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Existence and Properties of an Equilibrium State with Beam-Beam Collisions*

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Abstract

The equilbrium phase distribution of stored colliding electron beams is studied from the viewpoint of Vlasov-Fokker-Planck (VFP) theory. Numerical integration of the VFP system in one degree of freedom revealed a nearly Gaussian equilibrium with non-diagonal covariance matrix. This result is reproduced approximately in an analytic theory based on linearization of the beam-beam force. Analysis of an integral equation for the equilibrium distribution, without linearization, establishes the existence of a unique equilibrium at sufficiently small current. The role of damping and quantum noise is clarified through a new representation of the propagator of the linear Fokker-Planck equation with harmonic force.

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1 Introduction

The competition of damping and noise from synchrotron radiation in quanta results in a unique equilibrium state of a stored electron beam of sufficiently small current [1, 2]. Although very familiar, this is a noteworthy example of a macroscopic effect of quantal processes. At higher current, more complicated quasi-equilbrium states have been observed [3] and simulated [4]. These states have been called "sawtooth" modes; they are nearly periodic with very long period, comparable to the damping time. Again, the existence of such modes depends on the presence of damping and quantum noise.

An elegant and natural mathematical framework for study of equilibrium and non-equilibrium phase space distributions is based on the VFP equation, which is to say the ordinary Vlasov equation for self-consistent multi-particle motion, augmented with Fokker-Planck terms to account for damping and noise. Recently, stable, long-term numerical integration of the VFP equation for longitudinal motion has been achieved [4]. Calculations [4] with a realistic wake field gave good agreement with several aspects of observations at the SLAC damping rings [3].

Two counter-rotating stored beams, undergoing beam-beam collisions, can be treated by coupled VFP equations for the distribution functions of the two beams. This approach was formulated for the Vlasov part of the system by Chao and Ruth [5], in a model with one degree of freedom. In this paper we adopt their model, extended to include Fokker-Planck terms. Generalizations to more realistic models are certainly possible, and are on the agenda for future work.

The first order of business in the beam-beam problem is to determine the equilibrium distribution, understood as a distribution that is periodic in the machine azimuth s. Later, one will want to study stability of the equilibrium. If it is stable at small current, as expected, then the threshold current for instability and the character of trans-threshold, time-dependent motion are of interest. Surprisingly, the question of existence and character of an equilibrium state is rarely mentioned in the extensive beam-beam literature, although a few authors do recognize it as an open problem [6, 7, 8]. A first step [5] is to note that when the beam-beam force is linearized, the transverse motion for one beam has the familiar Courant-Snyder description, but with a "dynamic beta function". Consequently, there should be a conserved action , and the equilibrium distribution function in action-angle coordinates should be a function of action alone. This argument alone does not solve the equilibrium problem, however, since the dynamic beta function and the corresponding action are nonlinear functionals of the charge distribution of the opposing beam. Determination of the charge densities of the two beams so as to be self-consistent in the steady state is thus a remaining nonlinear problem. Certain aspects of this problem were treated by Furman, Ng, and Chao [6], and by Chao [7]. Another approach [8, 9] is to average the beam-beam force over a turn, neglecting its almost impulsive character. This leads to coupled integral equations of Haïssinski type [9], but seems unnecessarily rough as a physical model.

We have found it better to avoid both action-angle coordinates and averaging of the force. Also, we do not linearize the beam-beam force, except for an approximate analytic treatment which models in a simple way the important aspects of the full problem. We treat the full VFP system by numerical time-domain integration, and also by a nonlinear integral equation for the equilibrium distributions. We must defer to a longer report the comparison of our results to those of Furman *et al.* [6, 7].

2 Definitions and Equations

We treat vertical betatron motion with normalized phase-space variables (q, p) defined in terms of the vertical lattice function $\beta(s)$ and emittance ϵ as

$$q = y(\beta\epsilon)^{-1/2} , \quad p = (\beta y' - \beta' y/2)(\beta\epsilon)^{-1/2} , \tag{1}$$

where y is the vertical displacement and the prime denotes d/ds. The Hamiltonian is $H = (p^2+q^2)/2$ and the independent "time" variable of Hamilton's equations is the phase advance $\theta = \int_0^s du/\beta(u)$. The two beams may have dissimilar optics and intensities; we distinguish their properties by superscripts (1), (2).

