SLAC–PUB–8764
January 2001

Single-Particle Quantum Dynamics in a Magnetic Lattice *

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Abstract

We study the quantum dynamics of a spinless charged-particle propagating through a magnetic lattice in a transport line or storage ring. Starting from the Klein-Gordon equation and by applying the paraxial approximation, we derive a Schrödinger-like equation for the betatron motion. A suitable unitary transformation reduces the problem to that of a simple harmonic oscillator. As a result we are able to find an explicit expression for the particle wavefunction.

Contributed to 18th Advanced ICFA Beam Dynamics Workshop on
'Quantum Aspects of Beam Physics'
Capri, Italy, October 15-20, 2000

*Work supported by Department of Energy contract DE–AC03–76SF00515.

1 Introduction

Continuing progress in beam cooling techniques could lead in the future to regimes in which quantum effects will start to become important. At that point a fully quantum mechanical description of the beam dynamics will be required[1]. As a contribution to that description, in this paper we lay out a framework to compute the wavefunction for a single charged particle confined in a magnetic lattice transport line or storage ring. The expression we find may be useful as a basis to calculate transition rates in processes like synchrotron radiation emission[2] or intrabeam scattering as well as to assess the limitations to a machine performance caused by diffraction phenomena[3].

In our analysis we neglect spin effects so that the particle wavefunction is properly described by the Klein-Gordon (KG) equation. The KG equation can be solved in the paraxial approximation by exploiting the fact that in an accelerator the particle momentum is mostly longitudinal. The problem amounts to studying a non-relativistic quantum harmonic oscillator with a time-dependent restoring force – 'time' in this context is the location of the particle along the lattice. The solution can be found in the literature[4] and has already been applied in the field of quantum optics and ion traps. Here we present a method of solving the problem that involves a language more familiar to the accelerator physicist and emphasizes the correspondence with the classical motion. Previous work in this area using different methods includes that of Jagannathan and Kahn[5]. The methods employed here are more similar to those introduced by Fedele et al.[6] in their 'quantum-like' beam models.

In Sec. 2,3,4 we treat a particle dynamics in a straight channel. In Sec. 5, we discuss the coherent-state solutions and finally in Sec. 6,7 we extend the results to circular machines.

2 The Reduced Klein-Gordon Equation for a Particle in a Straight Transport Line

Neglecting spin effects a relativistic quantum particle can be described using a wavefunction that satisfies the KG equation. In general, if the charged particle is coupled to an external static magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ the KG equation reads

$$[\hat{E}^2/c^2 - m^2c^2 - (\hat{\boldsymbol{p}} - e\boldsymbol{A})^2]\psi(t, \boldsymbol{x}) = 0, \tag{1}$$

with the operators $\hat{E}$ and $\hat{\boldsymbol{p}}$ defined as usual as $\hat{E} = i\hbar\partial_t$ and $\hat{\boldsymbol{p}} = -i\hbar\nabla$.

In a straight transport line consisting of quads and drifts with the particle traveling along z one can choose a gauge for which the vector potential has the form $A_x = A_y = 0$ and $A_z = -(p_0/e)k(z)(x^2 - y^2)/2$ with p_0 being the design particle momentum and $k(z)$ the focusing function. Such a vector potential is consistent with Maxwell's equations through second order terms. In the absence of external focusing a solution of (1) representing a particle propagating along the $z-$axis with the design energy E_0 and momentum p_0 is given by the wavefunction

$$\psi = \psi_0 e^{i(p_0 z - E_0 t)/\hbar}, \tag{2}$$

where ψ_0 is a normalization constant. As the interaction with the confining magnetic field is turned on it is natural to look for solutions of (1) in the form

