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Comments on an Expanding Universe

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Abstract

Various results are obtained for a Friedmann-Robertson-Walker cosmology. We derive an exact equation that determines Hubble's law, clarify issues concerning the speeds of faraway objects and uncover a "tail-light angle effect" for distant luminous sources. The latter leads to a small, previously unnoticed correction to the parallax distance formula.

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I. Introduction

The standard Friedmann-Robertson-Walker the metric[1] in spherical-like coordinates r, θ and ϕ is given by

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)\left(\frac{dr^2}{(1-kr^2)} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right) \quad . \tag{1.1}$$

where $R(t)$ is the cosmic scale factor at time t, τ is proper time, and k determines whether three-space is a sphere ($k = +1$), a flat space ($k = 0$), or a hyperbolic sphere ($k = -1$).[2, 3, 4, 5]

The metric in Eq.(1.1) is a specific case of

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t)\left(g_{ij}(x) dx^i dx^j\right) \quad , \tag{1.2}$$

where g_{ij} is a function of the spatial coordinates only. In going from time t to a slightly later time t', each region of space stretches by the same factor $R(t')/R(t)$. Due to this stretching, faraway objects are carried away from any particular observer moving with recessional speeds v_r that increase with the distance:[6]

$$v_r = Hr + \ldots \quad , \tag{1.3}$$

where r is the distance to the object. The corrections to Eq.(1.3) vanish as $Hr/c \to 0$. Hence for $Hr/c \ll 1$, the linear Hubble law

$$v_r \approx Hr \quad , \tag{1.4}$$

is an excellent approximation. Hubble's constant is $H(t) = \dot{R}(t)/R(t)$.

In obtaining Eq.(1.4), it is assumed that the observer and the nearby objects do not have perculiar velocities. Throughout our work, we shall make use of a system of comoving observers. These comovers have values of the coordinates x that are fixed in time so that, according to the metric in Eq.(1.2), the distance between two nearby comovers increases by a factor of $R(t_2)/R(t_1)$ between times t_1 and t_2.

There are several commonly used distances to specify the spatial separation of a faraway object from Earth: the proper distance d_{prop}, the luminosity distance d_{lum} determined by apparent brightness, the parallax distance $d_{parallax}$, the angular size distance d_{as} obtained by measuring apparent width, and the time-of-flight distance d_{tof} given by $c(t_{obs}-t_{em})$ where t_{em} is the time at which a distant object emits a light ray and t_{obs} is the time at which it is observed on Earth. These distances agree with one another with increasing accuracy as the object approaches the Earth, but differ significantly when the object is very faraway.

Some of the above versions of distant violate the principle that instantaneous non-local measurements cannot be made. An example is proper distant. Speeds computed on the basis of proper distant are therefore unphysical.

Proper distant is an instantaneous measure of spatial separation that can be achieved only by engaging in a "conspiracy" of multiple observers. Let $d_{prop}(t)$ be the proper distance between an object (e.g., a luminous source) and an observer (e.g., an astronomer). Arrange in advance for a series of comoving observers to be positioned between the two and instruct them to measure at the common time t the distance to the next neighbor. See Figure 1. Let $\Delta x_{i+1,i}$ be the measured distance between observers i and $i+1$. Then arrange for the observers to get together later to sum their measurements:

$$d_{prop}(t) = \sum_i \Delta x_{i+1,i} \quad . \tag{1.5}$$

Since, at a latter time t', the distances $\Delta x_{i+1,i}$ all increase to $R(t')/R(t)\Delta x_{i+1,i}$,

$$d_{prop}(t') = \frac{R(t')}{R(t)} d_{prop}(t) \quad , \tag{1.6}$$

so that proper distance scales exactly with the cosmic scale factor.

Define $v_{prop}(t)$ to be the rate of change of proper distance with respect to time: $v_{prop}(t) \equiv \partial d_{prop}(t)/\partial t$. Then using d_{prop} as *the* definition of distance and *assuming* v_{prop} is the appropriate measure of speed, one would conclude that the Hubble law is exactly linear:

$$v_{prop}(t) = \frac{\partial d_{prop}(t)}{\partial t} = \frac{\dot{R}(t)}{R(t)} d_{prop}(t) = H(t) d_{prop}(t) \quad . \tag{1.7}$$

Indeed, any definition of distance that scales exactly with $R(t)$ as in Eq.(1.6) obeys such a linear Hubble law. Since $d_{prop}(t)$ can be made arbitrarily large, one finds, with these definitions of speed and distance, that distant objects travel faster than the speed of light c. An interesting result in ref.[7] states that the proper distance for which $H(t)d_{prop}(t) = c$ can actually occur within the particle horizon, that is, within the observable universe. It is sometimes stated that the Hubble law is exactly linear and that faraway objects can move away from Earth at a rate exceeding c.[2, 7, 8] However, the use of d_{prop} as *the* definition of distance is not physical as emphasized above in that it is impossible for an observer to make an instanteous measurement of it.