The Chao-Ruth model is intended to represent vertical motion of beams with width L_x much greater than height L_y. We calculate the force as though it came from uniform planes of charge normal to the y-axis, which is to say from a charge density of the form $\rho(x, y, z) = \sigma\lambda(y)$, where $\int_{-\infty}^{\infty} \lambda(y)dy = 1$ and σ is the total charge per unit area in (x, z)-space accounting for charge at all y. To get the electric field $E(y)$ we apply Gauss's Law to a semi-infinite cylinder running along the y-axis to $y = +\infty$, with a face perpendicular to the axis at y. We then do the same for a cylinder running from y to $-\infty$, and eliminate $E(\infty) = -E(-\infty)$ between the two resulting equations to find $E(y)$. An analogous calculation of $H(y)$ by Ampère's Law shows that the magnetic force is precisely equal to the electric force for an ultrarelativistic particle. The full Lorentz force on an ultrarelativistic particle is in the y-direction and has the following value in m.k.s. units:

$$e(\mathbf{E} + \mathbf{v} \times \mathbf{B})_y = \frac{e\sigma}{\epsilon_0} \int_{-\infty}^{\infty} \mathrm{sgn}(y - y')\lambda(y')dy' , \tag{2}$$

where $\mathrm{sgn}(x)$ is 1 for $x > 0$ and -1 for $x < 0$. Now suppose that the beam has width L_x and length L_z, and bears a charge $\pm eN$. We approximate the force it exerts on a particle in the other beam by (2) with $\sigma = \pm eN/L_xL_z$. This force acts only during the transit time of the particle through the oncoming beam. During that time the particle moves a distance $\Delta s = L_z/2$ in the lab frame, so that the force as a function of s with IP at $s = 0$ is

$$\frac{\pm e^2Nh(s)}{\epsilon_0 L_x L_z} \int_{-\infty}^{\infty} \mathrm{sgn}(y - y')\lambda(y', s)dy' , \tag{3}$$

where $h(s)$ is 1 for $0 < s \pmod{C} < L_z/2$ and 0 otherwise, where C is the circumference of the reference orbit. Since the transit time is tiny compared to a betatron period, it seems reasonable to concentrate this force at $s = 0$. To do that we replace $2h(s)/L_z$, made up of step functions of unit integral, with $\delta_C(s) = \sum_n \delta(s - nC)$, the periodic delta function of period C. The resulting force is smaller by a factor of 2 than that of Chao and Ruth [5], but agrees with later papers of Chao [7] and Forest [10].

Knowing the force we can find the kick in transverse momentum p_y, and translate that into the kick of $y' = p_y/P$, where P is the total momentum. For relativistic beams of opposite charge, dy'/ds for beam (1) has the value given in Eq.(4) of Ref.[[5]], reduced by a factor of 2.

We now turn to VFP theory with two phase-space distribution functions $f^{(i)}(q,p,\theta)$, $i = 1,2$, normalized to unit integral. The corresponding particle position densities $\lambda^{(i)}(q,\theta)$ are obtained by integrating over p. In general, θ is defined differently for the two beams, but we do not distinguish the θ's with a superscript. Just remember that each equation and each function has its own independent variable. Translating our results in terms of (y, y', s) to (q, p, θ), we find the coupled VFP equations for beams of opposite charge as follows:

$$\begin{aligned}
&\frac{\partial f^{(1)}}{\partial\theta} + p\frac{\partial f^{(1)}}{\partial q} - \Big[q + (2\pi)^{3/2}\xi^{(1)}\sum_n \delta(\theta - 2\pi\nu^{(1)}n)\,\cdot \\
&\cdot \int_{-\infty}^{\infty} \mathrm{sgn}(q-q')\int_{-\infty}^{\infty} f^{(2)}(q',p',\theta)dq'dp'\Big]\frac{\partial f^{(1)}}{\partial p} \\
&= 2\alpha^{(1)}\frac{\partial}{\partial p}\Big[pf^{(1)} + \frac{\partial f^{(1)}}{\partial p}\Big]\ , \quad (\text{and } 1 \leftrightarrow 2)\ .
\end{aligned} \tag{4}$$

The beam-beam parameter is $\xi^{(1)} = N^{(2)}\beta^{*(1)}r_e/((2\pi)^{1/2}\gamma^{(1)}\sigma_y^{(1)}L_x^{(2)})$. Here β^* is the beta function at the IP, $r_e = e^2/(4\pi\epsilon_0 mc^2)$ is the classical electron radius, γ is the Lorentz factor, and $\sigma_y = (\beta^*\epsilon)^{1/2}$ is the bunch height. The right hand side of (4) is the Fokker-Planck contribution, with damping constant $\alpha^{(1)} = 1/(2\pi\nu^{(1)}n_d^{(1)})$, where n_d is the number of turns in a damping time. Our phase space coordinates have been defined so that the damping and diffusion constants are equal.