$$\psi = \tilde{\psi}(x, y, z)e^{i(p_0 z - E_0 t)/\hbar}. \tag{3}$$

We can expect that the $z-$dependence in $\tilde{\psi}(x, y, z)$ results in variations taking place over distances of the order of the betatron wavelength λ_β. This defines the long-length scale of our problem, which should be compared with the short-length scale given by the De Broglie wavelength $\lambda_p = h/p_0$. After substituting (3) into (1), we use the fact $\lambda_\beta \gg \lambda_p$ to neglect $\partial_z^2\tilde{\psi}$ compared to $\partial_z\tilde{\psi}/\lambda_p$ and $(x^2 - y^2)\partial_z\tilde{\psi}$ compared to $(x^2 - y^2)\tilde{\psi}/\lambda_p$. Moreover, we can neglect the term containing $\partial_z k(z)$ because the distance over which $k(z)$ varies substantially – the magnet fringe field region – is also much longer than the De Broglie wavelength. Finally, we disregard terms more than quadratic in x and y because we are only interested in the linear approximation of the transverse dynamics. As a result we obtain the following Schrödinger-like equation for the amplitude $\tilde{\psi}(x, y, z)$:

$$i\hbar\frac{\partial\tilde{\psi}}{\partial z} = \left(-\frac{\hbar^2}{2p_0}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2p_0}\frac{\partial^2}{\partial y^2} + p_0\frac{k(z)}{2}(x^2 - y^2)\right)\tilde{\psi}, \tag{4}$$

where z is now interpreted as the independent 'time-like' variable. From Eq. (4) we can write off the effective Hamiltonian $\hat{\mathcal{H}}$ for the system

$$\hat{\mathcal{H}} = \frac{\hat{p}_x^2}{2p_0} + \frac{\hat{p}_y^2}{2p_0} + p_0\frac{k(z)}{2}(x^2 - y^2). \tag{5}$$

3 The Classical Motion

The classical Hamiltonian $\mathcal{H}$ corresponding to (5) leads to the Hill equations $x'' + k(z)x = 0$, and $y'' - k(z)y = 0$, [†] the solutions of which, *i.e.* $x = \sqrt{\epsilon_x\beta_x(z)}\cos[\varphi_x(z) + \varphi_{xo}]$ and $y = \sqrt{\epsilon_y\beta_y(z)}\cos[\varphi_y(z) + \varphi_{yo}]$, can be written in terms of the Courant-Snyder betatron functions $\beta_{x,y}(z)$ and phase functions $\varphi_{x,y}(z)$ defined by $\varphi'_{x,y} = 1/\beta_{x,y}$. In turn, the betatron functions $\beta_{x,y}$ are solutions of

$$\beta''_{x,y} - \frac{\beta'^2_{x,y}}{2\beta_{x,y}} \pm 2k(z)\beta_{x,y} - \frac{2}{\beta_{x,y}} = 0, \tag{6}$$

where the $+$ sign in front of the focusing function applies to β_x and the $-$ sign to β_y. The solutions are determined upon specification of the appropriate initial or boundary conditions. We know from the accelerator theory literature [7] that the betatron functions can be used to define canonical transformations that cast the original Hamiltonian into a simpler form. We also know that canonical transformations correspond in quantum mechanics to unitary transformations [8]. Therefore, we can use such a correspondence to build a suitable unitary operator that turns the quantum Hamiltonian into a simpler form as well. We desire a

[†]The prime $'$ means differentiation with respect to z.

Hamiltonian that has the $z-$dependence fully factored in order to simplify the solution of the corresponding KG equation.

Consider the classical case first. For simplicity we will focus only on the horizontal motion; extension to the vertical plane is trivial. The canonical transformation[7] that produces the desired Hamiltonian can be decomposed into a linear momentum kick that leaves x unchanged followed by a scaling. In particular the first transformation $(x, p_x) \to (x_1, p_{x1})$ is given by $x_1 = x$, and $p_{x1} = p_x - p_0 x \beta_x'(z)/2\beta_x(z)$, and has generating function $F_2(x, p_{x1}, z) = x p_{x1} + p_0 x^2 \beta_x'(z)/4\beta_x(z)$. The transformed Hamiltonian $\mathcal{H}_1$ in the new variables is

$$\mathcal{H}_1 = \mathcal{H} + \frac{\partial F_2}{\partial z} = \frac{{p^2}_{x1}}{2p_0} + \frac{\beta_x'}{2\beta_x} p_{x1} x_1 + \frac{p_0}{2\beta_x^2} x_1^2. \tag{7}$$

