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Quantum Ground State and Minimum Emittance of a Fermionic Particle Beam in a Circular Accelerator*

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Abstract

In the usual parameter regime of accelerator physics, particle ensembles can be treated as classical. If we approach a regime where $\epsilon_x \epsilon_y \epsilon_s \approx N_{particles} \lambda^3_{Compton}$, however, the granular structure of quantum-mechanical phase space becomes a concern. In particular, we have to consider the Pauli exclusion principle, which will limit the minimum achievable emittance for a beam of fermions. We calculate these lowest emittances for the cases of bunched and coasting beams at zero temperature and their first-order change rate at finite temperature.

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1 Dynamics

Consider an ultra-relativistic particle beam in a circular accelerator. Neglecting higher-order effects, the Hamiltonian can be written as a quadratic form in the usual phase-space coördinates $x, x', y, y', \sigma, \delta$.

However, this Hamiltonian is not appropriate for quantization, as energy and time have switched roles. Thus, we use the Hamiltonian of the system in the beam's frame of reference, which can be obtained by a series of canonical transformation from the lab frame[1]:

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{p_z^2}{2} - \frac{\gamma^2 \beta x p_z}{R} + \beta^2\gamma^2 \left(\kappa_x - \frac{\gamma^2}{R^2}\right)\frac{x^2}{2} + \beta^2\gamma^2\kappa_y\frac{y^2}{2} + \Phi_{RF}(z) \quad , \tag{1}$$

where γ is the relativistic factor, $\kappa_{x,y}$ the (possibly local) focusing strengths (in the case of magnetic quadrupoles, one has $\kappa_x = -\kappa_y$) and $\Phi(z)$ the external electric potential. The directions x, y, z are radial, transversal, and tangential, respectively. Note that we use units with $\hbar = c = k_B = m_0$ throughout, so all quantities are expressed in powers of the Compton length of the particles considered.

The longitudinal part of the Hamiltonian depends on the physical setup. The particles might either be confined by the nearly harmonic potential of the RF bucket, or we have case of a coasting beam, where the only constraints imposed on the longitudinal motion are the ones due to the periodicity of the problem. In the sequel, we will consider both cases.

2 Anisotropic Oscillator

Let us assume that the longitudinal motion is determined by the presence of an RF bucket. We can approximate the potential Φ by expanding it to 2nd order in z. For reasons of simplicity, we only take into account the $\mathrm{O}(z^2)$ term, i. e., we assume that the particle is on the orbit and is not losing energy.

We can obtain the stiffness of the longitudinal oscillator in the local canonical coördinates by Lorentz-boosting the (t, z) components of the momentum vector:

$$m_0 \begin{pmatrix} \gamma(1+\delta) \\ \gamma(1+\delta)\sqrt{1-(\gamma(1+\delta))^{-2}} \end{pmatrix} \rightarrow m_0 \begin{pmatrix} 0 \\ \delta \end{pmatrix} + \mathrm{O}(\delta^2) \tag{2}$$

Thus, for a bunched beam with dimensions σ_z, σ_δ, we have $\omega_l \gamma \sigma_l = \sigma_\delta$.

The longitudinal and radial part of the hamiltonian (1) have the form

$$H = \sum_{i=1}^{2} \frac{p_i^2}{2} + \frac{\omega_i^2 q_i^2}{2} - \mu q_1 p_2 \quad . \tag{3}$$

The associated infinitesimal symplectic transformation matrix (in (q_1, p_1, q_2, p_2)-space) reads

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & \mu \\ -\mu & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{pmatrix} \tag{4}$$

and its eigenvalues are determined by the equation

$$\lambda^4 + \lambda^2(\omega_1^2 + \omega_2^2) + \omega_2^2(\mu^2 - \omega_1^2) = 0 \quad , \tag{5}$$

which will have purely imaginary solutions for

$$(\omega_1^2 + \omega_2^2) > 4\omega_2^2(\mu^2 - \omega_1^2) > 0 \quad . \tag{6}$$

The first condition will always be fulfilled for realistic machines. The second one corresponds to the machine being below or above transition: if the second factor changes sign, the eigenfrequency can be mad real again by flipping the sign of $\omega_z^2 \propto \Phi''_{RF}$. However, the absolute sign of both the kinetic and the potential term will change, leading to the (for purposes of constructing the quantum-mechanical ground state) pathological case of a hamiltonian not limited from below. In the sequel, we will assume the machine is below transition.