Equation (4) has only a formal significance, since the θ- dependent factors multiplying the delta function actually change discontinuously at the IP where the delta function acts. Consequently, we cannot say how to evaluate those factors without further analysis. Actually, the correct change of the distribution function at the IP is easy to see. Let $f^{(1)}(q,p,0-)$ and $f^{(1)}(q,p,0+)$ represent the distributions just before and just after $\theta = 0 \pmod{2\pi\nu^{(1)}}$. Then by the usual argument from probability conservation [4] the distribution is changed by the inverse of the kick map; i.e., by the Perron-Frobenius operator for that map:

$$f^{(1)}(q,p,0+) = f^{(1)}(q, p - F(q,0-), 0-)\ , \tag{5}$$

where

$$F(q,0-) = -(2\pi)^{3/2}\xi^{(1)}\int \mathrm{sgn}(q-q')f^{(2)}(q',p',0-)dq'dp'\ . \tag{6}$$

For propagation of the distribution function between IP kicks, we have in (4) a linear Fokker-Planck equation with harmonic force. The propagator or fundamental solution of that equation is known [11], namely a function $\mathcal{K}(z,z',\theta)$, $z = (q,p)$ such that for any initial distribution $f(z,0)$ the solution at time θ is

$$f(z,\theta) = \int \mathcal{K}(z,z',\theta)f(z',0)dz'\ . \tag{7}$$

There are several equivalent representations of $\mathcal{K}$. The following form, derived from a probabilistic argument, is especially appealing:

$$\begin{aligned}\mathcal{K}(z,z',\theta) = &\quad \frac{1}{2\pi(\det\Sigma)^{1/2}}\exp[-(z-e^{A\theta}z')^T\Sigma^{-1}(z-e^{A\theta}z')/2]\ , \\ &\quad \Sigma = I - e^{A\theta}e^{A^T\theta}\ . \end{aligned} \tag{8}$$

Here T denotes transposition and $e^{A\theta}$ is the transfer matrix for the single-particle harmonic motion with damping. With damping constant α we have

$$e^{A\theta} = \quad e^{-\alpha\theta}\begin{pmatrix} \cos\Omega\theta + (\alpha/\Omega)\sin\Omega\theta & (1/\Omega)\sin\Omega\theta \\ -\Omega(1+(\alpha/\Omega)^2)\sin\Omega\theta & \cos\Omega\theta - (\alpha/\Omega)\sin\Omega\theta \end{pmatrix}\ , \tag{9}$$

$$\Omega = (1-\alpha^2)^{1/2}\ , \quad \det e^{A\theta} = e^{-2\alpha\theta}\ . \tag{10}$$

Let $\mathbf{K}$ denote the operator corresponding to the kernel $\mathcal{K}(z,z',\theta)$. The action of $\mathbf{K}$ has a simple expression in Fourier space. Writing $\hat{h}$ for the Fourier transform of h, we have

$$\widehat{\mathbf{K}h}(v) = \exp[-v^T e^{A\theta}\Sigma e^{A^T\theta}v/2]\hat{h}(e^{A^T\theta}v)\ . \tag{11}$$

3 Numerical Integration of the VFP Equation

The kick map (5) followed by the action of $\mathbf{K}$ gives the complete propagation of the distribution function over one turn, and thus specifies the meaning of the delta function in the VFP equation. For numerical work it is highly inefficient to use $\mathbf{K}$, however. Instead, we shall follow the method of [[4]], based on operator splitting. We write $\partial f/\partial\theta = \mathbf{L}_V(f) + \mathbf{L}_{FP}(f)$, where $\mathbf{L}_V$ and $\mathbf{L}_{FP}$ are the operators associated with the Vlasov and Fokker-Planck terms, respectively. We make a θ step under $\mathbf{L}_V$ alone followed by a θ step under $\mathbf{L}_{FP}$ alone, and so on. It turns out that in this problem the step for $\mathbf{L}_{FP}$ can be a full turn, owing to the small value of the damping constant α. For $\mathbf{L}_V$ we apply the Perron-Frobenius (PF) operator for the map $T = RK$, where K is the beam-beam kick and R the phase-space rotation through angle $2\pi\nu$. The PF operator is discretized on a grid in phase space, being defined at off-grid points by local polynomial interpolation [4]. The kick is calculated at grid points from values of the distribution on grid points. Then $f(T^{-1}(z))$ is computed for grid points z by interpolation to give an update of f at grid points. For $\mathbf{L}_{FP}$ we use a divided-difference discretization and a simple Euler step [4] with $\Delta\theta = 2\pi\nu$. In a typical run we use a 201×201 grid, and the calculation takes 6.5 hours for 30000 turns on a 400 MHz work station. The algorithm conserves charge to one part in 10^5 (or 10^4 at high current) over several damping times, and reproduces the known solution for zero current.