However, for metrics of the form in Eq.(1.2), there are definitions of distant that are physical and causal for which recessional speeds never exceed that of light. The basic idea is to use a dense network of comoving observers throughout the universe, who are allowed to make local measurements, that is, measurements in a small region centered about their positions. Non-local measurements are then achieved by communicating the local results to one another and by using relativistic dynamics. One needs to use relativistic dynamics because sizeable speeds enter for very distant objects. Any definition of distant that uses only local measurements and respects the principles of special relativity cannot lead to speeds of objects exceeding the speed of light.

In the process of carrying out our analysis, we also uncover an angle effect not previously noted in general relativity. The angle between two rays as measured by an observer in the vicinity of the rays but far from the source is not the same as the angle between the rays as emitted by the source. It is obvious that such an effect should exist: In special relativity, there is the "tail light" effect: The light from a receding source is observed to spread out. Since distant objects are moving away from one another in an expanding universe, the "tail light" effect should be present and, indeed, it is. This leads to a small correction to the standard formula for parallax distance.

One way to illustrate how angles can change with time is as follows: Consider two nearby comoving observers. The two agree to send out light rays in a direction perpendicular to the line between them. See Figure 2. Then since space is expanding, the angle between the light rays will initially be slightly greater than zero and seen to increase with time by any local observer.

The uncovering of the "tail light" effect was actually the main motivation for the current work. Recent redshift data of distant type Ia supernovae suggest that the expansion of the universe is accelerating.[9]. This is contrary to what most cosmologists had expected. Physically, distant supernovae appear to be dimmer than expected. The "tail light" effect could be the explanation if it has not been previously properly taken into account. However, using local comoving observer measurements, we obtain the standard formula for the luminosity distance of a light source. Hence, the "tail light" effect is not the origin of the unexpected faintness of distant supernovae. To account for an accelerating universe, a cosmological constant or some other dark energy contribution does need to be invoked.

II. An Exact (Differential) Hubble Law Equation

This section derives a new form of Hubble's law by determining the corrections to Eq.(1.3). It is straightforward to obtain an exact equation for the recessional velocity v as a function of distance r from the Earth. First establish a network of comoving observers. Each observer sees the comovers in its vicinity moving away according to the Hubble law in Eq.(1.4). See Figure 3.

Suppose that the recessional speed $v(r)$ at r has been determined using local measurements by comoving observers. Two speeds are involved in determining $v(r + \Delta r)$ at a slightly farther distance: (1) The comover at r observes that a comover Δr further out moves with a speed of $H\Delta r$ and (2) the comover at r is moving away from Earth with a speed of $v(r)$. Using the relativistic formula for the addition of velocities, one finds that that the comover's speed at $r + \Delta r$ is

$$v(r + \Delta r) = \frac{v(r) + H\Delta r}{1 + \frac{v(r)H\Delta r}{c^2}} \approx v(r) + H(1 - \frac{v^2(r)}{c^2})\Delta r \quad ,$$

or

$$\frac{dv(r)}{dr} = H(1 - \frac{v^2(r)}{c^2}) \quad . \tag{2.1}$$

Eq.(2.1) is a fundamental equation which can be integrated to obtain the exact recessional speed as a function of distance. For r small, the $v^2(r)/c^2$ term can be neglected and one recovers the linear Hubble law.

The formula for v implicitly defines a distance r by $dr/dt = v(t)$, which can be integrated out to determine r if $r(t_0)$ is known for some early time t_0 and if the history of the universe is provided, that is, an exact formula for $H(t)$.

If H is constant, which is the case when the expansion is exponential ($R(t) = \exp(Ht)$) and which might have happened in the early universe during inflation[10], one finds

$$v(r) = c \tanh(Hr/c) = c\frac{\exp(Hr/c) - \exp(-Hr/c)}{\exp(Hr/c) + \exp(-Hr/c)} \quad , \tag{2.2}$$

which yields a result for $v(r)$ that is always less than c and approaches c only for $r \to \infty$. It is sometimes misstated that, in an inflationary universe, superliminary speeds are achieved. For r small, one recovers $v(r) = Hr + \ldots$ from Eq.(2.2).

In a Friedmann-Robertson-Walker universe, H is not constant and one must integrate Eq.(2.1) taking into account the variation of H with time. An example of how the integration is performed is provided below.

Because only local measurements are made that respect the principles of special relativity in deriving Eq.(2.1), no object is viewed as having a recessional speed greater that c. Indeed, as $v(r)$ approaches c, the factor $(1 - v^2(r)/c^2)$ in Eq.(2.1) reduces the incremental increase in speed. In support of Eq.(2.1), we have been able to show that a number of results pertinent to an expanding universe are obtainable from the differential Hubble law. Here, we restrict ourselves to only one example: a derivation of the redshift as a Doppler effect.

The expansion of the universe causes the light from a distance source to be shifted to the red because the wavelength λ of light is stretched:

$$\lambda(t_{obs}) = \frac{R(t_{obs})}{R(t_{em})}\lambda(t_{em}) \quad , \tag{2.3}$$

where $\lambda(t_{obs})$ is the wavelength of the light at the time that it is observed and $\lambda(t_{em})$ is the wavelength at the time that it is emitted. Eq.(2.3) holds for any metric of the form in (1.2).

