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Nonlinear stage of the microbunching by coherent synchrotron radiation *

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1 Abstract

Coherent synchrotron radiation can lead to the microwave instability. In this note, we consider saturation of the most unstable mode close to the threshold of instability.

2 Effect of CSR on Beam Stability

Recent experiments [1] indicate that coherent synchrotron radiation (CSR) can be produced by microstructures within a bunch. The micro-bunching can be a result of microwave instability and has been discussed in our previous note [5]. We considered the case when CSR of a bunch as a whole is suppressed by screening and is caused by microstructures with longitudinal dimensions small compared with the bunch length. In this case, we could simplify consideration studying stability of a coasting beam with the line density equal to the line density of a bunch. Here we consider the nonlinear self-consistent regime of the instability.

The problem of a nonlinear regime of instability is a general problem. For the CSR driven instability of the coasting beam it is simplified due to simplicity of the kinematics of the beam and the frequency dependence of the CSR impedance. Our consideration is limited to saturation of the unstable mode neglecting interaction of the non-zero azimuthal modes. This limitation restricts beam current above but very close to the threshold of the instability. Nevertheless, consideration of this case can be instructive for understanding of the dynamics of instability above threshold.

3 Linear approximation

Consider a coasting beam with the line density $n_B = dN_{tot}/ds$ and relativistic factor γ_0 on a circular orbit in free space. Let us use (p, z) variables, where $p = (\Delta E/E)/\delta$ is energy offset $\Delta E/E$ of a particle in units of the rms energy spread δ, z is longitudinal shift of a particle with energy ΔE relative to position of the particle if it had $\Delta E = 0$, and $s = ct$. A particle with $z > 0$ is shifted toward the head of the train.

The beam is described by the longitudinal distribution function $\rho(p, z, s)$,

$$\rho(p, z, s) = \frac{1}{2\pi R_0}[\rho_0(p, s) + \sum_{n\neq 0} \rho_n(p) e^{inz/R - in\omega t - in\gamma_d s/c}], \quad \int dp\rho_0(p, s) = 1. \tag{1}$$

where $n\gamma_d$ is synchrotron radiation damping of the mode, $2\pi R_0$ is ring circumference, and $\omega_n = -\omega^*_{-n}$ is coherent frequency shift to be determined.

$\rho(p,z,t)$ is a solution of the Fokker-Plank equation

$$\frac{\partial\rho}{\partial s}-\eta\delta p\frac{\partial\rho}{\partial z}-\frac{N_{tot}r_0}{\gamma_0\delta}\frac{\partial\rho}{\partial p}\int dz'dp'W(z-z')\rho(p',z',s)=\gamma_d^0\frac{\partial}{\partial p}[\frac{\partial\rho}{\partial p}+p\rho], \quad (2)$$

where η is the slip factor, and γ_d^0 is the single particle radiation damping. Without screening effect, particles on the circular orbit with radius of curvature R interact due to the CSR with wake potential $W(z-z')$. The later (per unit length) is the differential operator [2] [3],

$$W(z-z')=\frac{2}{(3R^2)^{1/3}}\frac{1}{(z-z')^{1/3}}\frac{\partial}{\partial z'} \quad z'<z; W(z-z')=0, \quad z'>z. \quad (3)$$

Azimuthal harmonics are the solution of the Vlasov equation,

$$\frac{ic}{k\eta\delta_0\omega_0}\frac{\partial\rho_k}{\partial s}+(\Omega_k+p+\frac{\gamma_d}{\eta\delta\omega_0})\rho_k(p)=i\sum_m\frac{\lambda\zeta m^{1/3}}{k}\frac{\partial\rho_{k-m}}{\partial p}\int dp'\rho_m(p'), \quad (4)$$

which is obtained from Eq. (2) approximating the right-hand-side by the term $-\gamma_d\rho_n$. Such an approximation is valid for $p\simeq Re[\Omega]$ close to the maximum of $\rho_n(p)$. Here $Z(\Omega)=Z_0\zeta_0(\Omega/\omega_0)^{1/3}$ is impedance of the CSR given by the Fourier transform of the wake potential, $Z_0=120\pi$ Ohm, $\omega_0/2\pi$ is revolution frequency, $\zeta=(1/3^{1/3})[2\pi/\Gamma(1/3)+i\Gamma(2/3)]=1.626+i0.939$, and λ is linear density of the beam,

$$\lambda=\frac{n_Br_0}{\eta\gamma_0\delta^2}(\frac{R}{R_0})^{1/3}, \quad \Omega_k=\frac{\omega}{\eta\delta\omega_0}. \quad (5)$$