We show results for parameters suggested by PEP-II design values; namely, $\nu^{(1)} = 0.6342$, $\nu^{(2)} = 0.6387$, $n_d^{(1)} = 5014$, $n_d^{(2)} = 8579$, $\xi^{(1)}/\xi^{(2)} = 1.11$, where beam (1) is in the high energy ring (9 GeV) and beam (2) in the low energy ring (3.1 GeV). Keeping the ratio of beam-beam parameters at the stated value, we increase $\xi^{(1)}$ in steps, starting with a small value such as 0.01. The initial distribution for each beam is the Gaussian $f_0(z) = \exp(-z^T z/2)/2\pi$, $z = (q,p)$, which is the solution for zero current. Figure 1 shows

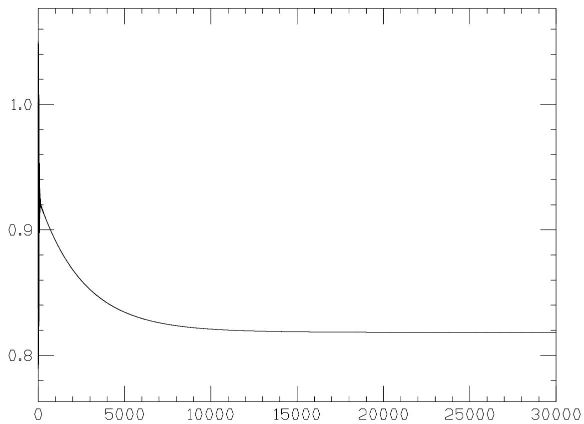

Figure 1: Normalized dimensionless beam size $\sigma_q^{(1)}$ *vs.* turn number, $\xi^{(1)} = 0.0280$.

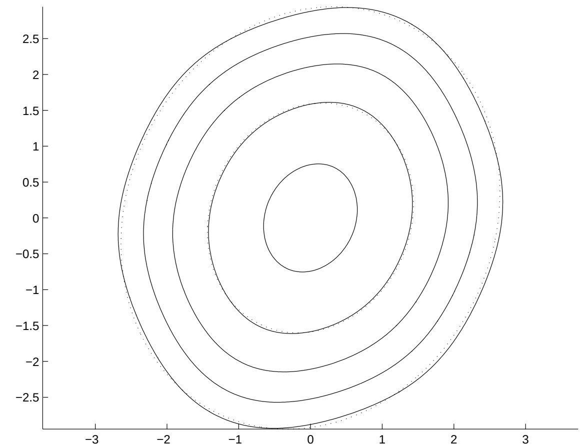

Figure 2: A contour plot of $-\log f^{(1)}$, where $f^{(1)}$ is the equilibrium distribution for beam (1) just before the IP. The dotted curves are for the Gaussian with the same covariance.

the normalized r.m.s. bunch size $\sigma_q^{(1)}$ for beam (1), just before the IP. It undergoes rapid oscillations in a region of transient behavior extending to about 150 turns, and then decreases slowly, reaching a steady state at about 2 damping times. Figure 2 is a contour plot of $-\log f^{(1)}$, where $f^{(1)}$ is the final distribution at $\xi^{(1)} = 0.028$. The solid lines are curves given by $-\log f^{(1)}(z) = c$, with $c = 2, 3, 4, 5, 6$. At smaller c the contours appear to be nearly elliptical, indicating a nearly Gaussian behavior.

To test the deviation from a Gaussian we compute the covariance matrix M of the final f, and look at the contour plot of $-\log g$, where g is the Gaussian with the same covariance, namely

$$g(z) = \frac{\exp(-z^T M^{-1} z/2)}{2\pi(\det M)^{1/2}} \ . \tag{12}$$

The dotted curves in Figure 2 represent $-\log g(z) = c$, for c equal to 3 and 6. They lie close to the corresponding solid curves, with more deviation at $c = 6$. Figure 3 shows a graph of the force in the equilibrium state.