The scaling $x_2 = x_1/\sqrt{\beta_x}$ and $p_{x2} = p_{x1}\sqrt{\beta_x}$, then removes the cross term and factors out the $z-$dependence in the Hamiltonian at same time. Such a transformation has generating function $F_2(x_1, p_{x2}) = x_1 p_{x2}/\sqrt{\beta_x}$. The resulting Hamiltonian reads

$$\mathcal{H}_2 = \mathcal{H}_1 + \frac{\partial F_2}{\partial z} = \frac{1}{\beta_x(z)} \left(\frac{p_{x2}^2}{2p_0} + \frac{p_0}{2} x_2^2 \right). \tag{8}$$

4 The Quantum Motion

First, let us recall how the quantum Hamiltonian transforms under unitary transformations. If the abstract vector $|\psi\rangle$ satisfies the Schrödinger equation $i\hbar\partial_z|\psi\rangle = \mathcal{H}|\psi\rangle$ the ket $|\psi'\rangle = U^{-1}|\psi\rangle$ transformed under unitary operator U^{-1} satisfies the Schrödinger equation $i\hbar\partial_z|\psi'\rangle = \mathcal{H}'|\psi'\rangle$, with the Hamiltonian $\mathcal{H}'$ given by

$$\mathcal{H}' = U^{-1}\mathcal{H}U - i\hbar U^{-1}\frac{\partial U}{\partial z}. \tag{9}$$

We are now ready to write the quantum equivalent of the canonical transformation introduced in the previous Section. The unitary operator U_1 generating the momentum kick is defined by $U_1^{-1}\hat{x}U_1 = \hat{x}$ and $U_1^{-1}\hat{p}_x U_1 = \hat{p}_x + p_0\beta_x'\hat{x}/(2\beta_x)$, where we have used ˆ to denote the quantum observables. Provided that $\beta_x(z)$ obeys Eq. (6), as in the classical case, it can be easily verified that

$$U_1 = \exp\left(i\frac{p_0}{\hbar}\frac{\beta_x'}{4\beta_x}\hat{x}^2 \right), \tag{10}$$

leads to the intermediate Hamiltonian

$$\hat{\mathcal{H}}_1 = \frac{\hat{p}_x^2}{2p_0} + \frac{\beta_x'}{4\beta_x}(\hat{p}_x\hat{x} + \hat{x}\hat{p}_x) + \frac{p_0}{2\beta_x^2}\hat{x}^2. \tag{11}$$

In turn, the scaling $U_2^{-1}\hat{x}U_2 = \hat{x}\sqrt{\beta_x}$ and $U_2^{-1}\hat{p}_x U_2 = \hat{p}_x/\sqrt{\beta_x}$ defines the unitary operator

$$U_2 = \exp\left(-\frac{i}{4\hbar}\log(\beta_x)(\hat{x}\hat{p}_x + \hat{p}_x\hat{x}) \right). \tag{12}$$

The transformed Hamiltonian $\hat{\mathcal{H}}_2$ is the quantum correspondent of $\mathcal{H}_2$:

$$\hat{\mathcal{H}}_2 = \frac{1}{\beta_x(z)}\left(\frac{\hat{p}_x^2}{2p_0} + \frac{p_0}{2}\hat{x}^2\right). \tag{13}$$

$\hat{\mathcal{H}}_2$ has the form of a Hamiltonian for a simple oscillator times a pure function of z. If we denote with

$$\phi_n(x) = \left(\frac{p_0}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n\left(x\sqrt{\frac{p_0}{\hbar}}\right)e^{-p_0x^2/2\hbar} \tag{14}$$

the eigenfunctions of the harmonic oscillator part we can easily verify that we can write the solutions of the Schrödinger equation $i\hbar\partial_z|\tilde{\psi}^{(2)}\rangle = \hat{\mathcal{H}}_2|\tilde{\psi}^{(2)}\rangle$ corresponding to the $n-$excited level of the betatron oscillations, as

$$\tilde{\psi}_n^{(2)}(x,z) = \phi_n(x)e^{-i(n+1/2)\varphi_x(z)}. \tag{15}$$

We recall that $\varphi_x(z) = \int dz/\beta_x(z)$.