Thus, expanding (1) to first non-trivial order in the canonical coördinates and applying the canonical transformation removing the mixed term in (3), we obtain the Hamiltonian of a 3-dimensional harmonic oscillator with corrected frequencies given by (5); the ground state is characterized by the occupation numbers $n_n^d \in \{0,1\}$ where $\sum n_d^i = N$ of the oscillator levels. For sake of generality, we consider the case of d dimensions. The ground state for a given particle number can be constructed by successively filling states with the lowest energy (we disregard spin here, which can be easily reintroduced by replacing $N \to 2N$ in the final formulae).

In $\frac{E}{\epsilon_F}$-space, the Fermi sea is just a unit d-simplex, in n_i-space, a d-simplex with axes of length $\frac{\omega_1}{\epsilon_F}, \ldots \frac{\omega_d}{\epsilon_F}$. Thus, the particle number for a ground state filled up to the Fermi energy ϵ_F, where we have disregarded the zero-mode energy $\frac{1}{2}\sum_i \omega_i$ of the oscillator,

$$N = \frac{{\epsilon_F}^d}{\omega_1 \cdots \omega_d d!} = \frac{1}{\Omega^d d!} \quad , \tag{7}$$

the volume of an unit d-simplex being $\frac{1}{d!}$ and $\Omega = \sqrt[d]{\omega_1 \cdots \omega_d}$.

The energy in the ith degree of freedom in that case is given by a sum over the d-simplex:

$$E_i = \sum_{i_1=1}^{\frac{\epsilon_F}{\omega_1}} \sum_{i_2=1}^{\frac{\epsilon_F - n_1\omega_1}{\omega_2}} \cdots \sum_{i_d=1}^{\frac{\epsilon_F}{\omega_d} - \frac{1}{\omega_d}\sum_{k=1}^{d-1} n_k\omega_k} \omega_i i_i \quad . \tag{8}$$

Replacing all the sums by integrals, we have

$$E_i = \frac{{\epsilon_F}^d}{\Omega^d} \int_0^1 \int_0^{1-q_1} \cdots \int_0^{1-\sum_{k=1}^{d-1} q_k} \epsilon_F q_i \mathrm{d}q_1 \cdots \mathrm{d}q_d = \frac{{\epsilon_F}^{d+1}}{\Omega^d (d+1)!} = \frac{N\epsilon_F}{d+1} \quad , \tag{9}$$

which, of courses, just expresses equidistribution of energy.

Finally, the emittance is given in terms of the averaged single-particle expectation value of the action. For a harmonic oscillator, the action is just $I = n$, where n is the excitation number of the energy level, so:

$$\varepsilon_i = \langle n_i \rangle = \frac{E_i}{N\omega_i} = \frac{\epsilon_F}{\omega_i(d+1)} = \frac{\Omega}{\omega_i(d+1)} \sqrt[d]{N d!} \quad . \tag{10}$$

and

$$\left.\begin{aligned} \varepsilon^{(d)} = \prod_{i=1}^{d} \varepsilon_i = N d! \left(\frac{2\pi}{d+1}\right)^d \\ \varepsilon^{(1)} = \pi N \quad , \quad \varepsilon^{(2)} = \frac{8}{9}\pi^2 N \quad , \quad \varepsilon^{(3)} = \frac{3}{4}\pi^3 N \end{aligned}\right\} \tag{11}$$

Thus, the projected emittances scale as $N^{\frac{1}{d}}$, as one would naïvely assume. Furthermore, due to the occurrence of the geometric mean of the frequencies in (10), the projected emittance in one dimension can be lowered by shallowing the potential in the other dimensions.