The threshold of instability of the equilibrium state is somewhere between $\xi^{(1)} = 0.0280$ and $\xi^{(1)} = 0.0373$. Figure 4 shows σ_q at the latter value. The fast oscillations at large time are reminiscent of what was found in longitudinal single beam motion with wake field [4].

4 Equilibrium with Linearized Force, without Radiation

For a first step in an analytical discussion we take two beams with equal properties and turn off the synchrotron radiation by putting $\alpha^{(i)} = 0$ in (4). We linearize the beam-beam force

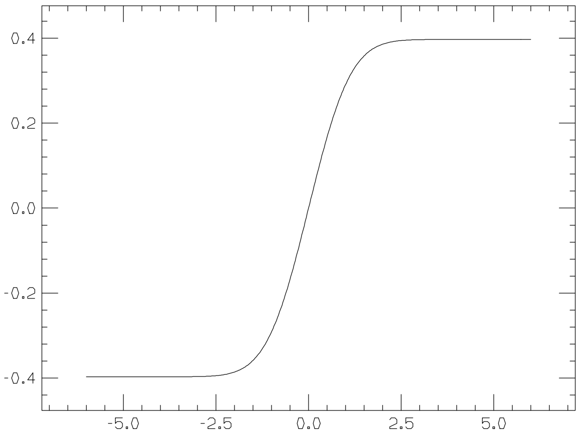

Figure 3: Beam-beam force $F(q)$ in the equilibrium state.

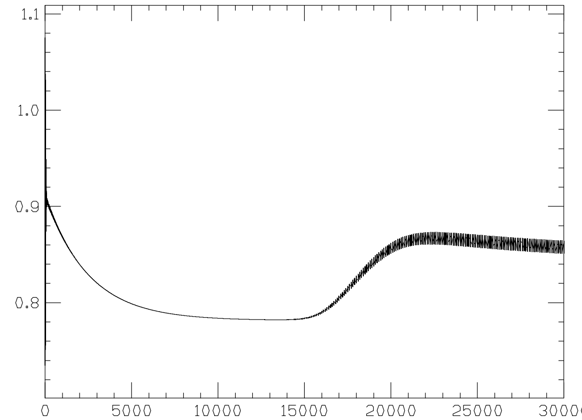

Figure 4: σ_q above the threshold of instability, at $\xi^{(1)} = 0.0373$.

as a function of q, but do not linearize the Vlasov equation in its dependence on f. By (6) the Taylor expansion of the force is

$$F(q) = -(2\pi)^{3/2}\xi\left[\left(\int_{-\infty}^{0} - \int_{0}^{\infty}\right)\lambda(q')dq' + 2\lambda(0)q + \mathcal{O}(q^2)\right] . \tag{13}$$

Motivated by the numerical results, we seek an equilibrium having the general Gaussian form (12) just before the beam-beam kick. Integrating this form over p to get λ we find $\lambda(q) = \exp(-q^2/(2m_{11}))/\sqrt{2\pi m_{11}}$, hence $F(q) = -4\pi\xi q/\sqrt{m_{11}} + \cdots$. We define $\eta = 4\pi\xi/\sqrt{m_{11}}$, and note that the condition of equilibrium, which is to say periodicity, is $g((RK)^{-1}z) = g(z)$, where R is the one-turn transfer matrix for the betatron motion, and K is the matrix transformation representing the kick:

$$R = \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix} , \quad K = \begin{pmatrix} 1 & 0 \\ -\eta & 1 \end{pmatrix} . \tag{14}$$

In other words, with $T = RK$ (hence $\det T = 1$), the equilibrium condition is

$$TMT^T = M , \tag{15}$$

where M must be symmetric and positive definite. For *any* matrix T with unit determinant, (15) has infinitely many symmetric solutions M, since the system regarded as three linear equations for three unknowns m_{11} , m_{22} , m_{12} has zero determinant. Provided that $t_{12} = \sin 2\pi\nu \neq 0$, all solutions can be expressed in terms of one parameter, m_{11}. In fact,

$$m_{22} = -\frac{t_{21}}{t_{12}} m_{11} , \quad m_{12} = m_{21} = \frac{t_{22} - t_{11}}{2t_{12}} m_{11} . \tag{16}$$

Also, the T that we actually have depends on M only through m_{11}, so that (16) gives the general solution of (15). From the definition of $T = RK$ we find

$$m_{22} = m_{11} + 4\pi\xi \cot 2\pi\nu \sqrt{m_{11}} , \quad m_{12} = 2\pi\xi\sqrt{m_{11}} . \tag{17}$$

Remarkably, the nonlinear equation (15) for M has been solved by linear means.

