupling constant" λ to $s^{-3}\lambda$, we see that

$$\tau(s^{-3}\lambda;\ \{s^{k-1}q_k\}) = \tau(\lambda;\{q_k\})\,. \tag{4.5}$$

Now, consider the function

$$F(\{q_n^+\},\{q_n^-\}) := \left[\mathrm{e}^{\frac{\lambda}{2}\sum_{k,l\geq 0}\sum_{i,j\in\{\pm\}} V_{kl}^{ij}\sqrt{\Delta_i}\sqrt{\Delta_j}\,\partial_{q_k^i}\partial_{q_l^j}}\ \tau(\lambda\Delta_+;\{q_n^+\})\tau(\lambda\Delta_-;\{q_n^-\})\right]. \tag{4.6}$$

Then, since the Gromov-Witten potentials of $\mathbb{P}^1$ for $g \geq 2$ all vanish, Givental's conjectural formula for $\mathbb{P}^1$ is

$$F(\{T_n^+\},\{T_n^-\}) = 1\,,$$

[2]There seems to be a misprint in the original formula for V_{kl}^{ij} in [G2], i.e. we believe that w and z should be exchanged, as in our expression here.

where it is understood that one sets $q_k^i = T_k^i$ after taking the derivatives with respect to q_k^i. Since T_n^+ and T_n^- are related by (4.3), let us rescale $q_k^- \mapsto (-1)^{k-1} q_k^-$ in (4.6). Then, since $\Delta_+ = -\Delta_-$, we observe from (4.5) that

$$\begin{aligned} F(\{T_n^+\},\{T_n^-\}) &= \left\{ \exp\left[\frac{\lambda}{2}\Delta_+ \sum_{k,l\geq 0} \Big(V_{kl}^{++}\partial_{q_k^+}\partial_{q_l^+} + i(-1)^{l-1}\, V_{kl}^{+-}\partial_{q_k^+}\partial_{q_l^-} + i(-1)^{k-1}\, V_{kl}^{-+}\partial_{q_k^-}\partial_{q_l^+} \right.\right. \\ &\left.\left. - (-1)^{k+l} V_{kl}^{--}\partial_{q_k^-}\partial_{q_l^-} \Big) \right] \tau(\lambda\Delta_+;\{q_n^+\})\tau(\lambda\Delta_+;\{q_n^-\}) \right\}\Bigg|_{q_n^+,q_n^-=T_n^+} . \end{aligned}$$

But, the V_{kl}^{ij} satisfy the relations $V_{kl}^{--} = -(-1)^{k+l} V_{kl}^{++}$ and $V_{kl}^{+-} = V_{lk}^{-+}$, so

$$\begin{aligned} F(\{T_n^+\},\{T_n^-\}) &= \left\{ \exp\left[\frac{\lambda}{2}\Delta_+ \sum_{k,l\geq 0} \Big(V_{kl}^{++}(\partial_{q_k^+}\partial_{q_l^+} + \partial_{q_k^-}\partial_{q_l^-}) + \right.\right. \\ &\left.\left. + 2i\,(-1)^{l-1}\, V_{kl}^{+-}\partial_{q_k^+}\partial_{q_l^-} \Big) \right] \tau(\lambda\Delta_+;\{q_n^+\})\tau(\lambda\Delta_+;\{q_n^-\}) \right\}\Bigg|_{q_n^+,q_n^-=T_n^+} . \end{aligned} \tag{4.7}$$

Now, consider the following transformations of the variables:

$$q_k^+ = x_k + y_k \qquad \text{and} \qquad q_k^- = x_k - y_k$$

so that

$$\partial_{q_k^+} = \frac{1}{2}\left(\partial_{x_k} + \partial_{y_k}\right) \qquad \text{and} \qquad \partial_{q_k^-} = \frac{1}{2}\left(\partial_{x_k} - \partial_{y_k}\right).$$

Then, in these new coordinates, (4.7) becomes

$$F(\{T_n^+\},\{T_n^-\}) = G(\{T_n^+\},\{0\}), \tag{4.8}$$

where the new function $G(\{x_k\},\{y_k\})$ is defined[3] by

$$G(\{x_n\},\{y_n\}) = \exp\left[\frac{\lambda}{4}\Delta_+ \sum_{k,l\geq 0} \left(V_{kl}\partial_{x_k}\partial_{x_l} + W_{kl}\partial_{y_k}\partial_{y_l} \right) \right] \tau(\lambda\Delta_+;\{x_n+y_n\})\tau(\lambda\Delta_+;\{x_n-y_n\}), \tag{4.9}$$

where

$$\begin{aligned} V_{kl} &:= V_{kl}^{++} + i(-1)^{l-1}\, V_{kl}^{+-} \ , \\ W_{kl} &:= V_{kl}^{++} - i(-1)^{l-1}\, V_{kl}^{+-} \ . \end{aligned}$$

Remark: The conjecture expressed in terms of (4.9), i.e. that $G(\{T_k^+\},\{0\}) = 1$, is now in a form which resembles the Hirota bilinear relations, which might be indicating some kind of an integrable hierarchy, perhaps of Toda-type.