Let us show how to obtain the redshift as a Doppler effect due to the recessional velocity. If a luminous source is receding from an observer at a speed v_{obs} then in special relativity

$$\lambda(t_{obs}) = \sqrt{\frac{c + v_{obs}}{c - v_{obs}}}\lambda(t_{em}) \quad . \tag{2.4}$$

To determine v_{obs}, one integrates Eq.(2.1) for the process in which light is emitted from a distant object and received by an observer on Earth. As the light propagates, the change in distance dr that it travels in dt is given by $dr = -cdt$. Using this in Eq.(2.1) gives

$$dv = -\left(1 - \frac{v^2}{c^2}\right) cH(t)dt \quad . \tag{2.5}$$

Separating variables and integrating from the initial time t_{em} to the final time t_{obs}, one obtains

$$\sqrt{\frac{c + v_{obs}}{c - v_{obs}}} = \exp(\int_{t_{em}}^{t_{obs}} H(t)dt) = \frac{R(t_{obs})}{R(t_{em})} \quad , \tag{2.6}$$

where the last equality holds because $H = dlog(R)/dt$. Substituting this result into Eq.(2.4) yields the result in Eq.(2.3). The derivation supports the validity of the differential Hubble law in Eq.(2.1) and illustrates how it is necessary to take into consideration the variation of the Hubble constant in integrating the equation.

It is sometimes stated that the redshift cannot be computed as a Doppler effect. The argument goes as follows. Suppose that one can vary the expansion factor $R(t)$ at will. Around the time of emission, adjust $R(t)$ so that it is constant. After emission, let $R(t)$ increase so that the universe expands and produces a redshift. Before observation, adjust $R(t)$ so that it is constant again. Then one might argue that, since the universe is not expanding during emission and observation, there is no relative velocity between emitter and observer during these processes, and hence no Doppler effect. There are several difficulties with this reasoning. First, it assumes that the relative speed between two distant objects can be instantaneously measured and hence is zero at the times of emission and observation. Second, the above derivation leading to Eq.(2.6) demonstrates unequivocally that the red shift *can* be computed as a Doppler effect for arbitrarily varying $R(t)$. It is clear from this computation that the recessional speed is "built up" during the entire period of light propagation and is not instantaneously produced. Third, changing $R(t)$ from a constant to a non-zero value creates an acceleration between the light and the observer (and also with the emitter). This acceleration generates a redshift as can can see as follows. Consider the line of comovers positioned between the source and final observer on Earth. Let each of these intermediate comovers absorb the light and instantly re-emit it. This has no effect on the light. During the periods for which $R(t)$ is unchanging, no redshift is generated. However, as soon as $R(t)$ increases, two nearby intermediate comovers achieve a relative velocity and the next one observes a redshift compared to the previous one that can be attributed to the acceleration of space or as a Doppler effect.

It is incorrect to incorporate both the Doppler effect and the stretching of space in determining $\lambda(t_{obs})/\lambda(t_{em})$: The redshift in general relativity in an expanding universe is due to the stretching of waves of light; Observers, however, have the option of viewing the the redshift as due a Doppler effect arising from the relative motion of sources and observers.

If a distant source has a peculiar velocity, then in general relativity there is both a cosmological redshift and a moving-source Doppler effect. It is easily checked that the use of (2.5) correctly produces the total wavelength shift as a single Doppler effect by using $v_{peculiar}$ for the initial speed and noting that the velocity addition formula in special relativity $v_3 = (v_1 + v_2)/(1 + v_1 v_2/c^2)$ can be written as

$$\sqrt{\frac{c+v_3}{c-v_3}} = \sqrt{\frac{c+v_1}{c-v_1}}\sqrt{\frac{c+v_2}{c-v_2}} \quad , \tag{2.7}$$

as can easily be checked.

III. The "Tail-Light" Effect

In this section, we compute the luminosity distance using a network of comoving observers because, among things, it allows us to uncover a "tail-light angle effect." In addition, since the luminosity distance is used in analyzing type Ia supernova data, a careful, detailed derivation of the formula is worth performing given that the tail-light effect has previously been overlooked.

If the absolute luminosity of an astrophysical object is known, then its apparent luminosity L_A can be used to determine a distance d_{lum} to the object[4]:

$$L_A \propto \frac{R^2(t_{em})}{R^2(t_{obs})} \frac{1}{d_{eff}^2} \quad , \tag{3.1}$$

where d_{eff}, the effective distance, is defined as

$$b_{obs} \equiv d_{eff} \phi_s \quad , \tag{3.2}$$

and b_{obs} is the impact parameter or distance measured by the observer of two nearby light rays emitted from the source with an angular separation at the source of ϕ_s. See Figure 4. The definition of luminosity distance d_{lum} is

$$d_{lum} \equiv \frac{R(t_{obs})}{R(t_{em})} d_{eff} \quad . \tag{3.3}$$

It remains to determine d_{eff}. Arrange for a set of equally spaced, intermediate comoving observers to be between the source and the receiver. Because the angle ϕ_s is typically extremely small, terms of order ϕ_s^2 may be neglected.

