

## Possible Quantum Mechanical Effect on Beam Echo\*

Alex Chao and Boaz Nash  
Stanford Linear Accelerator Center, Stanford University, CA 94309

### Abstract

The echo effect in charged particle beams provides a link between macroscopic measurable beam parameters and microscopic phase space motion of the beam. Since quantum mechanics dictates a granularization of the phase space, it influences how the phase space behaves microscopically, and thus potentially affect how the echo effect behaves macroscopically. In this study, we propose to examine the possible measurable macroscopic effects of quantum mechanics on beams through its echo effect.

*Presented at the 18th Advanced ICFA Workshop  
on Quantum Aspects of Beam Physics  
Capri, Italy  
October 15 - 20, 2000*

---

\*Work supported by Department of Energy contract DE-AC03-76SF00515.

## Possible Quantum Mechanical Effect on Beam Echo

Alex Chao, Boaz Nash

Stanford Linear Accelerator Center

The echo effect [1] in charged particle beams provides a link between macroscopic measurable beam parameters and microscopic phase space motion of the beam. Since quantum mechanics dictates a granularization of the phase space, it influences how the phase space behaves microscopically, and thus potentially affect how the echo effect behaves macroscopically. In this study, we propose to examine the possible measurable macroscopic effects of quantum mechanics on beams through the echo effect.

Figure 1 shows a schematic for the classical echo effect. We consider transverse motion in 1-D giving a 2-D phase space with coordinates  $(q, p)$ . The motion is taken to be that of a harmonic oscillator with frequency  $\omega$  with a perturbative non-linear term. Thus, a particle of mass  $m$  will follow a circular trajectory (in scaled  $(q, p/m\omega)$  phase space) with a frequency weakly dependent on initial amplitude. At  $t = 0$  the beam is subjected to a dipole kick resulting in a translation of the phase-space distribution. Subsequently, the beam distribution filaments because of the non-linearity and will fill up an annular region. This filamentation results in a diminishing of the beam centroid signal until it is essentially zero. After a sufficient amount of time  $\tau$ , the beam is subjected to a second, quadrupole kick which causes a squeezing of phase space. Subsequently, the squeezed filamented phase space continues to evolve in some sense undoing the filamentation process such that at a time  $2\tau$  the phase space becomes

bunched again, yielding a non-zero centroid signal. This beam centroid signal at time  $2\tau$  is called the echo.

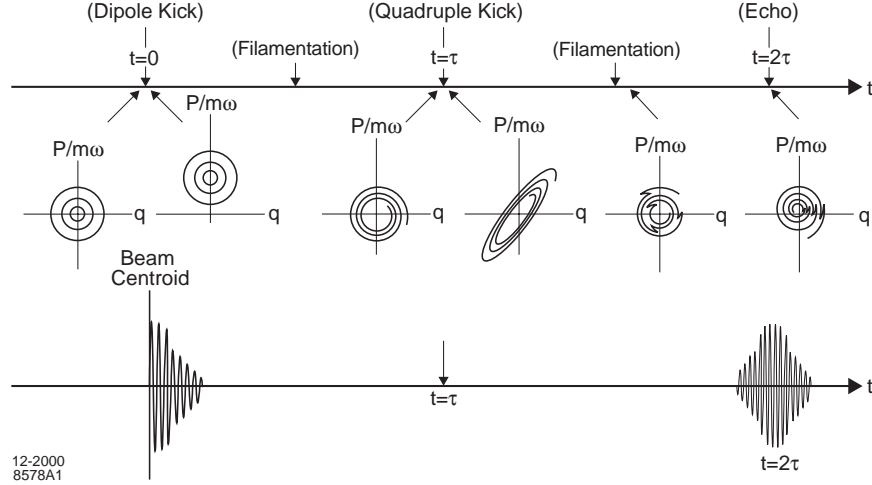


Figure 1: Schematic for the classical echo effect.

The mechanism of the echo effect depends on a detailed microscopic correlation of phase space dynamics over a long time period of  $2\tau$ . This property has led to the application of using the echo as a sensitive measure to detect slow diffusion of particle motion in phase space [1, 2]. The question being asked here is whether quantum mechanical requirement on phase space would also affect the echo signal over the long time  $2\tau$ .