Because the tau-functions are exponential functions, upon acting on them by the differential operators, we can factor them out in the expression of $G(\{x_k\},\{y_k\})$. We thus define

[3] We have simplified the expression by noting that the mixed derivative terms cancel because of the identity $V_{kl}^{+-} = (-1)^{k-l}\, V_{lk}^{+-}$.

DEFINITION 4.1 *$P(\lambda\Delta_+, \{x_k\}, \{y_k\})$ is a formal power series in the variables $\lambda\Delta_+$, $\{x_k\}$ and $\{y_k\}$ such that*

$$G(\{x_k\},\{y_k\}) = P(\lambda\Delta_+,\{x_k\},\{y_k\})\,\tau(\lambda\Delta_+,\{x_k+y_k\})\,\tau(\lambda\Delta_+,\{x_k-y_k\}). \qquad (4.10)$$

Hence, Givental's conjecture for $\mathbb{P}^1$ can be restated as

CONJECTURE 4.2 (Givental) *The generating function $G(\{T_k^+\},\{0\})$ is equal to one, or equivalently*

$$P(\lambda\Delta_+,\{T_k^+\},\{0\}) = \frac{1}{\tau(\lambda\Delta_+,\{T_k^+\})^2}. \qquad (4.11)$$

This conjecture can be verified order by order[4] in λ.

Let us check (4.11) up to order λ^2, for which we need to consider up to λ^6 expansions in the differential operators acting on the τ-functions. Let $h = \lambda\Delta_+$. The low-genus free energies for a point target space can be easily computed using the KdV hierarchy and topological axioms; they can also be verified using Faber's program [Fa]. The terms relevant to our computation are

$$\begin{aligned}
\frac{\mathcal{F}_0^{\rm pt}}{h} + \mathcal{F}_1^{\rm pt} + h\mathcal{F}_2^{\rm pt} \;=\;& \frac{1}{h}\Bigg[\frac{(q_0)^3}{3!} + \frac{(q_0)^3 q_1}{3!} + 2!\frac{(q_0)^3(q_1)^2}{3!\,2!} + 3!\frac{(q_0)^3(q_1)^3}{3!\,3!} + \frac{(q_0)^4 q_2}{4!} + 3\frac{(q_0)^4 q_1 q_2}{4!} + \\
&+ 12\frac{(q_0)^4(q_1)^2 q_2}{4!\,2!} + \frac{(q_0)^5 q_3}{5!} + 4\frac{(q_0)^5 q_1 q_3}{5!} + 6\frac{(q_0)^5(q_2)^2}{5!2!} + 30\frac{(q_0)^5 q_1 (q_2)^2}{5!\,2!} + \\
&+ \frac{(q_0)^6 q_4}{6!} + 10\frac{(q_0)^6 q_2 q_3}{6!} + 90\frac{(q_0)^6(q_2)^3}{6!\,3!} + \cdots\Bigg] + \\
&+ \Bigg[\frac{1}{24}q_1 + \frac{1}{24}\frac{(q_1)^2}{2!} + \frac{1}{12}\frac{(q_1)^3}{3!} + \frac{1}{4}\frac{(q_1)^4}{4!} + \frac{1}{24}q_0q_2 + \frac{1}{12}q_0q_1q_2 + \frac{1}{4}\frac{q_0(q_1)^2q_2}{2!} + \\
&+ \frac{q_0(q_1)^3q_2}{3!} + \frac{1}{6}\frac{(q_0)^2(q_2)^2}{2!\,2!} + \frac{2}{3}\frac{(q_0)^2q_1(q_2)^2}{2!\,2!} + \frac{10}{3}\frac{(q_0)^2(q_1)^2(q_2)^2}{2!\,2!\,2!} + \frac{1}{24}\frac{(q_0)^2q_3}{2!} + \\
&+ \frac{1}{8}\frac{(q_0)^2q_1q_3}{2!} + \frac{1}{2}\frac{(q_0)^2(q_1)^2q_3}{2!\,2!} + \frac{7}{24}\frac{(q_0)^3q_2q_3}{3!} + \frac{35}{24}\frac{(q_0)^3q_1q_2q_3}{3!} + \\
&+ 2\frac{(q_0)^3(q_2)^3}{3!\,3!} + 12\frac{(q_0)^3q_1(q_2)^3}{3!\,3!} + \frac{1}{24}\frac{(q_0)^3q_4}{3!} + \frac{1}{6}\frac{(q_0)^3q_1q_4}{3!} + 48\frac{(q_0)^4(q_2)^4}{4!\,4!} + \\
&+ \frac{59}{12}\frac{(q_0)^4(q_2)^2q_3}{4!\,2!} + \frac{7}{12}\frac{(q_0)^4(q_3)^2}{4!\,2!} + \frac{11}{24}\frac{(q_0)^4q_2q_4}{4!} + \frac{1}{24}\frac{(q_0)^4q_5}{4!} + \cdots\Bigg] + \\
&+ h\Bigg[\frac{7}{240}\frac{(q_2)^3}{3!} + \frac{29}{5760}q_2q_3 + \frac{1}{1152}q_4 + \frac{7}{48}\frac{q_1(q_2)^3}{3!} + \frac{7}{8}\frac{(q_1)^2(q_2)^3}{2!\,3!} + \\
&+ \frac{29}{1440}q_1q_2q_3 + \frac{29}{288}\frac{(q_1)^2q_2q_3}{2!} + \frac{1}{384}q_1q_4 + \frac{1}{96}\frac{(q_1)^2q_4}{2!} + \frac{7}{12}\frac{q_0(q_2)^4}{4!} + \\
&+ \frac{49}{12}\frac{q_0q_1(q_2)^4}{4!} + \frac{5}{72}\frac{q_0(q_2)^2q_3}{2!} + \frac{5}{12}\frac{q_0q_1(q_2)^2q_3}{2!} + \frac{29}{2880}\frac{q_0(q_3)^2}{2!} + \\
&+ \frac{29}{576}\frac{q_0q_1(q_3)^2}{2!} + \frac{11}{1440}q_0q_2q_4 + \frac{11}{288}q_0q_1q_2q_4 + \frac{1}{1152}q_0q_5 + \frac{1}{288}q_0q_1q_5 +
\end{aligned}$$