Figure 5(a) shows the initial emission of the two rays. Angles are display much larger than the actual case for clarity. Not surprisingly, observers disagree on what transpires as the rays move from the source to the intermediate comoving observer 1. According to observer 1, the light is emitted at a distance Δx at an angle ϕ_1, and it travels to the region of 1 while the source moves away at a recessional speed of v_{1s}. As the rays pass 1 at time t_1, the distance between the source and 1 becomes $R(t_1)/R(t_{em})\Delta x$ because space is expanding. The angle ϕ_1 is greater than ϕ_s because for a source that is moving away, one has the "tail light" effect of special relativity. See Figure 5(b). Using the standard formula for the relation between angles in special relativity for references frames moving with respect to one another,

$$\phi_1 = \sqrt{\frac{c + v_{1s}}{c - v_{1s}}} \phi_s = \frac{R(t_1)}{R(t_{em})} \phi_s \quad , \tag{3.4}$$

where the last equality follows from Eq.(2.6). The distance between the two rays b_1 as the light passes 1 is, according to observer 1,

$$b_1 = \Delta x \phi_1 = \frac{R(t_1)}{R(t_{em})} \Delta x \phi_s \quad . \tag{3.5}$$

According to the comoving observer at the source, the rays are emitted with an angular separation of ϕ_s, but, as the rays move from the source to 1, observer 1 moves away. The distance the light travels is $R(t_1)/R(t_{em})\Delta x$ and thus greater. Using this distance and angle, an observer at the source arrives at Eq.(3.5) for the impact parameter at 1, in accord with special relativity that moving observers agree on distances perpendicular to their relative motion. See Figure 5(c).

Now consider the process in which the rays travel from the region of intermediate comover 1 to the region of intermediate comover 2. It is convenient to consider a comoving observer at $1'$ where the upper ray passes near 1 and a corresponding comoving observer at $2'$ as in Figure 6(a). Because the observer at $1'$ is moving away from 1, the angle that the upper ray makes with a horizontal line perpendicular to the line running between 1 and $1'$ is less than ϕ_1. In fact, $1'$ is moving away with just the right speed to observe the angle as ϕ_s to order Δx^2. The situation for $1'$ and $2'$ in Figure 6 is thus similar to that of 1 and the source in Figure 5 except that the distance between $1'$ and $2'$ is $R(t_1)/R(t_{em})\Delta x$ instead of Δx. Let Δb_2 be the impact parameter seen by $2'$. Then

$$\Delta b_2 = \frac{R(t_2)}{R(t_1)}\frac{R(t_1)}{R(t_{em})}\Delta x\phi_s = \frac{R(t_2)}{R(t_{em})}\Delta x\phi_s \quad . \tag{3.6}$$

Since the primed comoving observers $1'$ and $2'$ are moving away from the unprimed observers 1 and 2 due to the expansion of the universe, the distance between them increases as the rays move from 1 to 2. Hence, the distance between primed and unprimed observers is $R(t_2)/R(t_1)b_1$ when the rays arrive in region 2 and the impact parameter b_2 at 2 is

$$b_2 = \frac{R(t_2)}{R(t_1)}b_1 + \frac{R(t_2)}{R(t_{em})}\Delta x\phi_s = \frac{R(t_2)}{R(t_{em})}2\Delta x\phi_s \quad . \tag{3.7}$$

As in the case of Figure 5, $1'$ and $2'$ disagree on what happens as the rays move from region 1 to 2 but agree on the value of b_2. See Figures 6(b) and 6(c).

The process in which rays go from comoving observer i to $i+1$ is similar to that of Figure 6 except the distance between i to $i+1$ is initially $R(t_i)/R(t_{em})\Delta x$ and the angle for the upper ray at i' is larger and equal to ϕ_i as seen by i. The comoving observer at i', however, sees the angle as ϕ_s. In place of Eq.(3.7), one has

$$b_{i+1} = \frac{R(t_{i+1})}{R(t_i)}b_i + \frac{R(t_{i+1})}{R(t_{em})}\Delta x\phi_s = \frac{R(t_{i+1})}{R(t_{em})}(i+1)\Delta x\phi_s \quad . \tag{3.8}$$

Equation (3.8) can be evaluated at the position of the receiver by setting $i = N-1$ for $N-1$ intermediate observers:

$$b_{obs} = \frac{R(t_{obs})}{R(t_{em})}\sum_i \Delta x\phi_s \quad . \tag{3.9}$$

Since $\sum_i \Delta x = N\Delta x = d_{prop}(t_{em})$, one concludes that

$$d_{eff} = \frac{R(t_{obs})}{R(t_{em})} d_{prop}(t_{em}) = d_{prop}(t_{obs}) \quad , \tag{3.10}$$

so that the luminosity distance in Eq.(3.3) is

$$d_{lum} = \frac{R(t_{obs})}{R(t_{em})} d_{prop}(t_{obs}) \quad , \tag{3.11}$$

which agrees with the standard result.[2, 4, 11] Although there is a "tail light" effect, it is not the reason why distant type Ia supernovae appear dimmer than expected.