We have yet to impose the condition that M be positive definite, which is equivalent to the conditions $m_{11} > 0$, $\det M > 0$ taken together. It is easy to check that $\det M > 0$ if and only if

$$\xi < \frac{\sqrt{m_{11}}}{2\pi}\left[\frac{1}{|\sin 2\pi\nu|} + \cot 2\pi\nu\right] . \tag{18}$$

Through (17) we have an infinite family of Gaussian equilibria parametrized by $m_{11} > 0$, provided that $\sin 2\pi\nu \neq 0$ and ξ satisfies (18). Actually, in the present case without radiation the Gaussian is irrelevant: only the invariance of the quadratic form $z^T M z$ under T was essential to the argument. Let us look for an equilibrium of the form $f(z) = \Phi(z^T M^{-1} z)/\mathcal{N}(M)$, where M is positive definite, Φ is an arbitrary positive function on the positive real line, and $\mathcal{N}$ is a normalization constant chosen to make $\int f(z)dz = 1$. Calculating the latter integral by a change of variable that diagonalizes M then scales by eigenvalues, we find $\mathcal{N}(M) = \pi\sqrt{\det M}\int_0^\infty \Phi(x)dx$. Furthermore, integration of $f(0,p)$ over p gives

$$\lambda(0) = \frac{2\int_0^\infty \Phi(x^2)dx}{\pi\sqrt{m_{11}}\int_0^\infty \Phi(x)dx} , \tag{19}$$

the same dependence on m_{11} as in the Gaussian case. It is now clear that the solution is again given by (17), if we replace ξ by $\kappa\xi$, where

$$\kappa = \frac{2\sqrt{2/\pi}\int_0^\infty \Phi(x^2)dx}{\int_0^\infty \Phi(x)dx} . \tag{20}$$

All we require of Φ is that the integrals in (20) exist.

An interesting exercise, which we leave to the reader, is to explore the behavior of M as ξ approaches the limit (18). The ellipses become long and thin, since only one eigenvalue vanishes. In the limit $\sin 2\pi\nu \to 0$ the bound on ξ depends on the sign of $\tan 2\pi\nu$ near the limit. It expands to infinity for $\tan 2\pi\nu > 0$ but shrinks to zero for $\tan 2\pi\nu < 0$.

The angles of inclination of axes of the ellipse depend only on the tune. The eigenvectors of M^{-1} are $v_1 = (\cos\pi\nu,\ -\sin\pi\nu)$, $v_2 = (\sin\pi\nu,\ \cos\pi\nu)$. Then just before the IP, one axis of the ellipse is displaced by an angle $-\pi\nu$ with respect to the q-axis. The beam-beam kick rotates this axis by $+2\pi\nu$, to compensate the lattice motion which rotates the ellipse by $-2\pi\nu$. This is equivalent to saying that the kick reflects the distribution in the q-axis.

5 Equilibrium with Linearized Force, Including Radiation

To include radiation we can follow the plan of the previous section, except for replacing the rotation R by propagation according to the linear Fokker-Planck equation. The latter is handled most easily in Fourier space, by means of formula (11). The presence of the exponential factor makes clear that we will be restricted to Gaussian distributions in this case. Again we assume that the equilibrium distribution has the form (12) just before the

IP. Translating the equilibrium condition into Fourier space and applying (11), we find that M must satisfy

$$M - TMT^T = e^{A\theta}\Sigma e^{A^T\theta} \ , \quad T = e^{A\theta}K \ , \quad \theta = 2\pi\nu \ . \tag{21}$$