By applying in sequence the transformations U_2 and U_1, we then recover the wavefunctions relative to the intermediate Hamiltonian $\hat{\mathcal{H}}_1$ and original Hamiltonian $\hat{\mathcal{H}}$. In particular, we have $|\psi_n^{(1)}\rangle = U_2|\psi_n^{(2)}\rangle$ or:

$$\tilde{\psi}_n^{(1)}(x,z) = \beta_x^{-\frac{1}{4}}\psi_n^{(2)}(x/\sqrt{\beta_x}, z), \tag{16}$$

i.e.

$$\begin{aligned}\tilde{\psi}_n^{(1)}(x,z) = \quad & \left(\frac{p_0}{\pi\hbar\beta_x(z)}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n\left(x\sqrt{\frac{p_0}{\hbar\beta_x(z)}}\right)\times \\ & \exp\left(-p_0x^2/[2\hbar\beta_x(z)]\right)\exp\left(-i(n+1/2)\varphi_x(z)\right).\end{aligned} \tag{17}$$

Finally, the solutions of the original Schrödinger equation (4) are ($|\psi_n\rangle = U_2|\psi_n^{(1)}\rangle$)

$$\tilde{\psi}_n(x,z) = \exp\left(i\frac{p_0}{\hbar}\frac{\beta_x'(z)}{4\beta_x(z)}x^2\right)\tilde{\psi}_n^{(1)}(x,z). \tag{18}$$

One can verify that indeed this is a solution of (4) by direct substitution.

5 Coherent-States

The wavefunctions (18) can be combined linearly to obtain localized wavepackets both longitudinally and transversally. For simplicity we will focus only on localization in the transverse plane, *i.e.* we consider only eigenstates of p_0. Of particular interest are those linear superpositions leading to coherent states. One way to introduce coherent states for a simple harmonic oscillator is to define them as eigenfunctions of the creation operator.[8] Here we can proceed in a similar way by using the creation operator a defined in terms of the observables $\hat{x}$ and $\hat{p}_x$ relative to Hamiltonian $\hat{\mathcal{H}}_2$:

$$a = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{p_0}{\hbar}} \hat{x} + i \frac{1}{\sqrt{p_0 \hbar}} \hat{p}_x \right). \tag{19}$$

The coherent states for the original system $\hat{\mathcal{H}}$ are then recovered by applying the unitary operators U_1 and U_2 introduced in Sec. 4. Equivalently, one can introduce coherent states by means of the displacement operator

$$D(\alpha) = e^{-i\varphi_x(z)/2} e^{\alpha a^\dagger - \alpha^* a}. \tag{20}$$

where α is the function $\alpha(z) = \alpha_0 \exp[-i\varphi_x(z)]$, with α_0 being a complex constant number. One can show that the coherent states $|\alpha_0\rangle$ result from applying $D(\alpha)$ to the harmonic oscillator ground state $|\alpha_0\rangle = D(\alpha)|\phi_0\rangle$. Use of the displacement operator allows one to quickly obtain the wavefunction representation $\psi^{(2)}_{\alpha_0}(x, z) = \langle x|\alpha_0\rangle$:

$$\psi^{(2)}_{\alpha_0}(x) = e^{-i\varphi_x(z)/2} e^{(\alpha^{*2} - \alpha^2)/4} e^{\sqrt{p_0/2\hbar}(\alpha - \alpha^*)x} \phi_0 \left(x - \sqrt{\frac{\hbar}{2p_0}} (\alpha + \alpha^*) \right), \tag{21}$$

where $\phi_0(x) = \langle x|\phi_0\rangle = (p_0/\pi\hbar)^{\frac{1}{4}} \exp(-p_0 x^2/2\hbar)$ is the wavefunction of the harmonic oscillator ground state. If we then act with U_2 and U_1 on $\psi^{(2)}_{\alpha_0}(x)$ we find the coherent states in the original variables

$$\psi_{\alpha_0}(x) = \exp\left(i \frac{p_0}{\hbar} \frac{\beta_x'(z)}{4\beta_x(z)} x^2 \right) \beta_x^{-\frac{1}{4}} \psi^{(2)}_{\alpha_0}(x/\sqrt{\beta_x}, z). \tag{22}$$