The angle ϕ_i measured by the ith observer at time t_i is

$$\phi_i \approx \left(\phi_s + \frac{H(t_i)b_i}{c}\right) = \phi_s \left(1 + H(t_i)R(t_i)\int_{t_{em}}^{t_i} \frac{ds}{R(s)}\right) \quad . \tag{3.12}$$

This is the "tail light" effect: $\phi_i > \phi_{i-1} > \phi_s$. Since the "tail light" effect can be quite significant for very distant astronomical luminous objects, one might wonder why it has not been detected experimentally. The reason is that, although ϕ_{obs} can differ by ϕ_s by a sizeable factor, both ϕ_{obs} and ϕ_s are miniscule and hence not directly measurable in practice. For example, suppose the mirror of a telescope is 1 meter so that $b_{obs} \sim 1$ meter and that a supernova is observed with a redshift of $z \approx 0.5$. Then $\phi_{obs}/\phi_s \approx 1.5$ but the order of magnitude of either angle is 10^{-26} radians.

It only takes a "small factor" within the framework of an $\Omega = 1$ Friedmann-Robertson-Walker universe to obtain agreement with the type Ia supernova observations. If the ith primed observer would have observed the angle of the upper ray as $R(t_i)/R(t_{i-1})$ times the angle observed by the $(i-1)$th primed observer (instead of an angle of ϕ_s), then one would have found a somewhat larger value of d_{eff} of $\frac{R(t_{obs})}{R(t_{em})}d_{tof}$ and d_{lum} would become $\frac{R^2(t_{obs})}{R^2(t_{em})}d_{tof}$, which turns out to fit the supernova data[9] perfectly for a flat-space universe. More specifically, in a $k = 0$ matter-dominated universe, $d_{eff} = d_{prop} = 2c(1 - 1/\sqrt{1+z})/H_0$, whereas $d_{eff}^{phen} = \frac{R^2(t_{obs})}{R^2(t_{em})}d_{tof} = 3c(1 + z - 1/\sqrt{1+z})/(2H_0)$. Since the use of d_{eff}^{phen} fits the supernova data so well, it can be used as a phenomenological parametrization in current and future studes of the acceleration of the universe.

IV. A Correction to the Formula for Parallax Distance

The definition of parallax distance is

$$d_{parallax} = \frac{b_{obs}}{\phi_{obs}} \quad , \tag{4.1}$$

where ϕ_{obs} and b_{obs} are respectively the observed angle and distance between two rays. See Figure 4. Because of the "tail light" effect, ϕ_{obs} is greater than ϕ_s, and the location of the source appears closer than otherwise would be the case.

Using the results for b_{obs} and ϕ_{obs} in Eqs.(3.9) and (3.12) of the last subsection, one finds

$$d_{parallax} = \frac{d_{prop}(t_{obs})(1 - kd_{prop}(t_{obs})^2/R(t_{obs})^2)^{-1/2}}{1 + H(t_{obs})d_{prop}(t_{obs})/c} \quad , \tag{4.2}$$

which differs from the standard result[4] by the factor in the denominator. It is important to note that since parallax measurements in astronomy are made at positions fixed to the center of the solar system, non-comoving observers are used. This is the reason why ϕ_{obs} should be used and not ϕ_s (compare unprimed and primed observers of the last subsection). Since parallax is currently only used for relatively nearby astrophysical objects, the denominator correction factor numerically does not significantly affect parallax distant measurements.

V. Conclusions

In this research, we clarified several issues concerning the physics of a Friedmann-Robertson-Walker cosmology and derived several new results. In particular, with the use of reasonable definitions, recessional speeds no longer exceed the speed of light. More importantly, we obtained a new, fundamental equation governing Hubble's law. There are statements in the literature that recessional speeds can exceed c and that the Hubble law is exactly linear, but they are based on definitions requiring non-local instantaneous measurements. We found a correction factor for parallax distance that had previously been overlooked. Another new result is that the light rays from a distant source spread out. This "tail light" effect, however, does not explain why recent distant type Ia supernovae appear dimmer than expected and therefore does not provide a way of avoiding the conclusion that the supernova data supports an accelerating expanding universe. Finally, we have uncovered a nice phenomenological fit for the type Ia supernova data.

Acknowledgments

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Figure Captions

Figure 1. Arrangement of Intermediate Comoving Observers to Compute the Proper Distance Between a Source and an Observer

Figure 2. The Spreading Out of Rays Emitted in the x-Direction by Separated Comoving Sources

Figure 3. The Recessional Velocity Vectors in a Region of a Comoving Observer

Figure 4. Rays Emitted at Small Angles by a Source and Measured by a Faraway Observer

The angle between the rays seen by the observer is larger than the angle at emission so that when the rays are projected back, they converge on a distance $d_{parallax}$, which is closer than one would obtain if the "tail light" angular effect were to be neglected.

