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Self-consistent particle distribution of a bunched beam in RF field

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Abstract

Analytical solution for self-consistent particle equilibrium distribution in RF field with transverse focusing is found. Solution is attained in approximation of high brightness beam. Distribution function in phase space is determined as a stationary function of the energy integral. Equipartitioning for beam distribution between degrees of freedom follows directly from the choice of stationary distribution function. Analytical expressions for r-z equilibrium beam profile and maximum beam current in RF field are obtained.

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1 INTRODUCTION

Emittance conservation and prevention of halo formation in a high brightness particle beam in RF accelerator are issues for existing and future high intensity accelerator projects. If the beam is matched with external focusing and accelerating field, its distribution function as well as beam emittance are conserved. Matched stationary beam does not exhibit halo formation. Finding matched conditions for the beam requires solutions of the self-consistent problem for beam distribution function in 6-dimensional phase space, which is typically possible only by numerical methods. In this paper we present analytical self-consistent solution for stationary bright bunched beam in RF field.

The problem of stationary self-consistent particle distribution in RF field was considered in several books and papers. Typical approximation to the solution of the problem is a uniformly-charged ellipsoid. Concept of ellipsoid gives the most simple way to estimate the maximum beam current in RF field. Meanwhile in general case ellipsoid is not a self-consistent solution for bunched beam in RF field. In Ref. [1] solution of one-dimensional problem in longitudinal phase space was found. Space charge density of a cylindrical bunch was found to be constant in every cross section of the bunch, but dependent on longitudinal coordinate. In Ref. [2] spatial particle distribution in 3-dimensional configuration space was calculated numerically. In this paper an analytical approximate solution for 3-D self-consistent particle equilibrium is attained.

2 SELF-CONSISTENT PROBLEM

Consider intense bunched beam of particles with charge q and mass m, propagating in a continuos focusing channel with applied accelerating RF field. Beam is supposed to be bunched at the frequency $\omega = 2\pi c/\lambda$, where c is the velocity of light and λ is a wavelength. Average longitudinal particle velocity of the beam is $\beta_s = v_s/c$, therefore distance between bunches is $\beta_s\lambda$. Particle motion is governed by single-particle Hamiltonian [1]:

$$H = \frac{p_x^2 + p_y^2}{2\,m\,\gamma} + \frac{p_z^2}{2\,m\,\gamma^3} + q\,U_{ext} + q\,\frac{U_b}{\gamma^2}\ , \qquad (2.1)$$

$$U_{ext} = \frac{E}{k_z} [\, I_o (\frac{k_z r}{\gamma}) \sin(\varphi_s - k_z \zeta) - \sin\varphi_s + k_z \zeta \cos\varphi_s] + G_t \frac{r^2}{2}, \quad (2.2)$$

where p_x and p_y are transverse particle momentum, $p_z = p - p_s$ is a deviation from longitudinal momentum of synchronous particle, $\gamma = (1 - v_s^2/c^2)^{-1/2}$ is an average beam energy, $\zeta = z - z_s$ is a deviation from position of synchronous particle, U_{ext} is a potential of external field, U_b is a space charge potential of the beam, E is an amplitude of accelerating field, φ_s is a synchronous phase, $k_z = 2\pi/(\beta_s\lambda)$ is a wave number, G_t is a gradient of focusing field, and r is a particle radius.

Space charge density distribution of a moving bunched beam has the form of $\rho = \rho\,(x, y, z - v_s t)$. Moving bunch creates an electromagnetic field with scalar potential $U_b = U_b\,(x, y, z - v_s t)$ and vector potential $\vec{A}_b = \vec{A}_b\,(x, y, z - v_s t)$, which obey wave equations [3]:

$$\Delta U_b - \frac{1}{c^2}\frac{\partial^2 U_b}{\partial t^2} = -\frac{\rho}{\varepsilon_o}, \quad (2.3)$$

$$\Delta \vec{A}_b - \frac{1}{c^2}\frac{\partial^2 \vec{A}_b}{\partial t^2} = -\mu_o \vec{j}, \quad (2.4)$$

where $\vec{j} = \rho\,\vec{v}_s$ is a current density of the beam. Current density has only longitudinal component

$$j_x = j_y = 0, \quad j_z = v_s\,\rho\,(x, y, z - v_s t), \quad (2.5)$$

and, therefore, vector potential has only longitudinal component A_z. In a moving coordinate system, where particles are static, the vector potential of the beam is zero, $\vec{A} = 0$. According to Lorentz transformation, longitudinal component of vector potential in laboratory system is $A_z = \beta_s U_b / c$. Therefore, for solution of the problem of self field of the bunch it is enough to solve only equation for scalar potential (2.3). Substitution of the value A_z into wave equation (2.4) gives the equation for scalar potential:

$$\frac{\partial^2 U_b}{\partial x^2} + \frac{\partial^2 U_b}{\partial y^2} + \frac{\partial^2 U_b}{\gamma^2 \partial \zeta^2} = -\frac{1}{\varepsilon_o}\,\rho\,(x, y, \zeta). \quad (2.6)$$

Equation (2.6) has to be solved together with Vlasov's equation for beam distribution function:

$$\frac{df}{dt} = \frac{1}{m\gamma}\left(\frac{\partial f}{\partial x} p_x + \frac{\partial f}{\partial y} p_y + \frac{\partial f}{\partial \zeta} p_z\right) - q\left(\frac{\partial f}{\partial p_x}\frac{\partial U}{\partial x} + \frac{\partial f}{\partial p_y}\frac{\partial U}{\partial y} + \frac{\partial f}{\partial p_z}\frac{\partial U}{\partial \zeta}\right) = 0 \ , \tag{2.7}$$

where $U = U_{ext} + \gamma^{-2} U_b$ is a total potential of the structure. Eqs (2.6), (2.7) define self-consistent distribution of a stationary beam which acts on itself in such a way, that this distribution is conserved.

3. BEAM EQUIPARTITIONING IN RF FIELD

General approach to find a stationary self-consistent beam distribution function is to represent it as a function of Hamiltonian f = f(H) and then to solve Poisson's equation. Because Hamiltonian is a constant of motion for stationary process, any function of Hamiltonian is also a constant of motion which automatically obeys Vlasov's equation. Convenient way is to use an exponential function $f = f_o \exp(-H / H_o)$:

$$f = f_o \exp\left(-\frac{p_x^2 + p_y^2}{2\, m\, \gamma H_o} - \frac{p_z^2}{2\, m\, \gamma^3 H_o} - q\frac{U_{ext} + U_b \gamma^{-2}}{H_o}\right). \tag{3.1}$$

Consider important consequence which follows immediately from Eq. (3.1). Let us rewrite distribution function (3.1) as

$$f = f_o \exp\left(-2\frac{p_x^2 + p_y^2}{p_t^2} - 2\frac{p_z^2}{p_l^2} - q\frac{U_{ext} + U_b \gamma^{-2}}{H_o}\right), \tag{3.2}$$

where $p_t = 2\sqrt{<p_x^2>} = 2\sqrt{<p_y^2>}$ and $p_l = 2\sqrt{<p_z^2>}$ are double root-mean-square (rms) beam sizes in phase space. Transverse, ε_t, and longitudinal, ε_l, rms beam emittances are:

$$\varepsilon_t = 2\frac{p_t}{mc}\sqrt{<x^2>} = 2\frac{p_t}{mc}\sqrt{<y^2>} \ , \tag{3.3}$$

$$\varepsilon_l = 2\frac{p_l}{mc}\sqrt{<\zeta^2>} \ . \tag{3.4}$$

Taking together Eqs. (3.1) - (3.4), the value of H_o can be expressed as a function of beam parameters:

$$16 \cdot H_o = \frac{m\, c^2}{\gamma}\frac{\varepsilon_t^2}{<x^2>} = \frac{m\, c^2}{\gamma}\frac{\varepsilon_t^2}{<y^2>} = \frac{mc^2}{\gamma^3}\frac{\varepsilon_l^2}{<\zeta^2>} \ . \tag{3.5}$$

Equation (3.5) can be rewritten as

$$\frac{\varepsilon_t}{R} = \frac{\varepsilon_l}{\gamma l}, \tag{3.6}$$

where $R = 2\sqrt{<x^2>}$ is a beam radius and $l = 2\sqrt{<\zeta^2>}$ is a half-size of the bunch length. Equation (3.6) expresses the equipartitioning condition for the beam in RF field [4]. From the above derivations it is clear, that equipartitioning is a consequence of stationarity of the collisionless beam distribution function, Eq. (3.1). If distribution function is stationary (time independent), equipartitioning is fulfilled. Opposite statement is not valid in general case: there are infinitely large number of distribution functions which obeys condition (3.6), but are not necessarily stationary. To find the stationary distribution function it is necessary to solve nonlinear Poisson's equation for unknown space-charge potential of the beam.

4. SPACE CHARGE FIELD OF THE BUNCH

Space charge density of the beam is obtained as an integral of beam distribution function over particle momentum:

$$\rho\,(x,y,\zeta) = q \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\, dp_x\, dp_y\, dp_z = \rho_o \exp\,(-\, q\frac{U_{ext} + U_b\gamma^{-2}}{H_o}), \tag{4.1}$$

where ρ_o is the space charge density in the center of the bunch. The value of ρ_o is unknown at this point due to unknown space charge potential of the beam, U_b. For further analysis let us introduce an average value of space charge density, $\bar{\rho}$, which is equal to the density of an equivalent uniformly-charged cylindrical bunch with the same beam radius, R, and the same half-bunch length, l, as that of unknown stationary bunch. Space charge density of the cylindrical bunch is

$$\bar{\rho} = \frac{Q}{V} = \frac{I\lambda}{2\pi R^2 l c} . \tag{4.2}$$

where $Q = I\lambda/c$ is a charge of the bunch, $V = \pi R^2 2l$ is a volume of the bunch and I is a beam current. Compare the value of $\bar{\rho}$, Eq. (4.2), with that for another distributions. Space charge density of a uniformly populated spheroid with semi-axises R and l is

$$\rho_s = \frac{3\,I\,\lambda}{4\pi\,R^2\,l\,c} = \frac{3}{2}\bar{\rho}\,. \qquad (4.3)$$

Bunch with Gaussian distribution

$$\rho = \frac{I\,\lambda}{(2\pi)^{3/2} c\,\sqrt{<x^2>}\,\sqrt{<y^2>}\,\sqrt{<\zeta^2>}}\;\exp\left(-\frac{x^2}{2<x^2>} - \frac{y^2}{2<y^2>} - \frac{\zeta^2}{2<\zeta^2>}\right), \qquad (4.4)$$

has a space charge density in its center

$$\rho_G = \frac{8}{(2\pi)^{3/2}}\,\frac{I\,\lambda}{c\,R^2 l} = \frac{8}{\sqrt{2\pi}}\,\bar{\rho}\,. \qquad (4.5)$$