The function α is related to the expectation values $\overline{x} = \langle \hat{x} \rangle_{\alpha_0}$ and $\overline{p}_x = \langle \hat{p}_x \rangle_{\alpha_0}$ for the coherent state:

$$\sqrt{\frac{\hbar}{2p_0}} (\alpha + \alpha^*) = \frac{\overline{x}}{\sqrt{\beta_x}}, \tag{23}$$

$$\sqrt{\frac{\hbar p_0}{2}} (\alpha - \alpha^*) = i\sqrt{\beta_x} \overline{p}_x - i p_0 \frac{\beta_x'}{2\sqrt{\beta_x}} \overline{x}. \tag{24}$$

It can be shown that two above equations indicate that $\overline{x}$ and $\overline{p}_x$ evolve according to the classical trajectory, as expected from Ehrenfest's Theorem [8].

For a simple harmonic oscillator the coherent states have the property that the wavepacket spread in both position and momentum is constant and has the minimum value consistent with the Heisenberg uncertainty principle. This is not true in our case because of the dependence of the betatron function on z and only where $\beta_x'(z) = 0$ the wavepacket spread is minimum. In particular, we have [with $(\Delta x)^2 \equiv \langle (\hat{x} - \overline{x})^2 \rangle_{\alpha_0}$, etc.], $(\Delta x)^2 = \hbar \beta_x(z)/2p_0$ and $(\Delta p_x)^2 = [\hbar p_0/2\beta_x(z)](1 + \beta_x'^2/4)$ and therefore $\Delta x \Delta p_x = (\hbar/2)\sqrt{1 + \beta_x'^2/4}$. On the other hand the quantum quantity corresponding to the unnormalized rms emittance evaluated on these states has the constant value

$$\varepsilon_x \equiv \left[(\Delta x)^2 \frac{(\Delta p_x)^2}{p_0^2} - \frac{1}{4} \langle \frac{(\hat{p}_x - \overline{p}_x)}{p_0} (\hat{x} - \overline{x}) + (\hat{x} - \overline{x}) \frac{(\hat{p}_x - \overline{p}_x)}{p_0} \rangle^2_{\alpha_0} \right]^{\frac{1}{2}} = \frac{\pi \lambda_c}{\gamma}, \tag{25}$$

where $\lambda_c = h/mc$ is the Compton wavelength and γ the relativistic factor.

6 Charged particle in uniform bending and periodic focusing

The calculation carried out in the previous Sections can be extended to include the dynamics of a charged particle in a storage ring. We assume a simplified model of storage ring for which in addition to a periodic focusing we now impose a bending provided by a uniform magnetic field of strength B_0. We assume that the magnetic field is pointing in the $y-$direction so that the classical equilibrium orbit for a charged particle is a circle of radius ρ_0 in the $x-z$ plane. We select the (classical) reference orbit to be centered at $x=z=0$. In cylindrical coordinates[‡] a vector potential $\boldsymbol{A}=(A_\rho,A_\phi,A_y)$ associated with the magnetic field that provides the desired confinement and focusing is given, through second order in terms of the deviations from the equilibrium orbit, by $A_\rho=A_y=0$ and

$$A_\phi=\frac{B_0}{2}\rho+b_2(\phi)\frac{\rho_0}{2\rho}[(\rho-\rho_0)^2-y^2]. \tag{26}$$

Our starting point is the KG equation, which now is expressed best in terms of cylindrical coordinates. Because the energy E of a particle does not depend on time we can still write the solution of the KG equation as

$$\Psi=e^{-iEt/\hbar}\hat{\Psi}(\rho,\phi,y) \tag{27}$$

as in Sec. 1 . To avoid possible confusion from now on we will use the capitalized letter Ψ to denote the quantum wavefunction for the system with bending. With this ansatz the KG equation becomes

$$\left[-\hbar^2\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\left(\frac{\hbar}{i\rho}\frac{\partial}{\partial\phi}-eA_\phi\right)^2-\hbar^2\frac{\partial^2}{\partial y^2}\right]\hat{\Psi}=p^2\hat{\Psi}, \tag{28}$$

where p is the particle mechanical momentum. We alert the reader that we will allow the possibility for the particle energy and momentum to be different from the design values E_0 and p_0. In analogy with Eq. (3) we make the following ansatz

$$\hat{\Psi}=e^{i\ell\phi}\tilde{\Psi}(\rho,\phi,y), \tag{29}$$

which spells out a decomposition of the wavefunction into fast (first term on the RHS) and slow (second term on the RHS) ϕ-varying component. That is, we are assuming $\ell\gg 1$ and $\partial_\phi\tilde{\Psi}\ll\ell\tilde{\Psi}$. On the basis of the exact solution of the problem (28) for the case with vanishing focusing ($b_2=0$), we expect that the particle mechanical momentum to be related to the quantum number ℓ by[8] $p^2=2\hbar|e|B_0(\ell+1/2)\simeq 2\hbar|e|B_0\ell$.