Note that a similar approach has been chosen elsewhere; [[2]] gives an estimate for ϵ_{min} from a similar reasoning, but ends up (due to a miscounting of the states) with a scaling different from our result.

3 Mixed Case: Longitudinally Free Particles

So far, we have assumed an anisotropic oscillator. But given the case of a particle moving freely longitudinally, the energy content of that degree of freedom will be given by the square of the (angular) momentum. (We might consider the boundary conditions imposed by a periodic box instead of a circular arrangement.) We treat the general case, i. e. a Hamiltonian

$$H = \sum_{\tilde{i}=1}^{\tilde{d}} \tilde{\omega}_{\tilde{i}}^2 \tilde{n}_{\tilde{i}}^2 + \sum_{i=1}^{d} \omega_i n_i \quad . \tag{12}$$

The integration over the free degrees of freedom runs over an ellipsoid; by rescaling to a unit sphere, we get

$$\begin{aligned} N = \frac{\epsilon_F{}^{\frac{\tilde{d}}{2}+d}}{\tilde{\Omega}^{\tilde{d}}\Omega^d} \int_{Sphere} \int_{Simplex}^{1-\tilde{q}^2} \mathrm{d}q \mathrm{d}\tilde{q} = \frac{\epsilon_F{}^{\frac{\tilde{d}}{2}+d}}{\tilde{\Omega}^{\tilde{d}}\Omega^d} \int_0^1 S_{\tilde{d}} q^{\tilde{d}-1} \frac{(1-\tilde{q}^2)^d}{d!} \mathrm{d}\tilde{q} \\ = \frac{S_d \epsilon_F{}^{\frac{\tilde{d}}{2}+d} B\left(d+1, \frac{\tilde{d}}{2}\right)}{2\tilde{\Omega}^{\tilde{d}}\Omega^d d!} = \frac{\pi^{\frac{\tilde{d}}{2}} \epsilon_F{}^{\frac{\tilde{d}}{2}+d}}{\tilde{\Omega}^{\tilde{d}}\Omega^d \Gamma\left(\frac{2d+\tilde{d}+2}{2}\right)} \quad , \end{aligned} \tag{13}$$

where $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$ is the surface of the d-dimensional unit sphere and B is the usual Beta function $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

We can readily write down the expectation values of energy in the different degrees of freedom:

$$\frac{E_{\tilde{i}}}{\varepsilon_F} = \left\langle \tilde{q}^2 \right\rangle = \frac{\int_0^1 \tilde{q}^{\tilde{d}+1} \sqrt{1-\tilde{q}^2}^d \mathrm{d}\tilde{q}}{\int_0^1 \tilde{q}^{\tilde{d}-1} \sqrt{1-\tilde{q}^2}^d \mathrm{d}\tilde{q}} = \frac{B\left(d+1, \frac{\tilde{d}}{2}+1\right)}{B\left(d+1, \frac{\tilde{d}}{2}\right)} = \frac{\tilde{d}}{2d+2+\tilde{d}} \tag{14}$$

and

$$\frac{E_i}{\varepsilon_F} = \langle q \rangle = \frac{\int_0^1 q^d (1-q)^{\frac{\tilde{d}}{2}} \mathrm{d}q}{\int_0^1 q^{d-1}(1-q)^{\frac{\tilde{d}}{2}} \mathrm{d}q} = \frac{B\left(d+1, \frac{\tilde{d}}{2}+1\right)}{B\left(d, \frac{\tilde{d}}{2}+1\right)} = \frac{2d}{2d+2+\tilde{d}} \tag{15}$$