Since different distributions give similar expressions for space charge density in the bunch center within the factor of $k \approx 1...3$, one can assume that unknown value of space charge density ρ_o in bunch center, Eq. (4.1), also differs from the average value of space charge density $\bar{\rho}$ within the same factor:

$$\rho_o = k\,\bar{\rho}\,. \qquad (4.6)$$

For further derivations introduce dimensionless variables:

$$V_{ext} = \frac{q\,U_{ext}}{H_o}, \qquad V_b = \frac{q\,U_b}{H_o}, \qquad \xi = \frac{r}{a}, \qquad \eta = \frac{\zeta}{a}, \qquad (4.7)$$

where a is a channel radius. The Poisson's equation (2.6) in cylindrical polar coordinates becomes

$$\frac{1}{\xi}\frac{\partial V_b}{\partial \xi} + \frac{\partial^2 V_b}{\partial \xi^2} + \frac{\partial^2 V_b}{\partial \eta^2 \gamma^2} = -q\,\frac{\rho_o\,a^2}{\varepsilon_o\,H_o}\exp - \left(V_{ext} + \frac{V_b}{\gamma^2}\right). \qquad (4.8)$$

Let us introduce the values of a bunching factor, $B \equiv 2l/(\beta\lambda)$, and dimensionless beam brightness, $b \equiv 2IR^2/(\beta\gamma I_c \varepsilon_t^2)$ (see Appendix 1) where $I_c \equiv 4\pi\varepsilon_o mc^3/q$ is a characteristic value of beam current. Parameter b is a ratio of "space charge term" to "emittance term" in KV envelope equation and is a measure of influence of space charge forces on beam dynamics. Regime with b>>1 corresponds to space charge dominated beam transport while regime with b<<1 corresponds to emittance dominated beam transport. Substitution of ρ_o, Eq. (4.6), and H_o, Eq. (3.5), with introduced values of B and b into Eq. (4.8) gives:

$$\frac{1}{\xi}\frac{\partial V_b}{\partial \xi} + \frac{\partial^2 V_b}{\partial \xi^2} + \frac{\partial^2 V_b}{\partial \eta^2 \gamma^2} = - \frac{8\,k\,b}{B}(\gamma\frac{a}{R})^2 \exp - (V_{ext} + \frac{V_b}{\gamma^2}) . \qquad (4.9)$$

Equation (4.9) is a nonlinear differential equation for unknown beam space charge potential, V_b, which appears in the left and right side of equation. In general case it can be solved only numerically. Below consider approximate analytical solution for high brightness beam following the method suggested in Ref. [5].

Unknown space charge potential of the beam can be represented as Fourier-Bessel series:

$$V_b = V_o + \sum_{n=o}^{\infty} \sum_{m=1}^{\infty} J_o(\upsilon_{om}\,\xi)[\, A_{nm} \cos(k_z n \eta a) + B_{nm} \sin(k_z n \eta a)] , \qquad (4.10)$$

where $J_o(\zeta)$ is a Bessel function, υ_{om} is a m-th root of the equation $J_o(\zeta) = 0$. Expansion (4.10) obeys Dirichlet boundary condition $V_b(1, \eta) = V_o$ at the perfect conductive surface of the channel and takes into account periodic function of potential due to train of the bunches.

To find the first approximation to solution of Poisson's equation, let us take only the first term in expansion of exponential function

$$\exp(-V_{ext} - V_b \gamma^{-2}) \approx 1 - V_{ext} - V_b \gamma^{-2}. \qquad (4.11)$$

Poisson's equation (4.9) then becomes:

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} [1 + \frac{\upsilon_{om}^2 + (k_z n a)^2 \gamma^{-2}}{8kb} B(\frac{R}{a})^2]\, J_o(\upsilon_{om}\xi)\, [A_{nm} \cos(k_z n \eta a) + B_{nm} \sin(k_z n \eta a)]$$

$$= (1 - V_{ext})\,\gamma^2 - V_o . \qquad (4.12)$$

Space charge potential, Eq. (4.10), is mostly represented by several low-order terms. For example, for the train of uniformly populated cylindrical bunches (see Appendix 2), the values of Fourier-Bessel coefficients drop quickly with numbers m, n:

$$A_{nm} \sim \frac{1}{n\, \upsilon_{om} [\upsilon_{om}^2 + (k_z n a)^2 \gamma^{-2}]} . \qquad (4.13)$$

For a space charge dominated beam, $b >> 1$, Eq. (4.12) can be simplified. Expression in square brackets in Eq. (4.12) is

$$1 + \frac{\upsilon_{om}^2 + (k_z n a)^2 \gamma^{-2}}{8 k b} B \left(\frac{R}{a}\right)^2 = 1 + \delta , \qquad (4.14)$$

where introduced parameter δ is:

$$\delta = \frac{\upsilon_{om}^2 + (k_z n a)^2 \gamma^{-2}}{8 k b} B \left(\frac{R}{a}\right)^2. \qquad (4.15)$$

Low-order roots of the Bessel function are $\upsilon_{o1} = 2.408$, $\upsilon_{11} = 3.832$, $\upsilon_{o2} = 5.52$. Product of $k_z a$ is usually close to unity:

$$k_z a = 2\pi \left(\frac{a}{\beta\lambda}\right) \approx 1 . \qquad (4.16)$$

Taking into account, that $B \leq 1$, $R / a \approx 0.5$, it is easy to see that the value of δ, Eq. (4.15), is much smaller than unity for a high brightness beam. It can be written as

$$\delta \approx \frac{1}{b_\varphi k} << 1, \qquad (4.17)$$

where b_φ is a dimensionless beam brightness of bunched beam:

$$b_\varphi = \frac{b}{B} \left(\frac{a}{R}\right)^2 = \frac{2}{\beta\gamma} \frac{I}{I_c B} \left(\frac{a}{\varepsilon_t}\right)^2 = \frac{2}{\beta\gamma} \frac{I_{peak}}{I_c} \left(\frac{a}{\varepsilon_t}\right)^2 , \qquad (4.18)$$

and $I_{peak} = I/B$ is a peak value of bunched beam current. Therefore, expression, Eq. (4.14), can be taken out of the sum in Eq. (4.12). With this approximation, Eq. (4.12) becomes:

$$(1+ \delta)(V_b - V_o) = (1 - V_{ext})\gamma^2 - V_o . \qquad (4.19)$$

Let us define constant V_o in such a way that the total potential of the structure vanishes at the bunch center:

$$V_{ext} (0, 0) + \frac{V_b (0, 0)}{\gamma^2} = 0 . \qquad (4.20)$$

External potential is equal to zero at beam center $V_{ext} (0, 0) = 0$, see Eq. (2.2), therefore condition (4.20) gives $V_b (0, 0) = 0$. Substitution of $V_{ext} (0, 0)$, $V_b (0, 0)$ into Eq. (4.19) defines constant $V_o = - \gamma^2/\delta$. Then, from Eq. (4.19) the self-consistent space charge dominated beam potential is:

$$V_b = -\frac{\gamma^2}{1+\delta} V_{ext} . \quad (4.21)$$

Equation (4.21) indicates that the particle distribution of the bright beam has such a shape that the space charge potential is opposite to the external potential. This fact is well known for a stationary distribution of a transported beam in a linear focusing channel [1], [2]. Equation (4.21) generalizes this statement for a 3-dimensional beam distribution. Space charge field of a stationary bunch always compensates for external field in the beam interior. This phenomenon is known from plasma physics as Debye shielding for nonneutral plasmas.

Lorentz force, $\vec{F} = \vec{E} + [\vec{v} \times \vec{B}]$, created by the beam is connected with space charge potential of the beam by equation [1, Eq. (2.43)]:

$$\vec{F}_b = -\frac{1}{\gamma^2} \text{grad } U_b . \quad (4.22)$$

Substitution of Eq. (4.22) into Eq. (4.21) gives relationship between Lorentz force of the beam and that of external field:

$$\vec{F}_b = -\frac{1}{1+\delta} \vec{F}_{ext} . \quad (4.23)$$

Second approximation to the self-consistent potential is given by holding one more term in the expansion of exponential function as $\exp(-V_{ext}-V_b\gamma^{-2}) \approx 1 - V_{ext}-V_b\gamma^{-2} + 0.5(V_{ext}+V_b\gamma^{-2})^2$, so that we have (see Ref. [5])

$$V_b = \gamma^2[1 + \delta - V_{ext} - \sqrt{(1+\delta - V_{ext})^2 - V_{ext}(V_{ext} - 2)}\,]. \quad (4.24)$$

With increasing of beam brightness, $\delta \to 0$, solution of Eq. (4.24) becomes close to that of Eq.(4.21).

In more general case Eq. (4.9) for a high brightness beam, $\delta \ll 1$, can be solved only numerically. Substitution of potential expansion, Eq. (4.10), into Eq. (4.9) together with parameter δ, Eq. (4.15), gives for Poisson's equation:

$$\delta(V_b - V_o) = \gamma^2 \exp\left(-V_{ext} - \frac{V_b}{\gamma^2}\right) . \quad (4.25)$$

Again, taking into account that in the beam center $V_{ext}(0, 0) = 0$, $V_b(0, 0) = 0$, the unknown constant in Eq. (4.25) is $V_o = -\gamma^2/\delta$. Using the value of V_o, Poisson's equation for a high brightness beam is as follows:

$$1 + \frac{\delta}{\gamma^2} V_b = \exp\left(-V_{ext} - \frac{V_b}{\gamma^2}\right). \qquad (4.26)$$

In Fig. 1 results of numerical solution of Poisson's equation for different values of beam brightness are presented. As seen, with increasing beam brightness, an exact numerical solution becomes close to linear relationship between space charge potential and external potential, Eq. (4.21).

Taking the first approximation to the space charge potential of the beam, Eq. (4.21), the Hamiltonian corresponding to the self-consistent bunch distribution is as follows:

$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + \frac{p_z^2}{2\, m\, \gamma^3} + q\left(\frac{\delta}{1+\delta}\right) U_{ext} . \qquad (4.27)$$

Equation (4.25) indicates that in the presence of intense, bright bunched beam ($\delta << 1$) the stationary longitudinal phase space of the beam becomes narrow in momentum spread, remaining, in the first approximation, the same in coordinate (see Fig. 2). This is in qualitative agreement with the study of Ref. [1], where the self-consistent problem for stationary bunched beam was solved numerically for longitudinal phase - space particle distribution. The resulting bunch was approximated by a hard-edged cylinder. Particle density was constant in every cross section of a cylinder, but depended on longitudinal position. Numerical results of the Ref. [1] indicated that in space-charge-dominated regime, separatrix of longitudinal phase space was substantially reduced in momentum, remaining almost unchanged in phase width, which is in good agreement with Eq. (4.25).