Here we have 3 inhomogeneous equations for $m_{11},\ m_{22},\ m_{12}$ (since 2 of the 4 equations are equivalent by symmetry). Multiplying on the right by $(T^T)^{-1}$ and recalling that $\det T = e^{-2\alpha\theta}$, we see that it is easy to eliminate m_{12} and m_{22}. First suppose that $t_{12} \neq 0$, which is to say $\sin\Omega\theta \neq 0$. Defining $\gamma = e^{2\alpha\theta}$ and $\sigma = e^{A\theta}\Sigma$ we find

$$\begin{aligned} m_{12} \quad &= \frac{1}{(1+\gamma)t_{12}}[(\gamma t_{22} - t_{11})m_{11} - \sigma_{11}] \ , \\ m_{22} \quad &= \frac{1}{\gamma t_{12}}[-t_{21}m_{11} + (\gamma - 1)t_{22}m_{12} - \sigma_{21}] \ , \end{aligned} \tag{22}$$

and the equation for m_{11} alone

$$\begin{aligned} &\left[\frac{1-\gamma^2}{\gamma}t_{12}t_{21} - \frac{1-\gamma}{1+\gamma}(t_{11} - t_{22}/\gamma)(\gamma t_{22} - t_{11})\right]m_{11} \\ &+\frac{1-\gamma}{1+\gamma}(t_{11} - t_{22}/\gamma)\sigma_{11} + (\sigma_{21}/\gamma - \sigma_{12} - \eta\sigma_{11})t_{12} = 0 \ . \end{aligned} \tag{23}$$

If $t_{12} = 0$, the equation for m_{11} is linear, and we get the complete M in explicit form:

$$\begin{aligned} m_{11} &= \frac{\pm\sigma_{11}}{2\sinh\alpha\theta} \ , \\ m_{12} &= \frac{\pm 1}{2\sinh\alpha\theta}[\sigma_{21} \mp e^{-\alpha\theta}\eta\ m_{11}] \ , \\ m_{22} &= \frac{\pm 1}{2\sinh\alpha\theta}[\eta\ \sigma_{21} + \sigma_{22} \mp 2\eta\cosh\alpha\theta\ m_{12}] \ . \end{aligned} \tag{24}$$

The sign is to agree with the sign of $\cos\Omega\theta = \pm 1$.

Now t_{11} and t_{21} are linear in η, and t_{22} and t_{12} are independent of η, where $\eta = 4\pi\xi/\sqrt{m_{11}}$. It follows that (23), after multiplication by $x = \sqrt{m_{11}}$, is a cubic equation for x. To lowest order in the damping constant α the coefficients of the cubic simplify. Cancelling an overall factor of α, we get the lowest order form of the equation, independent of α:

$$x^3 + 4\pi\xi\cot\theta x^2 - (1 + (2\pi\xi)^2)x - 2\pi\xi(\cot\theta - \cos 2\theta/\theta) = 0 \ . \tag{25}$$

Of course, this makes sense only if $\sin\theta \neq 0$, so that the expansion is useful only if $\sin\Omega\theta = \sin\theta + O(\alpha^2)$ is not too close to 0. With that restriction, (25) provides a good model of the exact polynomial (23). For typical values of α (around 10^{-5} in our numerical examples) we find that the roots of the two cubics agree to about 3 digits over a grid in the (ξ, ν) parameter space, including values of ν fairly close to 1/2, say $\nu = 0.505$.

For zero current the equation (25) reduces to $x^3 - x = 0$. The root $x = 1$ is the correct solution for zero current, corresponding to the unperturbed Gaussian. The roots $x = -1,\ 0$ are unphysical. To lowest order in the current parameter ξ, the roots of (25) are

$$x_\pm = \pm 1 - \pi\xi(\cot\theta + \cos 2\theta/\theta) \ , \quad x_0 = -2\pi\xi(\cot\theta - \cos 2\theta/\theta) \ . \tag{26}$$

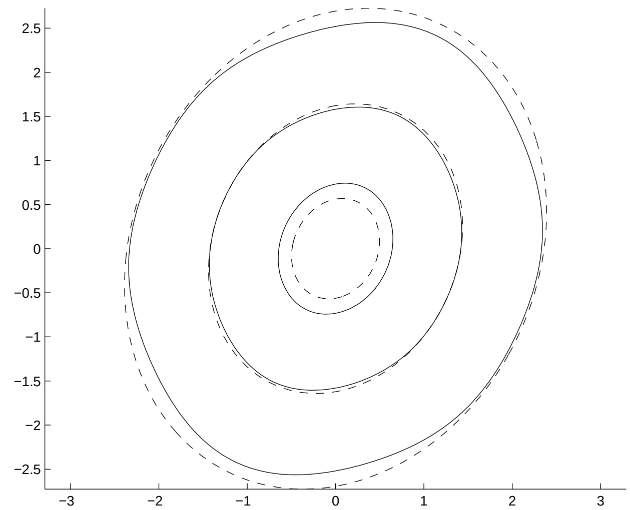

Figure 5: Some contours of equal probability for a VFP solution (solid line) and for the model with linearized force (dashed line)

Now x_0 may be positive or negative, but even if positive it corresponds to an unphysical solution, since the corresponding lowest order form of M is not positive definite.