For the longitudinal emittance, we need the expectation value of the action; for a particle in a box with periodic boundary conditions, this is again just $\langle xp \rangle = 2\pi n$ in the nth box eigenmode, so we need

$$\langle \tilde{q} \rangle = \frac{\int_0^1 \tilde{q}^{\tilde{d}} \sqrt{1-\tilde{q}^2}^d \mathrm{d}\tilde{q}}{\int_0^1 \tilde{q}^{\tilde{d}-1} \sqrt{1-\tilde{q}^2}^d \mathrm{d}\tilde{q}} = \frac{B\left(d+1, \frac{\tilde{d}+1}{2}\right)}{B\left(d+1, \frac{\tilde{d}}{2}\right)} \quad , \tag{16}$$

and the form factor for the product emittance is

$$\varepsilon^{(d,\tilde{d})} = (2\pi)^{d+\tilde{d}} N \frac{B^{\tilde{d}}\left(d+1, \frac{\tilde{d}+1}{2}\right) B^d\left(d+1, \frac{\tilde{d}}{2}+1\right)}{B^{\tilde{d}}\left(d+1, \frac{\tilde{d}}{2}\right) B^d\left(d, \frac{\tilde{d}}{2}+1\right)} \quad , \tag{17}$$

and, for real-world situations,

$$\varepsilon^{(2,1)} = \frac{5}{49} N \quad . \tag{18}$$

Putting in a real ring, we can express the rest-frame frequencies by the tune[3]: $\omega_x \approx \omega_y = \frac{\beta \nu_y}{\gamma L}$, where L is the length of the ring. The longitudinal momentum is quantized in units of $\frac{2\pi}{\gamma L}$, so $\omega_l = \frac{2\pi}{\sqrt{2}\gamma L}$:

$$\varepsilon_x = \langle q \rangle \frac{\varepsilon_F}{\omega_x} = \sqrt[5]{\frac{7200\pi^2 N^2}{16807 \gamma L \nu}} \quad . \tag{19}$$

For a ring with $L = 2\pi$m, $\gamma = 10$, $N = 10^{10}$, $\nu = 100\pi$ we get an emittance of ≈ 5.3 Compton wavelengths.

4 The ground state as a Fermi liquid

In our construction, we tacitly assume that the particle-particle interaction does not modify the particle content of the ground state. This corresponds precisely to the notion of a Fermi liquid (in our case, a highly anisotropic one), in which the free particle spectrum smoothly deforms into the quasi-particle spectrum of equal particle content when the interaction is switched on adiabatically. This naïve assumption of the existence of a Fermi surface may break down if we take into account particle-particle interactions. The behavior of an ultra-cold bunch above transition would be of special interest here, since it exhibits a negative-mass behavior in the longitudinal degree of freedom.

For the case of the system being below transition, we can make the following semi-quantitative argument for the existence of a Fermi liquid: The particle beam will have an average radial dimension given by the excursion of a particle on the Fermi edge in the radial or transverse oscillator potential:

$$m_{e^-}\bar{\omega}^2 \langle x \rangle^2 \approx \epsilon_F \quad . \tag{20}$$

In a "Mean Field" calculation, we estimate the effective transverse focusing strength $\bar{\omega}^2$ to be the sum of the external focusing and a space-charge tune depression due to a circular beam of radius $\sqrt{x^2}$:

$$\omega^2 = \omega_{ext}^2 - \omega_{sc}^2 = \omega_{ext}^2 - \frac{Ne^2}{\gamma L \langle x \rangle^2}. \tag{21}$$

The fact that the system is in the ground state allows one to eliminate ϵ_F; for the case of a coasting beam, we obtain a consistency condition:

$$\left(\frac{\bar{\omega}}{\omega_{ext}}\right)^2 \left[1 + \left(\frac{e^3 N}{\gamma L \omega_{ext}}\right)^{2/3} \left(\frac{\bar{\omega}}{\omega_{ext}}\right)^{-2/3}\right] = 1 \quad , \tag{22}$$

which has a solution $\frac{\bar{\omega}}{\omega_{ext}} < 1$ for for all N. The resulting effective frequency is shown in Fig. 1.