5 STATIONARY BUNCH PROFILE

Self consistent space charge density distribution of matched beam can be found from the Poisson's equation:

$$\rho(r,\zeta) = -\varepsilon_o \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_b}{\partial r}\right) + \frac{\partial^2 U_b}{\gamma^2 \partial \zeta^2}\right]. \qquad (5.1)$$

Substitution of Eq. (4.24) into Eq. (5.1) gives the stationary particle density distribution inside the bunch:

$$\rho(r,\zeta) = 2\gamma^2 G_t\, \varepsilon_o \left\{ 1 - \frac{\delta}{\sqrt{(1+\delta)^2 - 2\delta V_{ext}}} - \frac{\delta^2}{32\gamma} \frac{\varepsilon_t^2}{<x^2>} \left(\frac{mc^2}{G_t q\, a^2}\right) \frac{\left(\frac{\partial V_{ext}}{\partial \xi}\right)^2 + \left(\frac{\partial V_{ext}}{\gamma \partial \eta}\right)^2}{[(1+\delta)^2 - 2\delta V_{ext}]^{3/2}} \right\} . \qquad (5.2)$$

For a high brightness beam parameter $\delta << 1$, therefore, space charge density is close to constant within the bunch:

$$\rho(r,\zeta) \approx 2 \frac{\gamma^2}{1+\delta} G_t\, \varepsilon_o \ . \qquad (5.3)$$

From Eq. (4.21) it follows, that, in the first approximation, space charge potential of the beam is the same function of coordinates, as the external potential, with opposite sign. Therefore, equation $U_{ext}(r, \zeta)=$ const gives the family of equipotential lines of space charge field of the beam:

$$I_o\left(\frac{k_z r}{\gamma}\right)\sin(\varphi_s - k_z\zeta) - \sin\varphi_s + k_z\zeta\cos\varphi_s + \frac{G_t k_z}{2E} r^2 = \text{const} . \qquad (5.4)$$

In general case, bunch boundary does not create an equipotential surface, therefore Eq. (5.4) does not coincide with bunch profile. To treat the problem, consider uniformly populated bunch with boundary $R(\zeta)$, defined by nonlinear equation

$$I_o\left(\frac{k_z R}{\gamma}\right)\sin(\varphi_s - k_z\zeta) - \sin\varphi_s + k_z\zeta\cos\varphi_s + C(k_z R)^2 = \text{const} . \qquad (5.5)$$

Equation (5.5) differs from Eq. (5.4) by inserted parameter C, which will be used to adjust bunch shape in such a way, that self field of the bunch is approximately opposite to external field. The value of constant in right side of Eq. (5.5) can be determined from the condition, that longitudinal bunch size is, in the first approximation, the same as for zero - current mode. Therefore, at $R(\zeta) = 0$ one of the bunch boundary is $k_z\zeta = 2\varphi_s$ and the value of constant is

$$\text{const} = 2\varphi_s \cos\varphi_s - 2 \sin\varphi_s . \qquad (5.6)$$

Substitution of Eq. (5.6) into Eq. (5.5) gives expression for expected bunch profile:

$$I_o\left(\frac{k_z R}{\gamma}\right)\sin(\varphi_s - k_z\zeta) + \sin\varphi_s - (2\varphi_s - k_z\zeta)\cos\varphi_s + C(k_z R)^2 = 0 . \qquad (5.7)$$

Fig. 3 illustrates uniformly populated bunch with boundary, Eq. (5.7). Bunch profile in real space reminds separatrix shape in longitudinal phase space. Space charge field of the bunch in longitudinal direction is essentially nonlinear and repeats (with negative sign) the RF field inside the bunch. In transverse direction the space charge forces are close to linear function of coordinate and compensate external focusing forces. Shape and space charge forces of the bunch depend on parameter C. Consider that dependencies in more details.

For a long bunch, $\beta\lambda >> R_{max}$, the Bessel function can be approximated as $I(\chi) \approx 1 + \chi^2/4$, and equation (5.7) for bunch boundary becomes:

$$R(\zeta) = \frac{\beta\lambda}{2\pi}\sqrt{\frac{(2\varphi_s - k_z\zeta)\cos\varphi_s - \sin\varphi_s - \sin(\varphi_s - k_z\zeta)}{C + \frac{1}{4\gamma^2}\sin(\varphi_s - k_z\zeta)}} . \quad (5.8)$$

Transverse bunch size, R_{max}, is determined from equation $\frac{\partial R(\zeta)}{\partial \zeta} = 0$, which has an approximate solution $\zeta(R_{max}) \approx 0$. Substitution of this value into Eq. (5.8) gives for maximum beam size:

$$R_{max} = \frac{\beta\lambda}{2\pi}\sqrt{\frac{2(\varphi_s\cos\varphi_s - \sin\varphi_s)}{C + \frac{1}{4\gamma^2}\sin\varphi_s}} . \quad (5.9)$$

Exact value of $\zeta(R_{max})$ is slightly positive and maximum value of bunch profile is shifted to the head of the bunch. The phase length of a separatrix is approximately $3\varphi_s$ and full bunch length is $l_b = \beta\lambda 3\varphi_s/(2\pi)$. The ratio of transverse to longitudinal bunch sizes for a given value of synchronous phase is therefore:

$$\frac{R_{max}}{l_b} = \frac{1}{3|\varphi_s|}\sqrt{\frac{2(\varphi_s\cos\varphi_s - \sin\varphi_s)}{C + \frac{1}{4\gamma^2}\sin\varphi_s}} . \quad (5.10)$$

Let us compare space charge potential of the bunch with that of external RF filed. Consider for simplicity a non-relativistic case. Potential of arbitrary charged distribution at the point ζ_o at the axis is:

$$U_b(\zeta_o, 0) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho\, dV}{|\vec{r}|} = \frac{1}{4\pi\varepsilon_o}\int_0^{R(\zeta)}\int_0^{2\pi}\int_{\zeta_{min}}^{\zeta_{max}} \frac{\rho\,(r, \zeta')\; r\, dr\; d\zeta'\, d\varphi}{\sqrt{r^2 + (\zeta' - \zeta_o)^2}} \quad . \qquad (5.11$$

Introduce RF phase $\psi = - k_z\zeta$ instead of longitudinal coordinate ζ. After integration in Eq. (5.11) over radius and azimuth angle, the beam potential is:

$$U_b(\psi_o, 0) = U_o \int_{\psi_{min}}^{\psi_{max}} [\sqrt{k_z^2R^2(\psi) + (\psi - \psi_o)^2} - \sqrt{(\psi - \psi_o)^2}]\, d\psi \quad . \qquad (5$$

For typical values of parameter C = 1....5, bunch profile, Eq. (5.8), can be approximated as follow:

$$(k_zR)^2 \approx \frac{1}{C}[(\psi + 2\varphi_s)\cos\varphi_s - \sin\varphi_s - \sin(\psi + \varphi_s)] \quad .$$

Substitution of Eq. (5.13) into Eq. (5.12) gives the potential at the axis point ψ_o :

$$U_b(\psi_o,0)=U_o\int_{\varphi_s}^{-2\varphi_s}[\sqrt{\psi^2+\psi(\frac{\cos\varphi_s}{C}-2\psi_o)+\frac{2\varphi_s\cos\varphi_s-\sin\varphi_s-\sin(\varphi_s+\psi)}{C}+\psi_o^2}-\sqrt{(\psi-\psi_o)^2}]d\psi. \quad (5.14)$$

Because the value of synchronous phase in RF filed is negative $\varphi_s < 0$, integration in (5.14) has to be performed in the limits of (φ_s, - $2\varphi_s$). In Fig. 4 results of space charge potential of the bunch, Eq. (5.14) are presented. Also, an inverse autophasing potential is given:

$$V(\psi) = - U_{ext}(\psi, 0) = - \frac{E}{k_z}[\sin(\psi + \varphi_s) - \psi\cos\varphi_s] \quad .$$

As seen, the values of both potentials are close to each other. Therefore, uniformly populated bunch with boundary, Eq. (5.7), compensates for "restoring" autophasing force inside the bunch, which indicates good approximation of the bunch boundary by Eq. (5.7).

Parameter C can be expressed as a function of ratio of transverse, G_t^b, and longitudinal, G_z^b, gradients of space charge forces inside the bunch

$$C = C\left(\frac{G_t^b}{G_z^b}\right) . \qquad (5.16)$$

Figs. 5, 6 illustrate dependencies, Eq. (5.16), for different values of synchronous phase and beam energies. Components of electric field of a relativistic bunch in moving frame, E_x', E_y', E_z', were calculated via numerical solution of the Poisson's equation and then the Lorentz transform was applied to get components of electric field, E_x, E_y, E_z, in laboratory system:

$$E_x = \gamma E_x^{'}, \qquad E_z = \gamma E_z^{'}, \qquad E_z = E_z^{'} \,. \tag{5.17}$$

In laboratory system transverse field was reduced by the factor of $1/\gamma^2$ due to self magnetic field of the beam:

$$F_x = E_x - v_z B_y = \frac{E_x}{\gamma^2}, \qquad F_y = E_y + v_z B_x = \frac{E_y}{\gamma^2} \,. \tag{5.18}$$

Gradients of space charge field were calculated as derivatives of space charge forces in the vicinity of synchronous phase:

$$G_t^b = \frac{\partial F_x}{\partial x} = \frac{1}{\gamma^2}\frac{\partial E_x}{\partial x}, \qquad G_z^b = \frac{\partial F_z}{\partial \zeta} = \frac{\partial E_z}{\partial \zeta} \,. \tag{5.19}$$

According to Eq. (4.21), space charge field of a stationary bunch compensates for external accelerating and focusing field within the bunch. Therefore, if space charge forces are known, the opposite field defines required external filed. Gradients of external field are calculated from Eq. (2.2) in the vicinity of synchronous phase utilizing expansion

$$\sin(\varphi_s - k_z\zeta) \approx \sin\varphi_s - (k_z\zeta)\cos\varphi_s - \frac{1}{2}(k_z\zeta)^2\sin\varphi_s \,, \quad k_z\zeta << 1 \,, \tag{5.20}$$

$$I_o\left(\frac{k_z r}{\gamma}\right) \approx 1 + \frac{1}{4}\left(\frac{k_z r}{\gamma}\right)^2 . \tag{5.21}$$

Substitution of Eqs. (5.20), (5.21) into Eq. (2.2) gives for external potential:

$$U_{ext} = G_z \frac{\zeta^2}{2} + G_t \frac{r^2}{2}\left[1 - \frac{G_z}{2\,\gamma^2 G_t}\frac{\sin(\varphi_s - k_z\zeta)}{\sin\varphi_s}\right] \approx G_z \frac{\zeta^2}{2} + G_{t,\,eff}\frac{r^2}{2} \,, \tag{5.22}$$

where G_z is a longitudinal gradient of external field

$$G_z = 2\pi \frac{E\,|\sin\varphi_s|}{\beta\lambda} \,, \tag{5.23}$$

and $G_{t,\,eff}$ is an effective transverse gradient of external field, depressed due to RF defocusing:

$$G_{t,\,eff} = G_t\left(1 - \frac{G_z}{2\,\gamma^2 G_t}\right) . \tag{5.24}$$

Taking into account Eq. (4.21), the relationships between gradients of space charge field and that of external field are

$$G_z^b = -\left(\frac{1}{1+\delta}\right) \frac{2\pi\, E\,|\sin\varphi_s|}{\beta\lambda} , \tag{5.25}$$

$$G_t^b = -\left(\frac{1}{1+\delta}\right) \left[G_t - \frac{\pi\, E\,|\sin\varphi_s|}{\gamma^2\,\beta\lambda}\right] . \tag{5.26}$$

Eqs. (5.25), (5.26) together with dependencies, presented in Figs. 5, 6, uniquely define the shape of the stationary bunch for given values of accelerating field, E, focusing gradient, G_t, synchronous phase, φ_s, wavelength, λ, and beam energy, γ.