In the linear approximation, one neglects coupling of the non-zero harmonics and use unperturbed distribution function ρ_0. Then, the steady-state non-trivial solution,

$$\rho_m(p)=d_m\frac{i\zeta\lambda}{m^{2/3}}\frac{\partial\rho_0/\partial p}{\Omega_m+p}, \quad (6)$$

where d_m is an arbitrary in the linear approximation parameter and Ω_m is given by the dispersion equation

$$1=\frac{i\lambda\zeta}{m^{2/3}}\int_{-\infty}^{\infty}\frac{dp'\frac{\partial\rho_0(p')}{\partial p'}}{p'+\Omega_m}. \quad (7)$$

Comparison of Eqs. (6),(7) shows that $d_m=\int dp\rho_m(p)$ is amplitude of the density perturbation.

The non-trivial solution Eq. (6) for a Gaussian bunch exists [5] for λ larger than the threshold λ_m, $1.6\lambda_m/m^{2/3}=1$. The threshold is lower for lower harmonics m. However, m has to be large enough, $m>n=(R_0/R)(\pi R/2b)^{3/2}$, where b is the beam pipe half aperture. More accurate estimate for n depends on the geometry of the beam pipe. Hence, for the fastest growing mode with harmonic number n, $1.6\lambda_{th}=(R/R_0)^{1/3}(\pi R_0/2b)$. In the linear approximation for $\lambda>\lambda_{th}$, the amplitude of harmonics ρ_n grows exponentially.

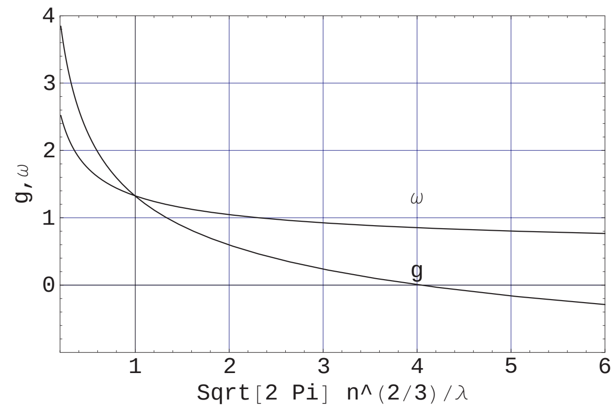

Figure 1: Solution of the dispersion equation Eq. (7) for a Gassian bunch. Here $\omega = \mathrm{Re}\Omega$, and $g = \mathrm{Im}\Omega$. Mode ρ_n is unstable for $g > 0$.

4 Quasi-linear approximation

The quasi-linear theory takes into account reverse effect of the growing mode on the distribution function $\rho_0(p)$. The Fokker-Plank equation for this mode takes form

$$\frac{\partial \rho_0}{\partial s} - \lambda\zeta\eta\delta\omega_0 n^{1/3}[\zeta \frac{\partial \rho_n^*}{\partial p} \int dp' \rho_n(p') + c.c.] = \gamma_d^0 \frac{\partial}{\partial p}[\frac{\partial \rho_0}{\partial p} + p\rho_0]. \tag{8}$$

Here we keep only contribution of the unstable azimuthal mode $\rho_{\pm n}$. With ρ_n found in the linear approximation,

$$\frac{\partial \rho_0}{\partial s} = \gamma_d^0 \frac{\partial}{\partial p}\{[1 + \frac{\Lambda \mathrm{Im}\Omega_n}{(\mathrm{Re}\Omega_n + p)^2 + (\mathrm{Im}\Omega)^2}]\frac{\partial \rho_0}{\partial p} + p\rho_0\}, \tag{9}$$

where

$$\Lambda = \frac{2\eta\delta\omega_0}{n^{1/3}\gamma_d^0}|\lambda\zeta|^2|d_n|^2. \tag{10}$$