Figure 5. The Computation of the Impact Parameter b_1 at the First Intermediate Observer; (a) The Initial Situation at Time t_{em} as Viewed by Observer 1; (b) The Process from the Viewpoint of Observer 1; (c) The Process from the Viewpoint of an Observer at the Source.

Figure 6. The Computation of the Impact Parameter at the Second Intermediate Observer; (a) The Situation When the Ray Passes 1 at Time t_1; (b) The Process from the Viewpoint of Observer $2'$; (c) The Process from the Viewpoint of Observer $1'$.

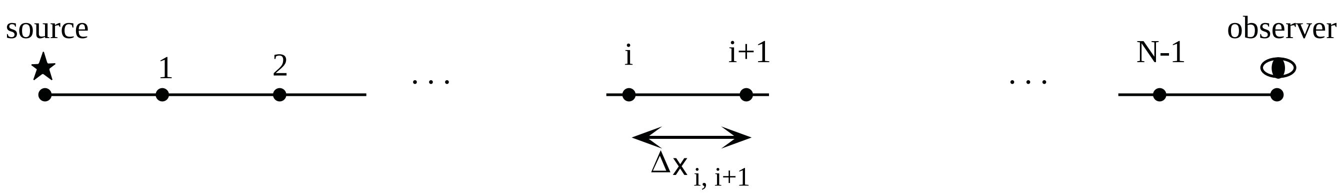


Figure 1

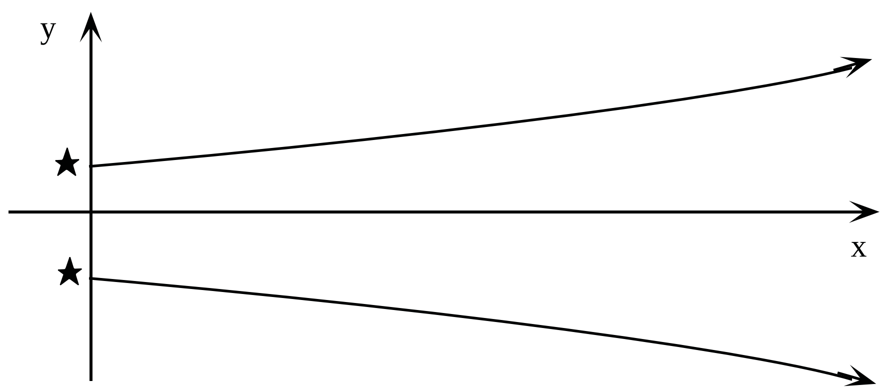


Figure 2

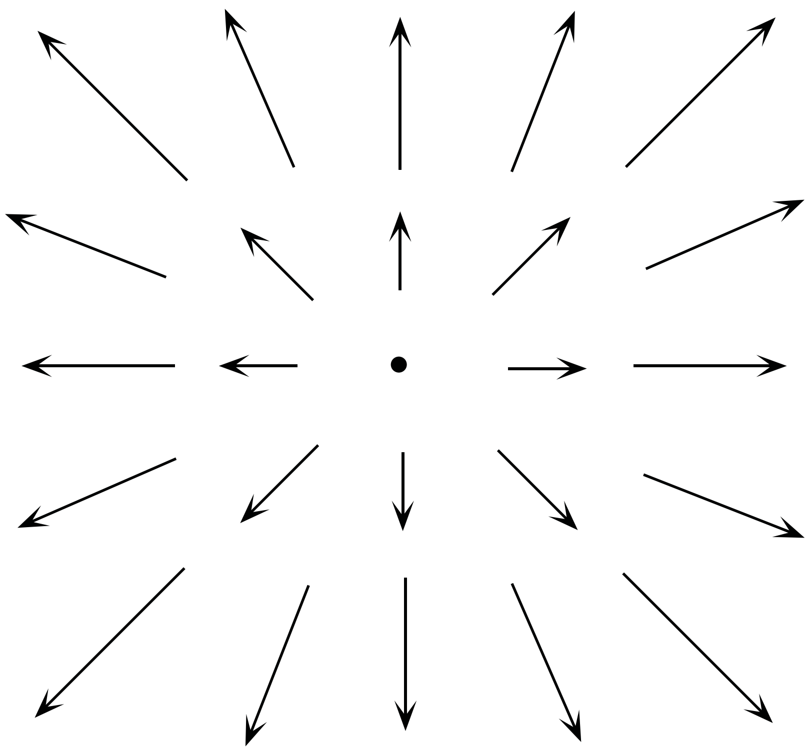


Figure 3

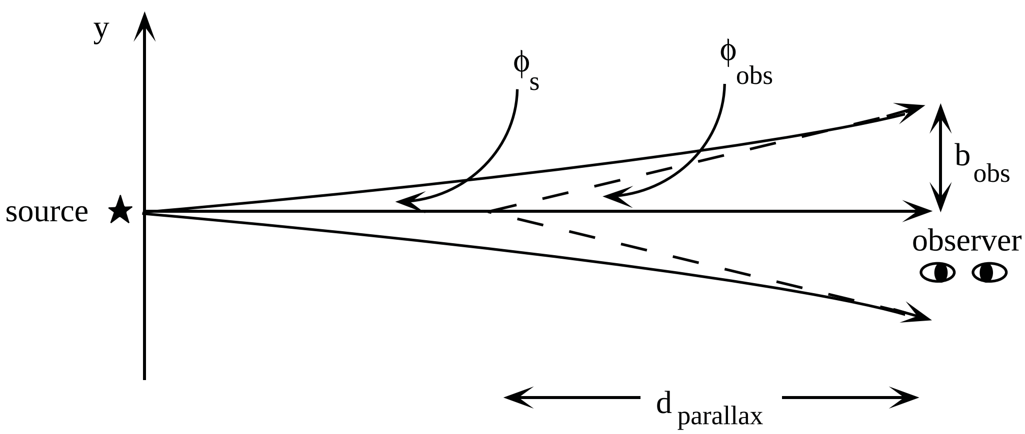


Figure 4

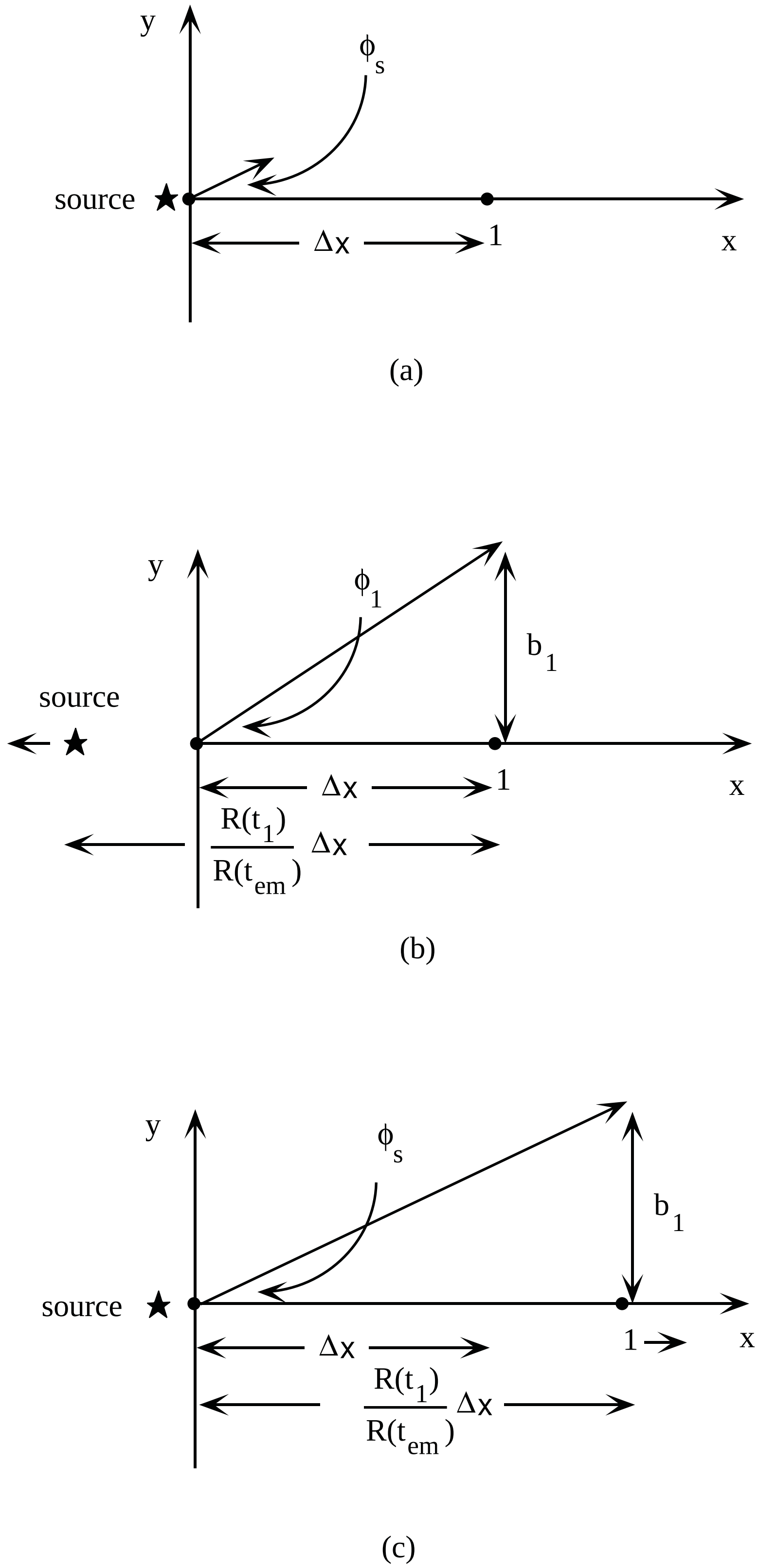
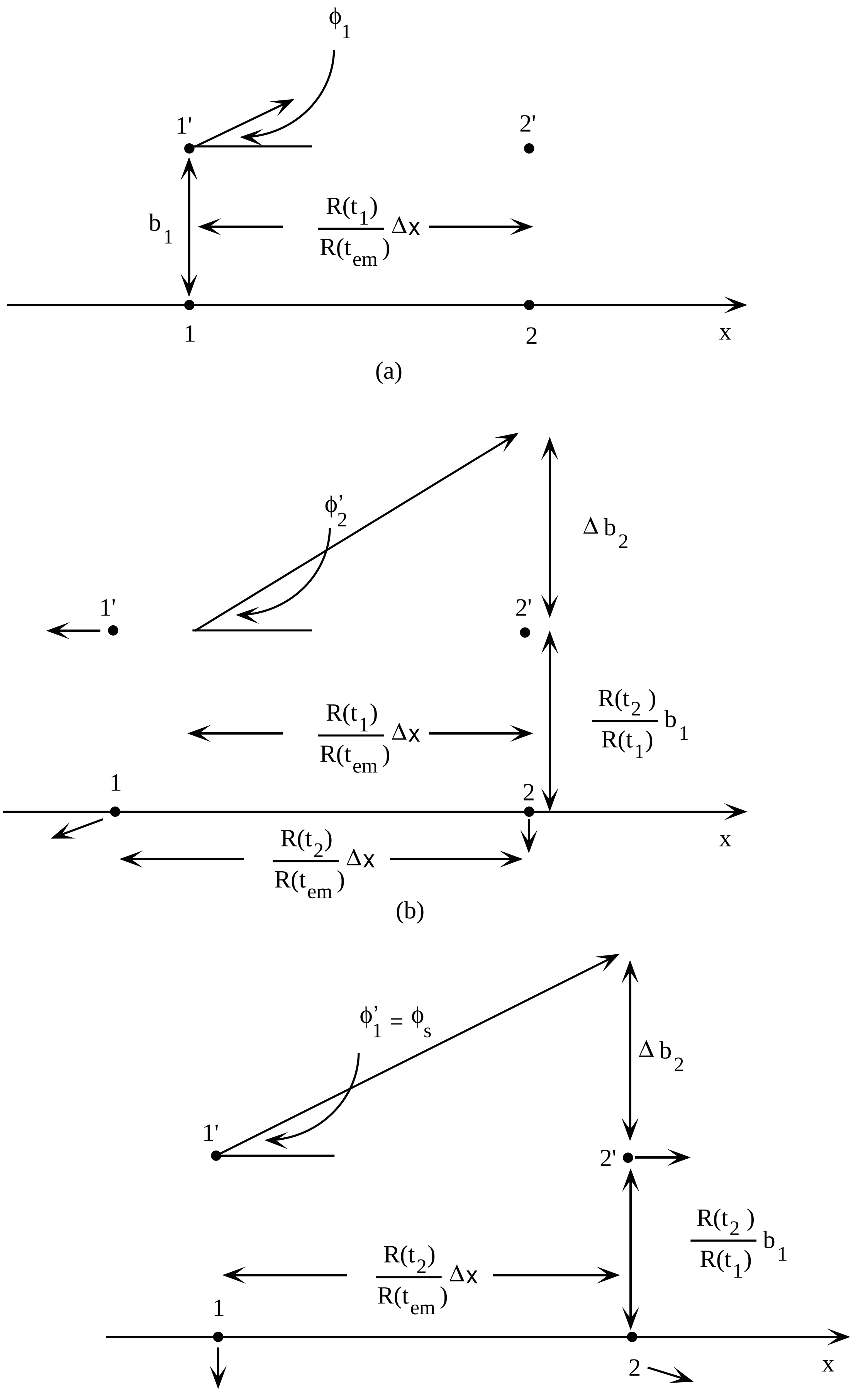

y
ϕ_s
source
Δx
1
x
(a)
y
ϕ_1
b_1
source
Δx
1
x
$\frac{R(t_1)}{R(t_{em})}\,\Delta x$
(b)
y
ϕ_s
b_1
source
Δx
1
x
$\frac{R(t_1)}{R(t_{em})}\,\Delta x$
(c)

Figure 5



ϕ_1
1'
2'
b_1
$\frac{R(t_1)}{R(t_{em})}\Delta x$
1
2
x
(a)
ϕ'_2
Δb_2
1'
2'
$\frac{R(t_1)}{R(t_{em})}\Delta x$
$\frac{R(t_2)}{R(t_1)} b_1$
1
2
x
$\frac{R(t_2)}{R(t_{em})}\Delta x$
(b)
$\phi'_1 = \phi_s$
Δb_2
1'
2'
$\frac{R(t_2)}{R(t_1)} b_1$
$\frac{R(t_2)}{R(t_{em})}\Delta x$
1
2
x
(c)

Figure 6