5 Finite Temperature

The above considerations were for the case of zero temperature. To generalize to finite temperatures, we follow the usual prescription and introduce a chemical potential. Let's treat the all-oscillator case first; the quantity we want to calculate is the logarithm of the partition function of the grand-canonical ensemble:

$$\begin{aligned}\log Z = \log \sum_{n_i \in \{0,1\}} & e^{-\beta \sum_i n_i \left(\sum_{k=1}^d \omega_k (i_k + \frac{1}{2}) - \mu\right)} \\ &= \sum_{i_k=0}^{\infty} \log\left(1 + e^{-\beta\left(\sum \omega_k (i_k + \frac{1}{2}) - \mu\right)}\right) \quad . \end{aligned} \tag{23}$$

Again, we transform the sum into an integral. The only non-trivial integration we have to do is the one perpendicular to the surfaces of constant energy E; the integration in all other directions gives the area $\frac{E^{d-1}}{(d-1)!}$ of that surface:

$$\log Z = \frac{1}{\Omega (d-1)!} \int_0^{\infty} \log\left(1 + e^{-\beta(E-\mu)}\right) E^{d-1} \mathrm{d}E \quad , \tag{24}$$

where we have subtracted the zero-point energy. We integrate by parts :

$$\log Z = \frac{\beta}{\Omega d!} \int_0^{\mu} \frac{E^d}{1 + e^{-\beta(E-\mu)}} \mathrm{d}E \quad . \tag{25}$$

For small temperatures, integrals of this style can be approximated using the Sommerfeld trick. We find