In the steady-state,

$$\rho_0 \propto e^{-\int_0^p \frac{p' dp'}{1+f(p')}}, \tag{11}$$

where

$$f(p) = \frac{\Lambda \mathrm{Im}\Omega_n}{(\mathrm{Re}\Omega_n + p)^2 + (\mathrm{Im}\Omega)^2}]. \tag{12}$$

For small Λ, the function $f(p)$ is small everywhere except the vicinity of the resonance $p = -\mathrm{Re}\Omega_n$. Therefore, $\rho_0(p)$ is substantially different from the unperturbed Gaussian distribution only in this vicinity, where it is flatten. This is the usual result of the quasi-linear theory.

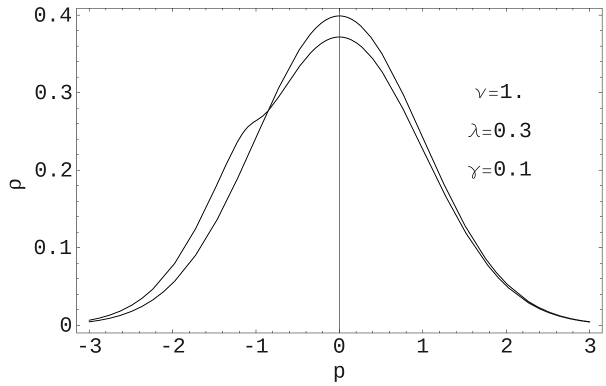

Figure 2: Modification of the Gaussian distribution by unstable mode.

The main effect of the change is that derivative $\partial\rho_0/\partial p$ at the pole of the dispersion equation goes to zero what changes the growth rate of the mode. With the modified ρ_0, the dispersion equation takes form

$$1 = -\frac{i\lambda\zeta}{n^{2/3}}\int\frac{p'dp'\rho_0(p')[p' + Re\Omega_n - iIm\Omega_n]}{(p' + \mathrm{Re}\Omega_n)^2 + (\mathrm{Im}\Omega_n)^2 + \Lambda(\mathrm{Im}\Omega_n)}. \tag{13}$$

Let us compare this equation with the dispersion relation of the linear case. Suppose that $\lambda > \lambda_{th}$ and Eq. (7) has the unstable solution with $Im[\Omega_n^{lin}] = \Gamma_n^{lin}$. Comparing Eq. (7) with Eq. (13) we can neglect the difference in ρ_0. In this case we can expect that Eq. (13) has a solution with $(\mathrm{Im}\Omega_n)^2 + \Lambda(\mathrm{Im}\Omega_n) = [\Gamma_n^{lin}]^2$. To get saturation of the harmonics ρ_n at some amplitude, the growth rate of the mode $Im\Omega$ has to be equal to synchrotron damping of the mode, $\mathrm{Im}\Omega_n = \gamma_d/(\eta\delta\omega_0)$.

That defines Λ,

$$\Lambda = \frac{[\eta\delta\omega_0\Gamma_n^{lin}]^2 - [\gamma_d]^2}{\gamma_d\eta\delta\omega_0}. \tag{14}$$

Hence, the amplitude of the mode at saturation

$$|d_n|^2 = \frac{n^{1/3}}{2(\eta\delta\omega_0)^2|\lambda\zeta|^2}\frac{\gamma_d^0}{\gamma_d}\{[\eta\delta\omega_0\Gamma_n^{lin}]^2 - [\gamma_d]^2\}. \tag{15}$$

5 Mode coupling

So far we neglected coupling between azimuthal harmonics. Let us check whether the coupling can become important in saturation. Consider Eq. (4) neglecting radiation dampling,

$$\frac{ic}{k\eta\delta_0\omega_0}\frac{\partial\rho_k}{\partial s} + (\Omega_k + p)\rho_k(p) = i\sum_m\frac{\lambda\zeta m^{1/3}}{k}\frac{\partial\rho_{k-m}}{\partial p}\int dp'\rho_m(p'). \tag{16}$$

Let us take into account coupling of ρ_k to ρ_0 and to the unstable mode $\rho_{\pm n}$ neglecting coupling to all stable modes. Then it is easy to see that only mode $\rho_{\pm 2n}$ is driven in this approximation. This can be understood noticing that unstable ρ_n describes particle trapped in a separatix generated by the wake field. Coupling to ρ_{2n} corresponds to period doubling of motion of trapped particles.