6. MAXIMUM BEAM CURRENT

Performed study allows us to determine the maximum beam current of bunched beam. The volume of the bunch is determined by

$$V = \pi \int_{z_{min}}^{z_{max}} R^2(\zeta)\, d\zeta = \frac{\beta\lambda}{2} \int_{\varphi_s}^{-2\varphi_s} R^2(\psi)\, d\psi . \tag{6.1}$$

For a long bunch ($\beta\lambda >> R_{max}$), an approximate bunch boundary, $R^2(\psi)$, is determined by Eq. (5.13). Integration in Eq. (6.1) gives for the bunch volume:

$$V = \frac{(\beta\lambda)^3}{8\pi^2 C} \left(3\varphi_s \sin\varphi_s - \frac{9}{2}\,\varphi_s^2 \cos\varphi_s + \cos\varphi_s - \cos 2\varphi_s\right) . \tag{6.2}$$

Total charge of the bunch is $Q = \rho\cdot V$ and beam current, $I = \frac{Q}{2\pi}\,\omega$, is

$$I = \frac{I_{max}}{1+\delta} , \tag{6.3}$$

$$I_{max} = I_c \left(\frac{\beta^3\gamma^2}{16\pi^3 C}\right) \left(\frac{G_t\, q\, \lambda^2}{m\, c^2}\right) \left[3\varphi_s \sin\varphi_s - \frac{9}{2}\,\varphi_s^2 \cos\varphi_s + \cos\varphi_s - \cos 2\varphi_s\right] , \tag{6.4}$$

where I_{max} is a maximum beam current for infinitely high brightness beam. The expression in square brackets in Eq. (6.4) is close to the cubic function of the synchronous phase, φ_s^3, (see Fig. 7), which exhibits that the maximum beam current is proportional to the cube of the synchronous phase. It is in qualitative agreement with analysis, based on well-known ellipsoidal approximation to bunched beam [1, 2, 6], see Section 7.

Substitution of Eq. (4.18), into Eq. (6.3) gives explicit expression for beam current:

$$I = I_{max} \left(1 - \frac{\varepsilon_t^2}{\alpha^2}\right), \qquad (6.5)$$

where α defines normalized acceptance of the channel in presence of transverse focusing and RF field:

$$\alpha = a \sqrt{\frac{\beta^2 \gamma}{8\pi^3 BC} \left(\frac{G_t q \lambda^2}{m c^2}\right) \left[3\varphi_s \sin\varphi_s - \frac{9}{2} \varphi_s^2 \cos\varphi_s + \cos\varphi_s - \cos 2\varphi_s\right]} \; . \qquad (6.6)$$

Eq. (6.4) gives a unique expression for the beam current limit (without separate transverse and longitudinal limits) for every combination of the values of E, G_t, φ_s and λ.

Fig. 8 illustrates dynamics of proton bunched beam with maximum possible current of I_{max} = 0.2A in the field with E = 20 kV/cm, φ_s = - 57.3^o, G_t = 280 kV/cm^2, δ =0, β = 0.01788, λ = 85.7 cm. Gradients of space charge forces of the beam obtained from Eqs. (5.25), (5.26) are G_z^b = - 69 kV/cm^2, G_t^b = - 245 kV/cm^2. The ratio of gradient is G_t^b / G_t^b = 3.6, which corresponds to the bunch with parameter C = 3.5 (see Fig. 6a). The value of beam current is defined from Eq. (6.4). Beam dynamics simulations show that bunch shape is approximately kept constant, while appearance of halo around bunch indicates that attained solution is approximate.

7 APPLICABILITY OF ELLIPSOID MODEL

Let us discuss applicability of the well known approximation of the bunch by uniformly populated ellipsoid. In derivations of self-consistent solution of beam distribution resulted in Eq. (4.21), there were no constraints on external potential, therefore Eq. (4.21) is valid for arbitrary external field. In the vicinity of synchronous particle, where external forces are approximately linear functions of coordinates, external potential is given by Eq. (5.22). Substitution of Eq. (5.22) into Eq. (4.21) gives for potential of stationary bunch:

$$U_b = -\frac{\rho}{2\varepsilon_o G_t}\left(G_z \frac{\zeta^2}{2} + \frac{G_{t,\,eff}}{2} r^2\right), \tag{7.1}$$

where ρ is given by Eq. (5.3). Potential, Eq. (7.1), corresponds to uniformly populated ellipsoid. In a moving system of coordinates, potential of ellipsoid, U_b', with space charge density $\rho' = \rho/\gamma$ is written as

$$U_b' = -\frac{\rho'}{2\varepsilon_o}\left(M\zeta'^2 + \frac{1-M}{2} r^2\right), \tag{7.2}$$

where $\zeta' = \zeta\gamma$ is a longitudinal deviation from the center of ellipsoid and M is a function of ratio of ellipsoid semi-axes:

$$M(R, \gamma l) = \frac{R^2\gamma l}{2}\int_o^\infty \frac{ds}{(R^2 + s)\,(\gamma^2 l^2 + s)^{3/2}} . \tag{7.3}$$

After transformation to laboratory system, the beam potential, $U_b = \gamma U_b'$, is

$$U_b = -\frac{\rho}{2\varepsilon_o}\left[M\gamma^2\zeta^2 + \frac{1-M}{2} r^2\right]. \tag{7.4}$$

Comparison of Eq. (7.1) and Eq. (7.4) gives for coefficient M

$$M(R, \gamma l) = \frac{G_z}{2\,\gamma^2 G_t} . \tag{7.5}$$

Coefficient M, Eq. (7.5), and space charge density ρ, Eq. (5.3), define family of ellipsoidal bunches with the same ratio of semi-axes R/l, which are in equilibrium with external field. Taking into account, that volume of ellipsoid is $V = (4/3)\pi R^2 l$, the maximum bunched beam current, $I_{max} = \rho V\omega/(2\pi)$, which can be carried by an ellipsoid is

$$I_{max} = I_c \frac{2}{3}\gamma^2\left(\frac{R^2 l}{\lambda^3}\right)\left(\frac{G_t\, q\, \lambda^2}{m\, c^2}\right). \tag{7.6}$$

Since bunch with current, Eq. (7.6), completely cancels external field, expression (7.6) gives both transverse and longitudinal current limit. Let us substitute gradient of focusing field, G_t, by the value of zero-current phase advance, σ_o, of betatron oscillations per period $S = N\beta\lambda$ of a pure focusing structure (without RF field):

$$\sigma_o = \sqrt{\frac{q\,G_t}{m\gamma}}\,\frac{S}{\beta c}\ . \tag{7.7}$$

In presence of RF field effective focusing gradient is $G_{t,\,eff} = G_t(1 - M)$, see Eqs. (5.24), (7.5). Therefore, zero-current phase advance per period, $\sigma_{o,t}$, including both the focusing and RF defocusing term is defined by:

$$\sigma_{o,t}^2 = \sigma_o^2\,(1 - M)\,. \tag{7.8}$$

Phase width of bunch, which contains most of the particles, can be approximately taken as $2\varphi_s$ and, therefore, half of bunch length is

$$l = \beta\lambda\varphi_s/(2\pi)\,. \tag{7.9}$$

Substitution of expressions (7.7) - (7.9) into Eq. (7.6) gives for the current limit

$$I_{max} = \frac{4}{3}\frac{mc^2}{Z_o\,q}\,\beta\gamma^3\,\frac{\varphi_s\,\sigma_{ot}^2}{(1 - M)\,N^2}\left(\frac{R}{\lambda}\right)^2\ , \tag{7.10}$$

where $Z_o = (c\varepsilon_o)^{-1} = 376.73\Omega$ is the impedance of the free space. Expression (7.10) is the well known transverse current limit [6, Eq. (9.55)]. Let us show that Eq. (7.6) gives also longitudinal current limit. Substitution of parameter M, Eq. (7.5), and amplitude of accelerating field E, Eq. (5.23), into Eq. (7.6) gives for current limit:

$$I_{max} = \frac{8\pi^2}{3Z_o}\,\frac{E\,\sin\varphi_s}{\beta M}\,\frac{R^2 l}{\lambda^2}\ , \tag{7.11}$$

which is well-known expression for longitudinal current limit in RF filed [6, Eq(9.52)]. Usually parameter M can be approximated as $M \approx R/(3\gamma l)$. For that approximation the current limit, Eq. (7.11), is:

$$I_{max} = \frac{2\,\beta\gamma}{Z_o}\,E\,\varphi_s^2\,|\sin\varphi_s|\,R\ . \tag{7.12}$$

For small absolute values of synchronous phase one can assume $|\sin\varphi_s| \approx |\varphi_s|$, and the current limit, Eq. (7.12), is proportional to the cube of synchronous phase [1, 2, 6], which is consistent with derivations of Section 6.

Performed analysis shows that approximation of bunched beam by uniformly populated ellipsoid is valid for small bunches, $R << \beta_s\lambda$, $l << \beta_s\lambda$, while more general analysis results in bunch shape, described by Eq. (5.7).

8 SUPPRESSION OF BEAM EMITTANCE GROWTH IN RF LINAC DUE TO RF DEFOCUSING

In RF linac bunched beam is formed from injected continuous beam as a result of longitudinal particle oscillations around synchronous particle. This process is accompanied with emittance growth and halo formation which was a subject of detailed study of many papers (see Refs. [1], [2], [4], [6] - [10] and cited references there).