Similarly, for non-zero current and the exact solution of equation (23) we find one positive root which corresponds to a positive definite M, one negative root that is clearly unphysical, and one root near zero that can be either positive or negative, but is unphysical even if positive since the corresponding M is not positive definite. This result was seen in a numerical exploration over a fine grid in (ξ, ν) space. This outcome is quite satisfactory: as expected from experience in the single-beam problem, inclusion of radiation reduces the infinite family of Vlasov solutions to a single solution of Gaussian type.

With radiation we have found no analog of the constraint (18). Nevertheless, the beam can get larger than the beam pipe at large current or in near-resonant conditions ($\sin \Omega\theta$ small). This is seen in the small-α form of M for $\sin \Omega\theta = 0$. By (24) we find

$$m_{11} = 1 + \mathcal{O}(\alpha) \ , \quad m_{12} = -\frac{\eta}{2\alpha\theta} + \mathcal{O}(1) \ , \quad m_{22} = \frac{\eta^2}{2(\alpha\theta)^2} + \mathcal{O}(1/\alpha) \ . \tag{27}$$

Since $m_{22} = \mathcal{O}(\alpha^{-2})$, the ellipses are long and thin in the p direction. Although the equilibrium exists mathematically even at a resonance, it may be unrealizable in the machine.

Figure 5 compares the result of the present linearized model with numerical integration of the VFP equation for equal beams with $\nu = 0.6364$, $\xi = 0.0266$, $n_d = 5000$. The angle of tilt of the near-elliptical curves is given quite well by the linearized theory, but in other respects the agreement is somewhat rough. The covariance matrix of the VFP solution is $m_{11} = 0.8297$, $m_{22} = 1.016$, $m_{12} = 0.1075$, whereas that of the linearized model is $m_{11} = 0.9095$, $m_{22} = 1.185$, $m_{12} = 0.1593$. It is interesting that the Gaussian determined by the covariance matrix of the VFP solution gives a better fit to the VFP solution than the Gaussian from the theory with linearized force.

6 Integral Equation for the Equilibrium state

We write coupled integral equations for the equilibrium distributions $f^{(1)}$, $f^{(2)}$ just *after* the IP. The difference in convention compared to the above discussion (*after* rather than *before*) arises from a technical point in the analysis. The equations are

$$g^{(i)}(q, p + (2\pi)^{3/2}\xi^{(i)} \int \mathrm{sgn}(q - q')g^{(j)}(z')dz') = f^{(i)}(z) \ , \quad i \neq j$$
$$g^{(i)}(z) = \int K^{(i)}(z, z', 2\pi\nu^{(i)})f^{(i)}(z')dz' \ , \tag{28}$$

with

$$\int f^{(i)}(z)dz = 1 \ . \tag{29}$$

It is essential that the normalization constraint (29) be regarded as part of the definition of the mathematical system.

The physical meaning of (28) should be obvious: we start just after the IP, propagate for one turn by the linear Fokker-Planck operator, then apply the beam-beam kick, and require that the result be equal to what we started with.

It is possible to analyze these equations without any approximations, using methods of functional analysis. By applying the implicit function theorem in an appropriate Banach space, one can show that there is a solution, unique in that space, at sufficiently small $\xi^{(i)}$. The proof will be published elsewhere. In accord with the linearized theory with radiation, a non-resonance condition is not required. The method of proof is readily generalized to the beam-beam problem with two degrees of freedom.

The system (28), (29) is analogous to the Haïssinski equation for longitudinal motion with wake field, but certainly quite different in form. It lives on phase space, and it depends on both the damping and diffusion constants (the ratio of the two being buried in our choice of variables). The principal dependence is on the ratio, however, as the discussion of the previous section and numerical VFP solutions indicate. The Haïssinski equation lives on q-space, and it depends only on the ratio of damping and diffusion constants.

7 Conclusion

We have found a fairly complete depiction of the equilibrium state in a beam-beam interaction model with one degree of freedom. It remains to work out relations to the dynamic beta function description, and to extend the theory to 2 or 3 degrees of freedom. In higher dimensions we expect the equilibria to be generally similar to what we have found, but time-dependent phenomena at high current should be much richer than in 1 degree of freedom. Preliminary results on high-current motion suggest that many interesting results can be expected from numerical integration of the VFP equation in the time domain.

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