[4] This procedure is possible because when $q_0 = q_1 = 0$, only a finite number of terms in the free-energies and their derivatives are non-vanishing. In particular, the genus-0 and genus-1 free energies vanish when $q_0 = q_1 = 0$.

$$+\frac{245}{12}\frac{(q_0)^2(q_2)^5}{2!\,5!}+\frac{11}{6}\frac{(q_0)^2(q_2)^3q_3}{2!\,3!}+\frac{109}{576}\frac{(q_0)^2q_2(q_3)^2}{2!\,2!}+\frac{17}{960}\frac{(q_0)^2q_3q_4}{2!}+$$
$$+\frac{7}{48}\frac{(q_0)^2(q_2)^2q_4}{2!\,2!}+\frac{1}{90}\frac{(q_0)^2q_2q_5}{2!}+\frac{1}{1152}\frac{(q_0)^2q_6}{2!}+\cdots\Bigg].$$

This expression gives the necessary expansion of $\tau(\lambda\Delta_+;\{x_k\pm y_k\})$ for our consideration, and upon evaluating $G(\{T_k^+\},\{0\})$, we find

$$P(h,\{T_k^+\},\{0\})=1-\frac{17}{2359296}e^{-3t^1/2}h+\frac{41045}{695784701952}e^{-3t^1}h^2+\mathcal{O}(h^3). \tag{4.12}$$

At this order, the expansion of the right-hand-side of (4.11) is

$$\tau(h,\{T_k^+\})^{-2}=1-2\,\mathcal{F}_2^{\rm pt}\,h+2\left[(\mathcal{F}_2^{\rm pt})^2-\mathcal{F}_3^{\rm pt}\right]h^2+\mathcal{O}(h^3). \tag{4.13}$$

At $q_n=T_n^+$, $\forall\,n$, the genus-2 free energy is precisely given by

$$\mathcal{F}_2^{\rm pt}=\frac{1}{1152}T_4+\frac{29}{5760}T_3\,T_2+\frac{7}{240}\frac{T_2^3}{3!}=\frac{17}{4718592}e^{-3t^1/2}, \tag{4.14}$$

and the genus-3 free energy is

$$\begin{aligned}\mathcal{F}_3^{\rm pt}\;=\;&\frac{1}{82944}T_7+\frac{77}{414720}T_2T_6+\frac{503}{1451520}T_3T_5+\frac{17}{11520}(T_2)^2T_5+\frac{607}{2903040}(T_4)^2\\&+\frac{1121}{241920}T_2T_3T_4+\frac{53}{6912}(T_2)^3T_4+\frac{583}{580608}(T_3)^3+\frac{205}{13824}(T_2)^2(T_3)^2\\&+\frac{193}{6912}(T_2)^4T_3+\frac{245}{20736}(T_2)^6\\=\;&-\frac{656431}{22265110462464}\,e^{-3t^1}\,.\end{aligned}$$

Thus, we have

$$\tau(h,\{T_k^+\})^{-2}=1-\frac{17}{2359296}\,e^{-3t^1/2}\,h+\frac{41045}{695784701952}\,e^{-3t^1}\,h^2+\mathcal{O}(h^3), \tag{4.15}$$

which agrees with our computation of $P(\lambda,\{T_k^+\},\{0\})$ in (4.12).

It would be very interesting if one could actually prove Givental's conjecture, but even our particular example remains elusive and verifying its validity to all orders seems intractable using our method.

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