ρ_{2n} satisfies the following equation:

$$\frac{ic}{2n\eta\delta_0\omega_0}\frac{\partial\rho_{2n}}{\partial s}+(\Omega_{2n}+p)\rho_{2n}=\frac{i\lambda\zeta}{(2n)^{2/3}}\frac{\partial\rho_0}{\partial p}\int dp'\rho_{2n}(p')+\frac{i\lambda\zeta}{2n^{2/3}}\frac{\partial\rho_n}{\partial p}\int dp'\rho_n(p'). \tag{17}$$

In the steady-state,

$$(\Omega_{2n}+p)\rho_{2n}(p)=i\frac{\lambda\zeta}{(2n)^{2/3}}\frac{\partial\rho_0}{\partial p}\int dp'\rho_{2n}(p')+i\frac{\lambda\zeta}{2n^{2/3}}\frac{\partial\rho_n}{\partial p}\int dp'\rho_n(p'). \tag{18}$$

Neglecting the last term driven by the coupling to ρ_n one would get dispersion equation for the mode $2n$ which has the same structure as dispersion equation for the mode n but with λ replaced by $\lambda/2^{2/3}=\lambda/1.58$. Therefore, the mode $2n$ is stable for λ above λ_{th} of the n-th mode but for $\lambda<1.58\lambda_{th}$. In this range, we can retain only the second driving term. Then, using $\int dp\rho_n(p)=d_n$, we get

$$\rho_{2n}(p)=i\frac{\lambda\zeta d_n}{(2n)^{2/3}}\frac{\partial\rho_n/\partial p}{(\Omega+p)}. \tag{19}$$

Using solution Eq. (6) for ρ_n, we get for the amplitude $d_{2n}=\int dp\rho_{2n}$,

$$|d_{2n}|=|d_n|^2|\frac{\lambda\zeta}{4n^{2/3}}|F[\frac{n^{2/3}}{\lambda}]|, \tag{20}$$

where

$$F[\frac{n^{2/3}}{\lambda}]=\frac{\lambda\zeta}{\sqrt{2\pi}n^{2/3}}\frac{\partial^2}{\partial\Omega^2}\int\frac{dp(\partial\rho_0/\partial p)}{\Omega+p}. \tag{21}$$

Function $|F|$ is plotted in Fig.3. It is of the order of $F\simeq 2.5$ at $\sqrt{2\pi}n^{2/3}/\lambda$ where mode ρ_n become unstable. Hence, $|d_{2n}|\simeq|d_n|^2$ and the excitation of the mode ρ_{2n} is negligible for amplitudes $|d_n|<<1$.

6 Appendix: Effect of screening

Here we reproduce the known result of screening on the CSR impedance. For the coasting beam in a free space, the real part of the CSR impedance in this case is given by the spectral density of single particle synchrotron radiation power $P(n)=P(\omega)/\omega_0$,

$$(e^2/\pi R)Re[Z(\omega)]=P(\omega). \tag{22}$$

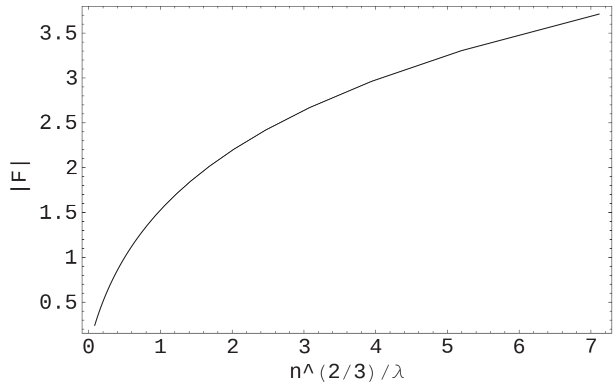

Figure 3: Function $|F|$, see text.

For a bunched beam with the bunch length l_b and bunch population N_B the CSR of a bunch as a whole is possible only for harmonics $n < R/l_b$. The previous results has to be modified in two ways. First, the linear density $n_B = N_B/l_b$ in Eq. (5). Another modification is needed if the bunch is moving in the beam pipe with finite aperture b. In this case, CSR is suppressed [4] at low harmonics $n <\simeq (R/b)^{3/2}$.