One of the main reason for beam emittance growth is a dependence of transverse oscillation frequency on phase of particle in RF field (RF defocusing). From Hamiltonian, Eq. (4.27), the equation of transverse particle motion is

$$\frac{d^2x}{dt^2} + \frac{\delta}{1+\delta}(\Omega_r^2 - \frac{1}{2}\Omega^2\frac{\sin\varphi}{\sin\varphi_s})\, x = 0\,, \qquad (8.1)$$

where Ω_r is a transverse oscillation frequency without RF field

$$\Omega_r^2 = \frac{q\, G_t}{m\gamma}\,, \qquad (8.2)$$

and Ω is a longitudinal oscillation frequency:

$$\Omega^2 = \omega^2\frac{qE\lambda}{mc^2}\frac{\sin\varphi_s}{2\pi\beta\gamma^3}\,. \qquad (8.3)$$

During formation of bunches from continuos beam, particles perform longitudinal oscillations with large amplitude, which affect transverse oscillations. In Fig. 9 the phase space trajectory of the particle in RF accelerating field is presented. It is clear, that phase space ellipse in phase space is deformed due to RF defocusing. The effective emittance of the beam after many particle oscillations in RF field can be significantly larger than the initial beam emittance. Qualitative analysis of beam emittance growth for zero-current mode was done in Ref. [1]:

$$\frac{\varepsilon}{\varepsilon_o} = 1 + \frac{\Phi \, \mathrm{ctg}\varphi_s}{4\frac{\Omega_{r,s}^2}{\Omega^2} - 1}, \tag{8.5}$$

where Φ is an amplitude of phase oscillations and $\Omega_{r,s}$ is a transverse oscillation frequency of synchronous particle:

$$\Omega_{r,s}^2 = \Omega_r^2 - \frac{\Omega^2}{2}. \tag{8.6}$$

From Eq. (8.5) it follows that emittance growth due to RF defocusing is most essential at the stage of beam bunching, where amplitude of phase oscillations is large, and for small values of ratio $\Omega_{r,s}/\Omega$. According to Eq. (4.27), in space-charge -dominated regime, transverse and longitudinal oscillation frequencies are depressed in the same proportion, therefore dependence of emittance growth on ratio $\Omega_{r,s}/\Omega$ has to be qualitatively the same as for zero-current mode.

This effect was studied numerically for drift tube linear accelerator with solenoid focusing. Radial oscillation frequency in absence of RF and space charge forces is a the Larmor frequency, $\omega_L = qB/(2m\gamma)$, where B is the magnetic field of solenoid. The lower value of magnetic field is limited by transverse stability constraint $\omega_L > \Omega$ [1]. With increasing of the ratio of ω_L/Ω, the dependence of transverse oscillation frequency on RF filed is damped and emittance growth is expected to be suppressed.

In Figs. 10-13 results of beam dynamics study in RF proton linac for energy 3 MeV and beam current of 250 mA are presented. Drift tube accelerator structure consists of prebuncher, buncher and acceleration section. Synchronous phase is changing monotonously from 90^o to 30^o. Accelerating gradient increased gradually from zero to the final value of 2.5 Mev/m. Initial beam with KV distribution, injection energy of 150 keV and normalized emittance of 0.12 π cm mrad was chosen to be matched with constant solenoid field. Transmission efficiency obtained in simulation was 90% .

From results of simulation it is clear that emittance growth is occurred mainly at the stage of beam bunching, where amplitude of phase oscillation is large. Emittance growth is saturated after beam is bunched. For small value of magnetic field, ω_L/Ω= 1.25, close to transverse stability limit, the strong emittance growth (up to 100%) was observed. With increasing of magnetic field, the emittance growth was seriously reduced and finally can be made close to zero, see Fig. 10. It is also accompanied with suppression of halo formation in phase space (see Fig. 11) and in real space (see Fig. 12). Final beam profile (see Fig. 13) resembles that, obtained in self-consistent analysis of Section 5. Performed study indicates that emittance growth due to RF defocusing can be controlled by an appropriate choice of focusing gradient with respect to RF field.

5 CONCLUSIONS

An approximate self-consistent solution for a bunched beam in an uniform focusing channel with applied RF acceleration field was obtained. Analytical derivations were performed in the limit of a high brightness beam, when space charge forces are dominated. Nonlinear equation for stationary beam profile as well as expression for space charge limited beam current are derived. Applicability of ellipsoidal model to bunched beam in RF field is discussed.

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APPENDIX 1. Dimensionless beam brightness

Consider KV envelope equation for round beam [1]:

$$R'' - \frac{\epsilon_t^2}{R^3} + k(z)\,R - \frac{2\,I}{I_c\,\beta^3\gamma^3 R} = 0\,. \tag{A.1}$$

Equation (A.1) contains two defocusing terms: one is proportional to square of beam emittance and another one is proportional to beam current. Ratio of that two terms gives estimation, which factor is dominated in beam transport:

$$b = \frac{2}{(\beta\gamma)^3}\,\frac{I}{I_c}\,\frac{R^2}{\epsilon_t^2} = \frac{2}{(\beta\gamma)}\,\frac{I}{I_c}\,\frac{R^2}{\varepsilon_t^2}\,. \tag{A.2}$$

Transport with b >> 1 corresponds to space-charge dominated regime, while b<<1 corresponds to emittance- dominated regime. The value of b is proportional to the ratio of beam brightness, I / ε^2, to normalized value, I_c / R^2. Therefore, parameter b has a meaning of a dimensionless beam brightness. Additional factor of $2 / (\beta\gamma)$ indicates that significance of the space charge forces drops with increasing of beam energy.

APPENDIX 2. Space charge field of the train of cylindrical bunches.

Consider train of uniformly populated cylindrical bunches of length $2l$, radius R and space charge density ρ_o (see Fig. 14). Space charge potential of the beam obeys Poisson's equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_b}{\partial r}\right) + \frac{\partial^2 U_b}{\gamma^2 \partial \zeta^2} = -\frac{1}{\varepsilon_o}\rho\,(r, \zeta)\,. \tag{A.3}$$

Let us expand both space charge density and unknown potential by Fourier-Bessel series:

$$\rho(r, \zeta) = \sum_{n=o}^{\infty}\sum_{m=1}^{\infty} \rho_{nm}\, J_o(\upsilon_{om}\frac{r}{a}) \cos\left(\frac{2\pi n\zeta}{L}\right), \tag{A.4}$$

$$U_b(r, \zeta) = \sum_{n=o}^{\infty}\sum_{m=1}^{\infty} U_{nm}\, J_o(\upsilon_{om}\frac{r}{a}) \cos\left(\frac{2\pi n\zeta}{L}\right). \tag{A.5}$$

Expansions, Eqs. (A.4), (A.5) obey Dirichlet boundary condition at the ideal cylindrical surface of the tube r = a and periodic condition in longitudinal direction $U_b(r, \zeta) = U_b(r, \zeta + L)$. Coefficients ρ_{nm} are obtained by multiplying of Eq. (A.4) by $J_o(\upsilon_{om'}\frac{r}{a})\cos\left(\frac{2\pi n'\zeta}{L}\right)$ r dr dζ and integration in the limits of (0, a), (-L/2, L/2):

$$\rho_{nm} = \rho_o \frac{4}{\upsilon_{om}}\left(\frac{R}{a}\right)\left(\frac{2l}{L}\right)\left[\frac{J_1(\upsilon_{om}\frac{R}{a})}{J_1^2(\upsilon_{om})}\right]\left[\frac{\sin\left(\frac{2\pi n l}{L}\right)}{\left(\frac{2\pi n l}{L}\right)}\right]. \tag{A.6}$$

In derivation of Eq. (A.6) the condition of orthogonality of the Bessel function was used:

$$\int_o^a J_o(\upsilon_{om'}\frac{r}{a})\, J_o(\upsilon_{om}\frac{r}{a})\, r\, dr = \begin{cases} 0, & m \neq m' \\ \frac{a^2}{2} J_1^2(\upsilon_{om}), & m = m' \end{cases}. \tag{A.7}$$

Substitution of Eqs. (A.4), (A.5) into Poisson's equation gives the following algebraic relationship between coefficients series:

$$U_{nm} = \frac{\rho_{nm}}{\varepsilon_o \left[\left(\frac{2\pi n}{\gamma L}\right)^2 + \left(\frac{\upsilon_{om}}{a}\right)^2 \right]} \quad . \tag{A.8}$$

Therefore, space charge field of the train of the bunches is defined as follow:

$$U_b(r, \zeta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4\,\rho_o}{\varepsilon_o\, \upsilon_{om} \left[\left(\frac{2\pi n}{\gamma L}\right)^2 + \left(\frac{\upsilon_{om}}{a}\right)^2 \right]} \left(\frac{R}{a}\right)\left(\frac{2l}{L}\right) \frac{J_1(\upsilon_{om}\,\frac{R}{a})}{J_1^2(\upsilon_{om})} \frac{\sin\left(\frac{2\pi n l}{L}\right)}{\left(\frac{2\pi n l}{L}\right)} J_o(\upsilon_{om}\,\frac{r}{a}) \cos\left(\frac{2\pi n \zeta}{L}\right). \tag{A.9}$$

Figure captions

Fig. 1. Results of the numerical solution of Eq. (4.26) for a self-consistent potential of a high brightness beam, $\gamma = 1$: (a) $\delta = 0.3$; (b) $\delta = 0.2$; (c) $\delta = 0.1$.

Fig. 2. Separatrix of longitudinal particle motion: a) low brightness beam, b<<1, b) high brightness beam, b>>1.

Fig. 3. Approximate stationary self-consistent particle distribution in RF field, $\varphi_s = -60^o$. C = 3.8: a) RF field, b) particle distribution, c) space charge field of the beam, d) resulting field.

Fig. 4. Comparison of potential functions of beam and RF filed: (dotted line) space charge potential of bunched beam distribution at the axis, $\varphi_s = -60^o$, C = 3.8; (solid line) inverse external potential at the axis, $-U_{ext}(\psi, 0)$.

Fig. 5. Coefficient C in bunch shape for $\varphi_s = -30^0$ as a function of ratio of transverse and longitudinal gradients of space charge field of the beam: a) $\gamma = 1$, b) $\gamma = 3$, c) $\gamma = 6$.

Fig. 6. Coefficient C in bunch shape for $\varphi_s = -60^0$ as a function of ratio of transverse and longitudinal gradients of space charge field of the beam: a) $\gamma = 1$, b) $\gamma = 3$, c) $\gamma = 6$.

Fig. 7. Function $f(\varphi_s) = 3\varphi_s \sin\varphi_s - \frac{9}{2}\varphi_s^2 \cos\varphi_s + \cos\varphi_s - \cos 2\varphi_s$ in maximum beam current.