The simplest example of quantum mechanical effect on beam observables can be found by replacing the point-particles by wavepackets in an equilibrium beam distribution in phase space with rms size  $\sigma$ . If all wavepackets are displaced

quantum mechanical ground states, then the beam emittance is found to be

$$\epsilon = \begin{cases} m\omega\sigma^2, & \text{classical} \\ m\omega\sigma^2 + \frac{1}{2}\hbar, & \text{quantum mechanical} \end{cases} \quad (1)$$

This is a quantum mechanical contribution to beam emittance. However, the magnitude of the quantum mechanical contribution is too microscopic to be measured (unless one has an extremely cold beam). It is hoped that a quantum mechanical effect on echo could be more macroscopic.

In obtaining Eq.(1), we have assumed that particles do not interact with one another. This excludes any space charge Columb interactions. It also ignores the effect of the exclusion principle. The latter is particularly important when the beam is extremely cold [3].

It is possible that the *centroid* motion would not be affected by quantum mechanical considerations when one ignores the exclusion principle. However, as demonstrated by Eq.(1), quantum mechanics does affect the beam's *second moments*. It is therefore possible that the first quantum mechanical effect on echo occurs to the higher order echo when the beam is first kicked by a quadrupole kick, followed by a sextupole kick, and the echo signal is to appear by observing the second moment of the beam.

We would like to model the foregoing classical narrative of the echo effect using quantum mechanical wavepackets instead of classical point-particles. Thus, we must translate each component of such an echo experiment into quantum mechanical language. Because transverse motion is non-relativistic, we can use the Schrödinger equation to describe the propagation of single-particle wavepackets [7, 8]. The picture of an evolving distribution in phase space gives way to

an evolving Wigner function [5, 8] from which macroscopic parameters such as beam centroid position or beam emittance can be derived.

There are two “active” elements to the echo experiment described above. The beam is first given a dipole kick, or displaced, and then given a quadrupole kick, or squeezed. These two transformations can be described via two unitary operators, namely, the *displacement operator* and the *squeeze operator* [4]. In the absence of non-linearities, the effect of the displacement and squeeze operators on the time dependence of the quantum state of each particle can be obtained by standard quantum mechanical treatments. However, in order to model the filamentation process in the beam, we need to treat the case including non-linearity, ideally, by generalizing the linear analysis. We know that  $\langle q \rangle$  and  $\langle p \rangle$  will follow the classical non-linear equations of motion and perform the classical filamentation. What happens to the shapes of the wavefunctions over time, however, is not readily apparent, and constitutes the main focus of the subsequent study.

In classical mechanics one replaces the detailed granularity of individual particles in phase space with a time dependent phase space density function in order to talk about macroscopic beam properties. Examples of such beam properties are emittance and beam centroid position. One calculates such quantities in quantum mechanics using the Wigner function.

Given the initial wavefunctions  $\psi_j(q, 0)$  for all particles  $j = 1, 2, \dots, N$ , we can compute  $\psi_j(q, t)$ . We propose to write the answer as  $\psi(q, t; q_0, p_0)$  where  $(q_0, p_0)$  are the initial centroid coordinates of an injected particle. We then compute

the statistical Wigner function as

$$W(q, p, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dq_0 \psi_0(q_0, p_0) \times \int_{-\infty}^{\infty} dy \psi^*(q + \frac{y}{2}, t; q_0, p_0) \psi(q - \frac{y}{2}, t; q_0, p_0) e^{-ipy/\hbar} \quad (2)$$

For the case of the harmonic oscillator, it turns out that there is a shortcut to obtaining the time-dependent Wigner function. One can show that the Wigner function as defined above satisfies

$$W(q, p, t) = W(\bar{q}, \bar{p}, 0) \quad (3)$$

where

$$\bar{q} = q \cos \omega t + \frac{p}{m\omega} \sin \omega t \quad (4)$$

$$\bar{p} = p \cos \omega t - m\omega q \sin \omega t. \quad (5)$$

Thus, the Wigner function simply rotates with frequency  $\omega$ .<sup>†</sup>

As an example, we calculate the Wigner function corresponding to a Gaussian distribution of “squeezed state” [4] particles. At  $t = 0$  we let

$$\psi(q, 0; q_0, p_0) = \left(\frac{b^2}{\pi}\right)^{\frac{1}{4}} e^{-\frac{b^2}{2}(q-q_0)^2 + \frac{ip_0}{\hbar}(q-q_0)}, \quad (6)$$

where  $b$  is real positive number. For  $b \neq \beta = \sqrt{\frac{m\omega}{\hbar}}$  we have a squeezed state.