This result can be easily understood [7]. The n-th harmonics radiates within the angle θ with the plane of motion of the order of $\theta \simeq n^{-1/3}$. Such a harmonics has transverse $k_\perp = n(\omega_0/c)\theta$ and can propagate in the beam pipe only if $k_\perp$ is above the cut-off π/b. Hence, only modes with $n > (\pi R/b)^{3/2}$ can be radiated within the beam pipe. This condition and the condition $n < l_b/R$ can be fulfilled only for sufficiently short bunches $l_b < b\sqrt{\pi b/R}$. For longer bunches, CSR can exist only due to microbunching with longitudinal dimensions smaller than l_b. The impedance of the CSR which takes into account the screening effect can be derived from the spectrum of single particle synchrotron radiation [6]. For $\theta >> 1/\gamma_0$,

$$P(\omega,\theta) = \frac{ce^2}{R^2}\frac{n^2}{6\pi^3}[\theta^4 K^2_{2/3}(\frac{n}{3}\theta^3) + \theta^4 K^2_{1/3}(\frac{n}{3}\theta^3)]. \qquad (23)$$

The real part of the impedance can be obtained from Eq. (14), integrating $P(\omega,\theta)$ over the solid angle $d\Omega = 2\pi d\theta$ with the condition $|\theta| > \pi R/(nb)$,

$$Re[Z(n\omega_0)] = Z_0\frac{2n^2}{3\pi}\int_{\pi R/(nb)} d\theta[\theta^4 K^2_{2/3}(\frac{n}{3}\theta^3) + \theta^4 K^2_{1/3}(\frac{n}{3}\theta^3)]. \qquad (24)$$

or, changing variables,

$$Re[Z(n\omega_0)]/Z_0 = n^{1/3}\Phi(y), \quad y = \frac{1}{3n^2}(\frac{\pi R}{b})^3, \qquad (25)$$

where

$$\Phi[y] = \frac{2}{(3)^{1/3}\pi}\int_y^\infty dx x^{2/3}[K^2_{2/3}(x) + K^2_{1/3}(x)]. \qquad (26)$$

Function $\Phi(y)$ is shown in Fig. 4. For small y (large aperture b) $F \to F(0) = 1.62$ and gives impedance used above. In the screening regime of large y, $F(y)$ goes to zero as $F(y) \to (2/3^{1/3})y^{-1/3}e^{-y}$.

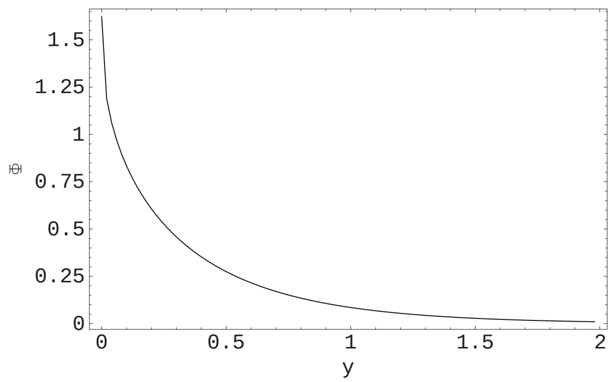

Figure 4: Screening effect, function $\Phi(y)$, see text

Fig. 4 shows that impedance is suppressed above $y \simeq 1$, i.e. for harmonics $n <\simeq (\pi R/b)^(3/2)$ which confirm the simple estimate given above.

7 Conclusion

For given beam pipe aperture, CSR of a bunch as a whole is possible only for relatively short bunches. Experiment [1] indicates that CSR is still possible in this case if there is a micro-structures within the bunch. The CSR itself can produce such a bunch modulation provided that linear bunch density is above a threshold value λ_{th}. For Gaussian bunch, $1.6\lambda_{th}/n^{2/3} = 1$. The most unstable mode has harmonic number $n = (\pi R/b)^{3/2}$. Dynamics of the instability above the threshold may lead to steady state saturation of the mode amplitude and modification of the density profile of the bunch. Eq. (20) gives the estimate of the equilibrium amplitude of modulation set by the reverse effect of the growing mode on the distribution function.

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