Fig. 8. Dynamics of bunched proton beam with current I = 0.2 A in the field with parameters E = 20 kV/cm, $G_t = 280$ kV/cm^2, $\lambda = 85.7$ cm, $\varphi_s = -60^o$: a) tc/λ, b) $tc/\lambda = 10$, c) $tc/\lambda = 30$.

Fig. 9. Phase space trajectory of particle in standing wave RF accelerator.

Fig. 10. Beam emittance growth (up) and phase trajectories of particles (bottom) in RF accelerator: 1) $\omega_L/\Omega = 1.25$, 2) $\omega_L/\Omega = 1.75$, 3) $\omega_L/\Omega = 2.5$, 4) $\omega_L/\Omega = 3.5$.

Fig. 11. (Left column) initial beam emittance, (right column) final beam emittance in RF accelerator: 1) $\omega_L/\Omega = 1.25$, 2) $\omega_L/\Omega = 1.75$, 3) $\omega_L/\Omega = 2.5$, 4) $\omega_L/\Omega = 3.5$.

Fig. 12. (Left column) initial x-y beam distribution, (right column) final x-y beam distribution in RF accelerator: 1) $\omega_L/\Omega = 1.25$, 2) $\omega_L/\Omega = 1.75$, 3) $\omega_L/\Omega = 2.5$, 4) $\omega_L/\Omega = 3.5$.

Fig. 13. (Left column) initial beam profile, (right column) final beam profile in RF accelerator: 1) $\omega_L/\Omega = 1.25$, 2) $\omega_L/\Omega = 1.75$, 3) $\omega_L/\Omega = 2.5$, 4) $\omega_L/\Omega = 3.5$.

Fig. 14. On space charge potential calculation of the train of cylindrical bunches.

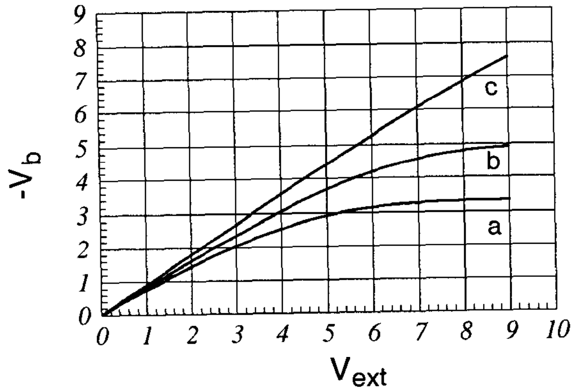
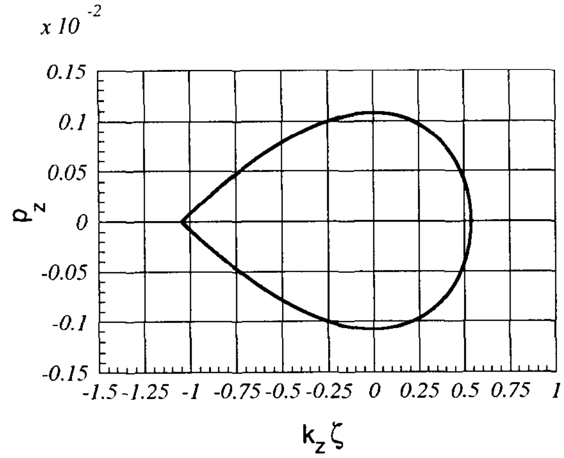


Fig. 1.

a)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0.15
0.1
0.05
0
-0.05
-0.1
-0.15
p z
-1.5 -1.25 -1 -0.75 -0.5 -0.25 0 0.25 0.5 0.75 1
k z ζ

b)

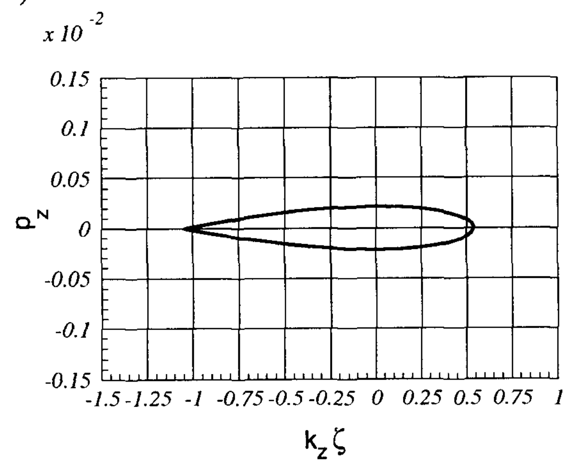

x 10 -2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
p z
-1.5 -1.25 -1 -0.75 -0.5 -0.25 0 0.25 0.5 0.75 1
k z ζ

Fig. 2.

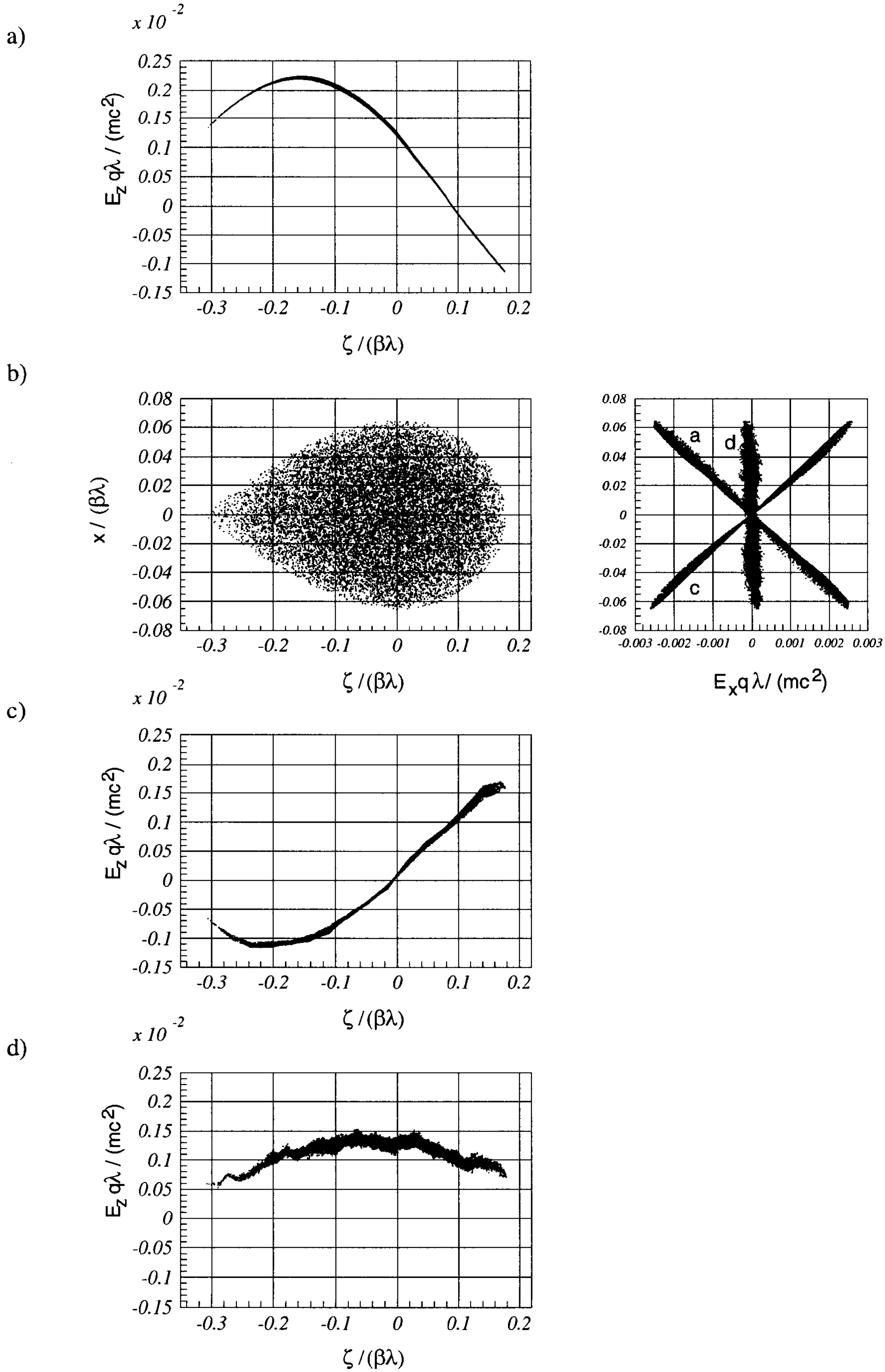

a)
x 10 -2
E_z qλ / (mc²)
ζ / (βλ)
b)
x / (βλ)
ζ / (βλ)
a
d
c
E_x q λ / (mc²)
x / (βλ)
c)
x 10 -2
E_z qλ / (mc²)
ζ / (βλ)
d)
x 10 -2
E_z qλ / (mc²)
ζ / (βλ)

Fig. 3

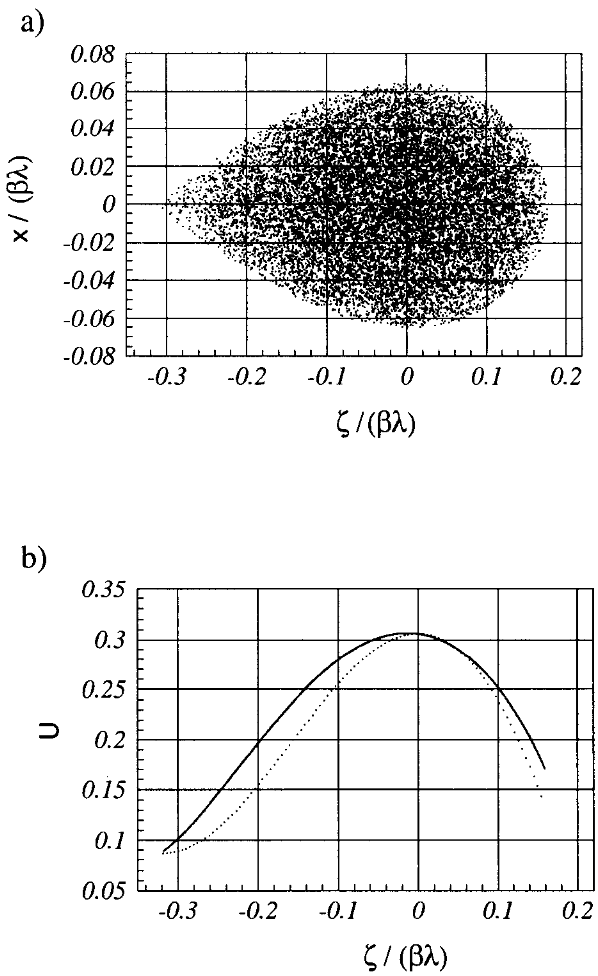

a)
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
x / (βλ)
-0.3
-0.2
-0.1
0
0.1
0.2
ζ / (βλ)
b)
0.35
0.3
0.25
0.2
0.15
0.1
0.05
U
-0.3
-0.2
-0.1
0
0.1
0.2
ζ / (βλ)

Fig. 4.