We can now proceed as in Sec. 1 and neglect the second order derivatives $\partial_\phi^2\tilde{\Psi}$ compared to $\ell\partial_\phi\tilde{\Psi}$ and the term $\partial_\phi A_\phi$ compared to ℓA_ϕ. As a result the KG equation then reads

$$i\hbar\left(\frac{\hbar\ell}{\rho}-eA_\phi\right)\frac{2}{\rho}\frac{\partial\tilde{\Psi}}{\partial\phi}=\left(-\hbar^2\frac{\partial^2}{\partial\rho^2}-\hbar^2\frac{\partial^2}{\partial y^2}+V(\rho,y)-p^2\right)\tilde{\Psi}, \tag{30}$$

[‡]defined by $x=\rho\cos\phi, z=\rho\sin\phi, y$.

where $V(\rho, y)$ is an effective potential that has the form

$$V(\rho, y) = -\frac{\hbar^2}{4\rho^2} + \frac{1}{\rho^2}(\hbar\ell - eA_\phi\rho)^2 \simeq \frac{1}{\rho^2}(\hbar\ell - eA_\phi\rho)^2. \tag{31}$$

The last equality holds because we are assuming $\ell \gg 1$. At this point we expand the effective potential V around its point of minimum, $\rho = \rho_{\min} = \sqrt{2\hbar\ell/|e|B_0}$. By requiring $\rho_{\min} = \rho_0$ (*i.e.* we want the expansion of V to be centered on the classical reference orbit) the equation above identifies $\ell_0 = [\rho_0^2|e|B_0/2\hbar]^{\frac{1}{2}} = p_0\rho_0/2\hbar$, as the quantum number relative to the state of an on-momentum particle. We write $\rho = \rho_0 + \xi$ and in the expansion for V we keep only terms quadratic in the variables ξ, y, and $\delta = (\ell - \ell_0)/2\ell_0 = \Delta p/p_0$. After some algebra we then obtain the reduced KG equation in the form of the following Schrödinger-like equation

$$\begin{aligned} i\hbar\frac{\partial\tilde{\Psi}}{\partial s} = \Bigg[& - \frac{\hbar^2}{2p_0}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2p_0}\frac{\partial^2}{\partial y^2} + \frac{p_0}{2}\left(\frac{1}{\rho_0^2} + k(s)\right)x^2 \\ & - \frac{p_0}{2}k(s)y^2 - p_0\frac{\delta}{\rho_0}x + \frac{p_0}{2}\delta^2\Bigg]\tilde{\Psi}. \end{aligned} \tag{32}$$

In writing (32) we have rescaled the independent variable ϕ according to $\phi = s/\rho_0$, written the focusing function as $k(s) = eb_2(s/\rho_0)/p_0$, and finally re-christen ξ as x. In conclusion, the desired solution of the KG equation (28) around the reference orbit is

$$\hat{\Psi} = e^{i\ell s/\rho_0}\tilde{\Psi}(\xi, y, s)/\sqrt{\rho_0 + \xi} \simeq e^{i\ell s/\rho_0}\tilde{\Psi}(\xi, y, s)/\sqrt{\rho_0}. \tag{33}$$

with $\tilde{\Psi}$ given by the solution of (32).