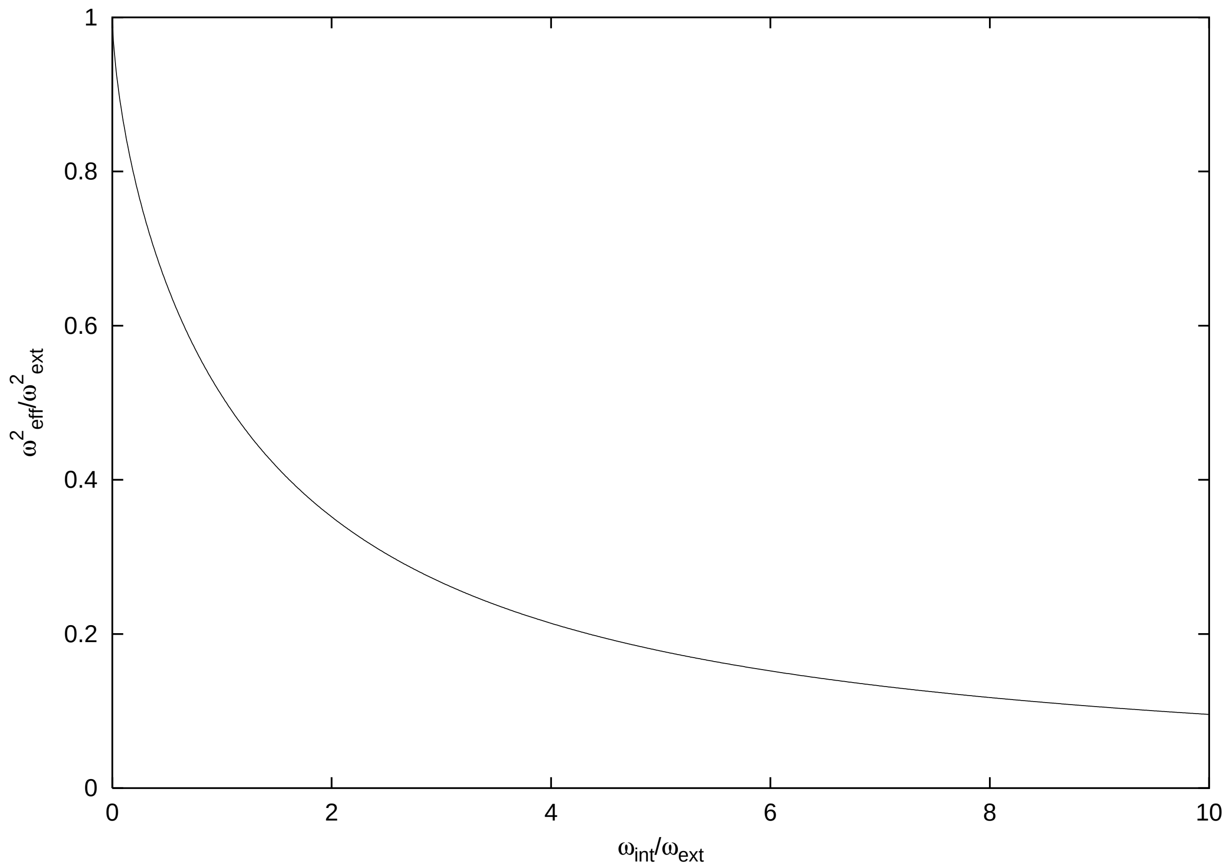

Figure 1: Effective focusing strength of a coasting beam in its ground state; $\omega_{int} = e^3 N/(\gamma L)$

$$\log Z = \frac{\beta}{\Omega d!}\left(\frac{\mu^{d+1}}{d+1} + \frac{d\mu^{d-1}\pi^2}{6\beta^2} + \dots\right) \quad . \tag{26}$$

As

$$\left.\begin{aligned} \langle\varepsilon_i\rangle &= -\frac{1}{N}\beta\frac{\partial}{\partial\omega_i}\log Z \\ \langle E\rangle &= -\frac{1}{N}\frac{\partial}{\partial\beta}\log Z \end{aligned}\right\} \quad , \tag{27}$$

we can write the temperature-dependent contributions to emittance and energy:

$$\left.\begin{aligned} \Delta\langle E\rangle = \frac{1}{N}\frac{\mu^{d-1}\pi^2}{6\beta^2\Omega(d-1)!} &= \frac{d\pi^2}{6\beta^2\sqrt[d]{\Omega N d!}} \\ \Delta\langle\varepsilon_i\rangle &= \frac{1}{\omega}\Delta\langle E\rangle \end{aligned}\right\} \quad , \tag{28}$$

where we have used the zero-temperature E_F as chemical potential.

6 Limitations and Prospects

So far, we have considered two limiting cases of a circular setup: An infinitely extending harmonic potential and a free particle subject to periodic boundary conditions. While this is a realistic approach for the transverse degrees of freedom, the longitudinal dynamics is more complex. In the case of high longitudinal densities, two factors limit our model:

1. The longitudinal bucket is anharmonic and limited from above. While the former fact is probably benign, the latter poses the question of how the increasing spectral density at the upper boundary of the bucket affects transversal emittance in the case of an almost full RF bucket; this has to be further investigated. An appropriate approach would be to treat the periodic chain of RF bucket as a periodic potential, so the longitudinal eigenfunction would be Bloch functions.

2. The particle-particle interaction affects the stiffness of the transverse oscillators. In the limiting case of zero temperature and sufficiently strong focusing forces in a smooth lattice, the ground state is believed to be a crystalline state. The most simple realization of a crystalline state would be a one-dimensional electron chain, which is known as a 1-D Wigner crystal[4] in the context of solid state physics (in a Wigner crystal, the neutralizing field is provided by the ions of the crystal lattice, which are "smeared out" homogeneously to form the "jellium", whereas in our case stability is achieved by external focusing elements). This case is highly degenerate: we expect the electrons to be in well-localized states with equal potential energy, the only quantum effect being the phononic oscillation of the electron lattice[5, 6, 7]; consequently, the transverse emittance can be at its quantum-mechanical minimum of 1 Compton wavelength. Modeling the crossover behavior from free-particle eigenstates as in this paper to localized states and the transition from finite to zero emittance requires further investigation.

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