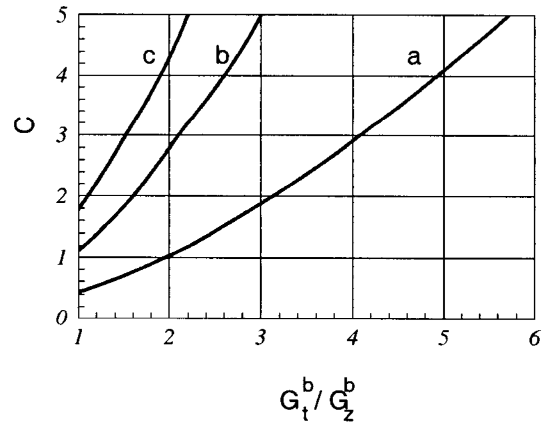


Fig. 5.

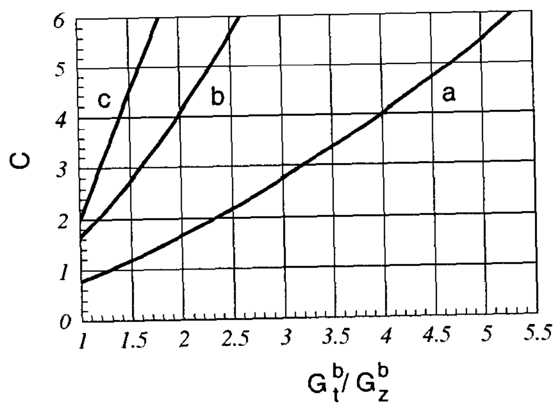


Fig. 6.

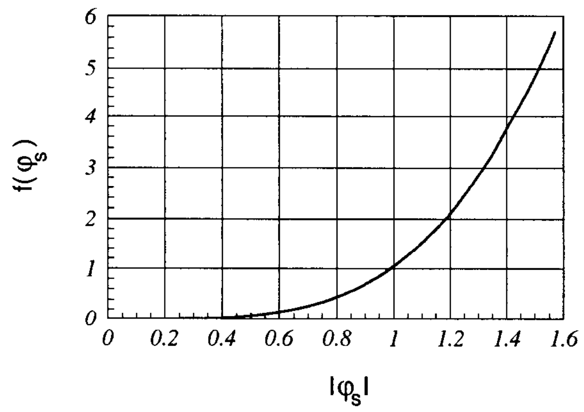


Fig. 7.

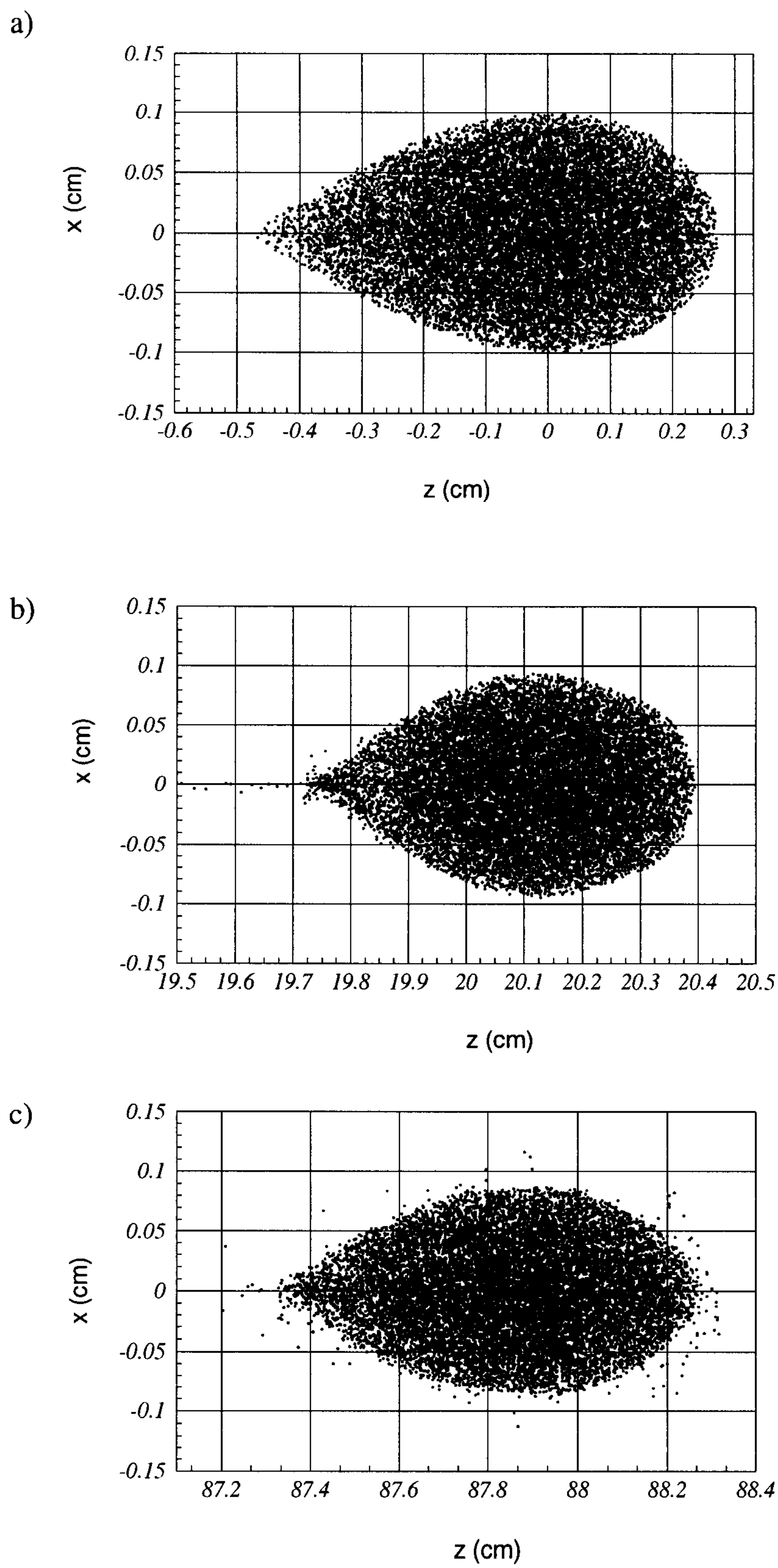

a)
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
x (cm)
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
z (cm)
b)
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
x (cm)
19.5
19.6
19.7
19.8
19.9
20
20.1
20.2
20.3
20.4
20.5
z (cm)
c)
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
x (cm)
87.2
87.4
87.6
87.8
88
88.2
88.4
z (cm)

Fig.8.

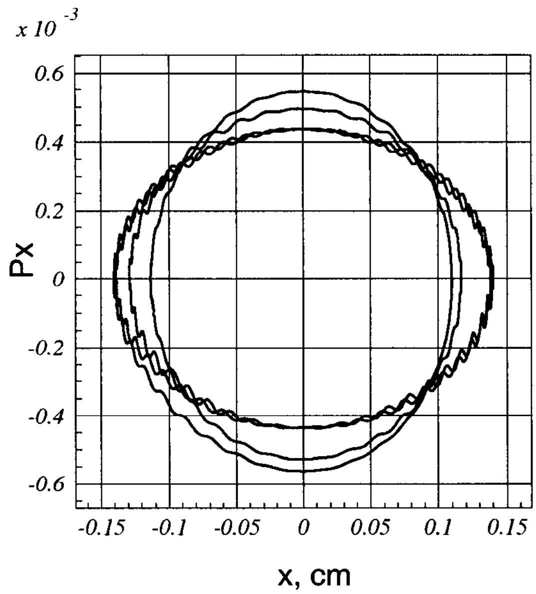


Fig. 9.

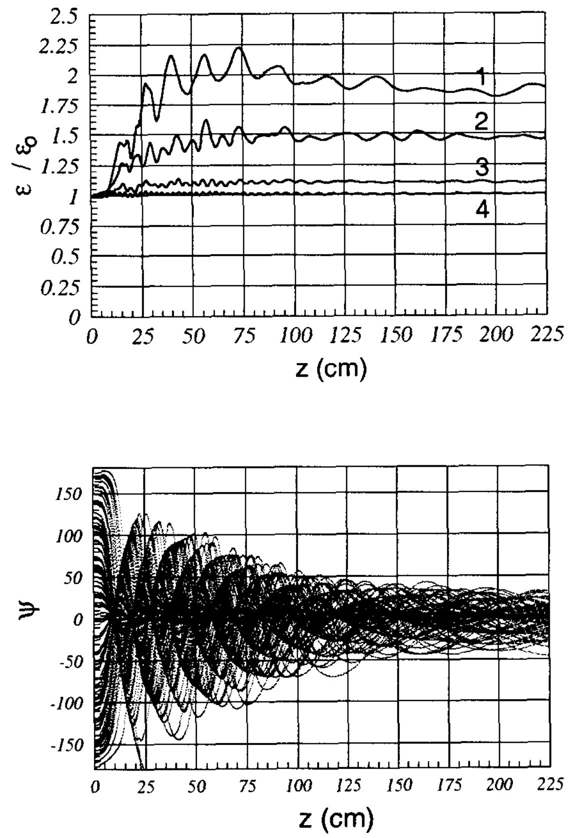
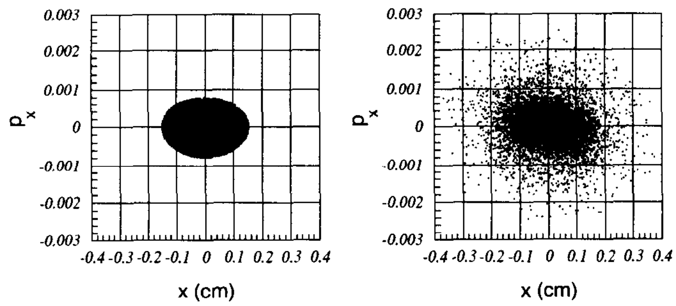
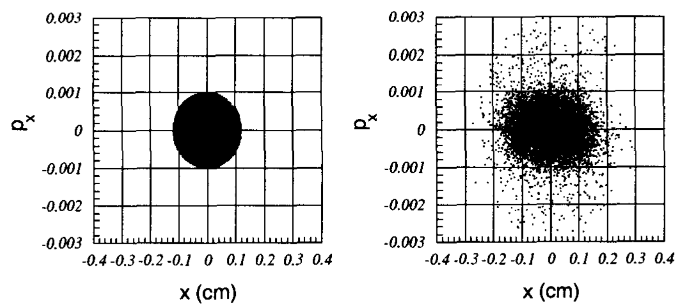

2.5
2.25
2
1.75
1.5
1.25
1
0.75
0.5
0.25
0
$\varepsilon / \varepsilon_0$
1
2
3
4
0 25 50 75 100 125 150 175 200 225
z (cm)
150
100
50
0
-50
-100
-150
ψ
0 25 50 75 100 125 150 175 200 225
z (cm)

Fig. 10.