7 Treatment of Dispersion

The Schrödinger equation (32) differs from (4) because of the coupling term δx. A way to solve Eq. (32) is to first introduce a suitable unitary transformation that remove the coupling. In complete analogy with the classical case[7] such a transformation consists of one translation in position and one in momentum. The first, $U_1^{-1}\hat{x}U_1 = \hat{x} + \delta D$ and $U_1^{-1}\hat{p}_xU_1 = \hat{p}_x$ is generated by $U_1 = \exp\left(-i\delta\hat{p}_xD(s)/\hbar\right)$. The second transformation is generated by $U_2 = \exp\left(i\delta\hat{x}p_0D'(s)/\hbar\right)$, yielding $U_2^{-1}\hat{x}U_2 = \hat{x}$ and $U_2^{-1}\hat{p}_xU_2 = \hat{p}_x + p_0\delta D'(z)$. In both cases $D(s)$ is the dispersion function defined as a solution of the inhomogeneous equation

$$D'' + k_x(s)D = \frac{1}{\rho_0}. \tag{34}$$

By virtue of (34) the transformed states $|\tilde{\Psi}_n^{(2)}\rangle = U_2U_1|\tilde{\Psi}_n\rangle$ obey the Schrödinger equation $i\hbar\partial_z|\tilde{\Psi}_n^{(2)}\rangle = \hat{H}_2|\tilde{\Psi}_n^{(2)}\rangle$ with

$$\hat{H}_2 = \frac{1}{2p_0}\hat{p}_x^2 + \frac{1}{2}p_0k_x(s)\hat{x}^2 + \frac{p_0\delta^2}{2}\left(k_xD^2 - D'^2 - \frac{2D}{\rho_0}\right). \tag{35}$$

This Hamiltonian has the same form as Hamiltonian (5) apart from the last purely $s-$dependent term on the RHS. Therefore, a solution of the resulting Schrödinger equation is given by

$$\tilde{\Psi}_n^{(2)}(x,s) = \tilde{\psi}_n(x,s) \exp\left[-\frac{i}{\hbar}\frac{p_0\delta^2}{2}\int_0^s [1 + k_x(t)D^2(t) - D'^2(t) - \frac{2D(t)}{\rho_0}]dt\right] \tag{36}$$

where $\tilde{\psi}_n(x,s)$ is the same as in Eq. (18) with s replacing z. By undoing the transformations U_1 and U_2 we can finally obtain the solutions $|\tilde{\Psi}_n\rangle = U_1U_2|\tilde{\Psi}_n^{(2)}\rangle$ of the original Schrödinger equation (32):

$$\tilde{\Psi}_n(x,s) = e^{i\delta p_0(x-\delta D)D'/\hbar}\tilde{\Psi}_{2,n}(x-\delta D, s). \tag{37}$$

Next we combine Eq.'s (33) and (37) and upon including the vertical degree of freedom we finally recognize that the wavefunction corresponding to the n_x and n_y transverse levels of excitation reads

$$\Psi(x,y,s) = \frac{C}{\sqrt{\rho_0}}e^{i\ell s/\rho_0}\tilde{\Psi}_{n_x}(x,s)\tilde{\psi}_{n_y}(y,s), \tag{38}$$

with $\tilde{\Psi}_{n_x}(x,s)$ given by Eq. (37) and $\tilde{\psi}_{n_y}(y,s)$ by Eq. (18); we have introduced the constant C to guarantee a proper normalization. Enforcing periodicity upon the wavefunction $\Psi(x,y,0) = \Psi(x,y,2\pi\rho_0)$ yields the following quantization condition on the particle momentum (through first order in $\Delta p = p_0\delta$)

$$\Delta p = \frac{\hbar}{\rho_0}\left[(m-\ell_0) + \left(n_x + \frac{1}{2}\right)\nu_x + \left(n_y + \frac{1}{2}\right)\nu_y\right], \tag{39}$$

where ν_x and ν_y are the tunes and m, n_x, n_y are integers.

In conclusion, we have succeeded in deriving an explicit expression for the wavefunction of a quantum particle confined in a storage ring or transport line. We have stressed the correspondence between classical and quantum motion by showing that they can both be described in terms of the functions $\beta_{x,y}(z)$ and $D(z)$, which obey the same equations in both cases. With the appropriate choice of the boundary conditions these can be identified as the lattice functions one is familiar with from accelerator theory.

We would like to acknowledge C. Hill, S. De Martino, F. Illuminati for useful discussions during the Workshop and in particular R. Fedele for pointing out a mistake in one of our equations. We have also benefited from many discussions with A. Kabel. Work supported by the US Dept. of Energy.

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