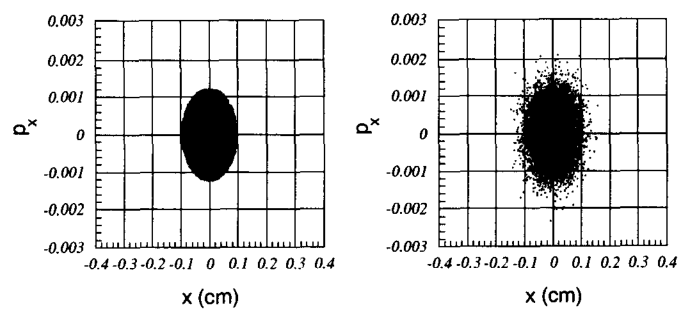
a)



b)



c)



d)

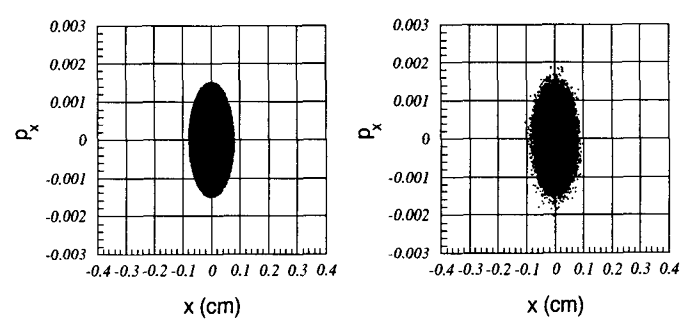


Fig. 11.

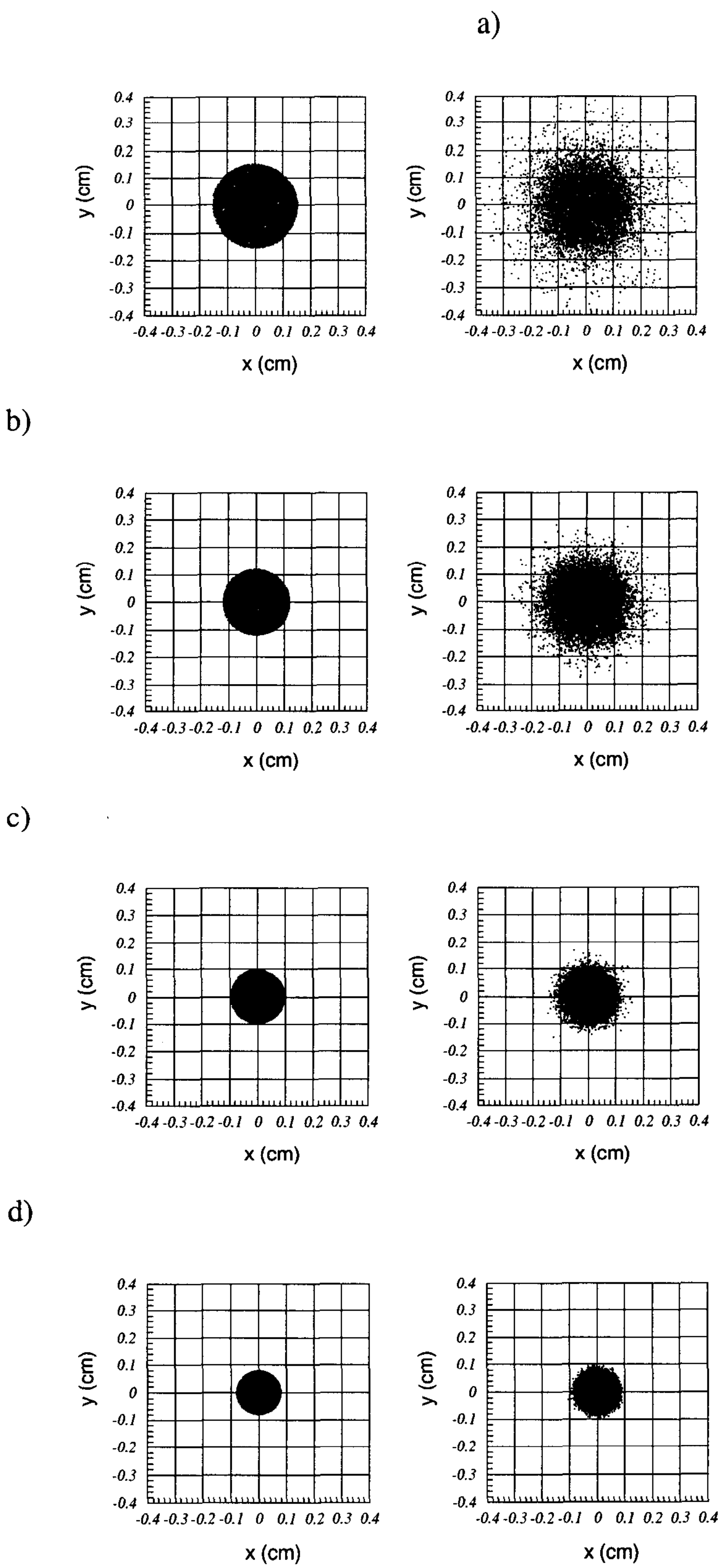


Fig. 12.

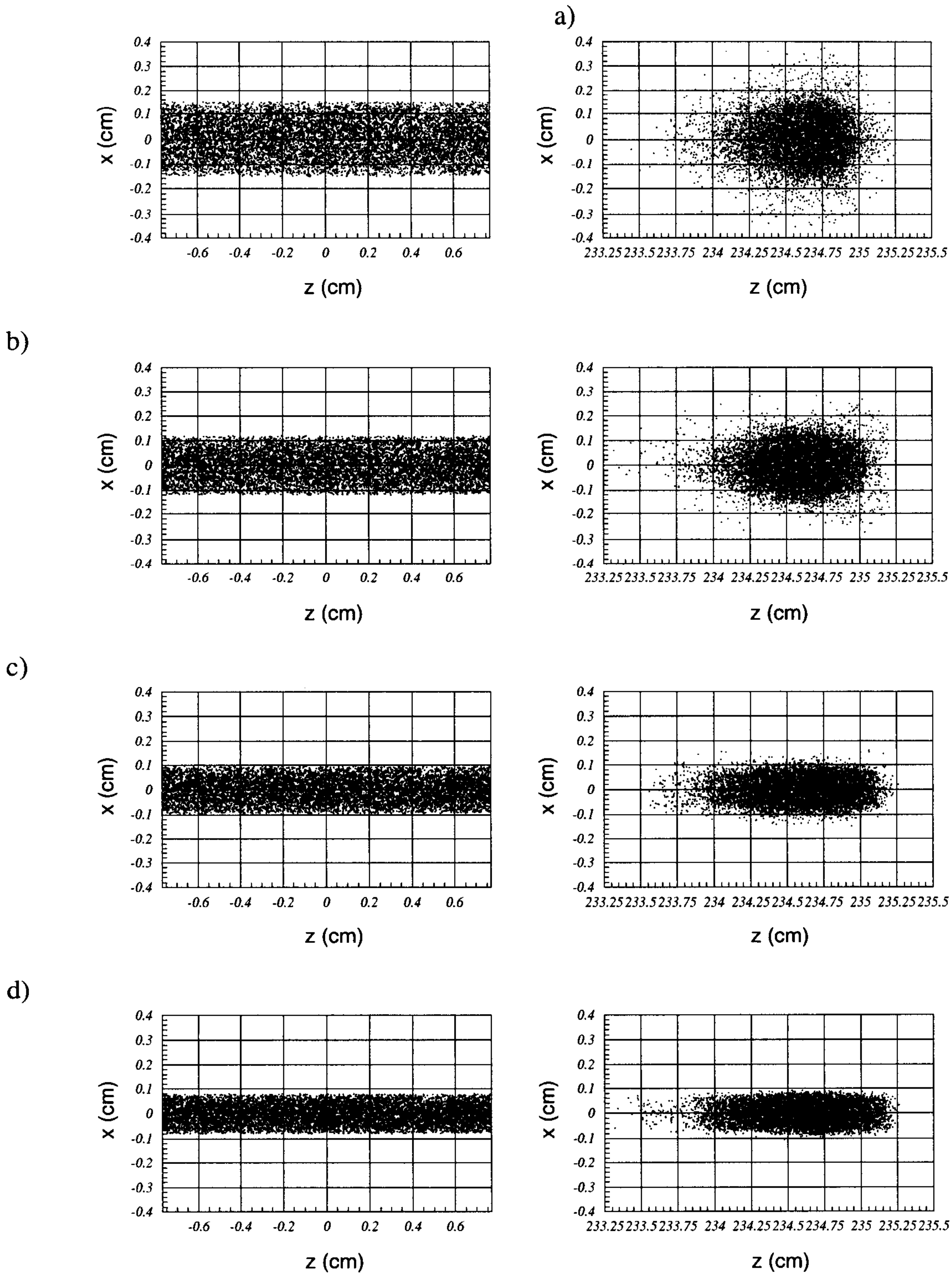

a)
b)
c)
d)
x (cm)
z (cm)
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
233.25
233.5
233.75
234
234.25
234.5
234.75
235
235.25
235.5

Fig. 13.

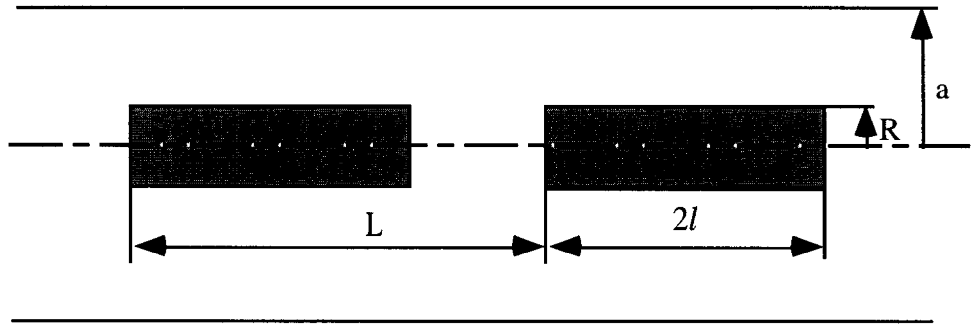


Fig. 14.