Because of Eq. (3), we just use the  $t = 0$  wave-functions to compute the Wigner function. First, we compute

$$\int_{-\infty}^{\infty} dy \psi^*(q + \frac{y}{2}, t; q_0, p_0) \psi(q - \frac{y}{2}, t; q_0, p_0) e^{-ipy/\hbar} = 2e^{-b^2(q-q_0)^2 - \frac{\beta^4}{b^2}(\frac{p+p_0}{m\omega})^2} \quad (7)$$

<sup>†</sup>The obvious generalization for the non-linear case where  $q$  and  $p$  would simply follow the classical motion is, unfortunately, not generally true. See [6] and references therein, but note their slightly different definition of the Wigner function in comparing our Eq. (3).

To compute the  $t = 0$  Wigner function, we need to multiply by the classical distribution and integrate over  $q_0$  and  $p_0$ . For our example, we choose a centered Gaussian distribution

$$\psi_0(q_0, p_0) = \frac{1}{2\pi\sigma_q\sigma_p m\omega} e^{-\frac{q_0^2}{2\sigma_q^2} - \frac{p_0^2}{2\sigma_p^2 m^2 \omega^2}}. \quad (8)$$

Computing the  $t = 0$  Wigner function, and putting in the time-evolution, the result is

$$W(q, p, t) = \frac{1}{\pi\hbar\sqrt{(2\sigma_q^2 b^2 + 1)(2\sigma_p^2 \frac{\beta^4}{b^2} + 1)}} e^{-\frac{b^2}{2\sigma_q^2 b^2 + 1} \bar{q}^2 - \frac{\left(\frac{\beta^4}{b^2}\right)}{2\sigma_p^2 \left(\frac{\beta^4}{b^2} + 1\right)} \frac{\bar{p}^2}{m^2 \omega^2}} \quad (9)$$

which is a rotating bi-Gaussian. The Wigner function for the displaced Gaussian classical distribution which occurs in the discussion of the echo effect can be similarly computed.

For the case  $b = \beta$ , the Wigner function reduces to

$$W(q, p, t) = \frac{1}{\pi\hbar\sqrt{(2\sigma_q^2 \beta^2 + 1)(2\sigma_p^2 \beta^2 + 1)}} e^{-\frac{\beta^2}{2\sigma_q^2 \beta^2 + 1} \bar{q}^2 - \frac{\beta^2}{2\sigma_p^2 \beta^2 + 1} \frac{\bar{p}^2}{m^2 \omega^2}} \quad (10)$$

Taking the limit in which  $\sigma_q = \sigma_p = \sigma$ , the Wigner function gains rotational symmetry in  $(q, \frac{p}{m\omega})$  so that the time-dependence goes away:

$$W(q, p, t) = \frac{1}{\pi\hbar(1 + 2\sigma^2 \beta^2)} e^{-\frac{\beta^2}{1 + 2\sigma^2 \beta^2} (q^2 + \frac{p^2}{m^2 \omega^2})} \quad (11)$$

One can show that the quantum emittance given by Eq.(1) follows from this equation.

A generalization of the calculation is needed when a small non-linearity is added.

We thank P. Chen, A. Dragt, A. Kabel, M. Venturini for many useful discussions.

## References

- [1] G. Stupakov, SSCL Report 579 (1992).
- [2] L.K. Spenzouris, J.-F. Ostiguy, P.L. Colestock, Phys. Rev. Lett. 76, 620 (1996).
- [3] A. Kabel, this conference.
- [4] L. Mandel and E. Wolf, "Optical Coherence and Quantum Optics", Cambridge, 1995.
- [5] A. Dragt, p.655-657, Proc. Quantum Aspects of Beam Physics, World Scientific, 1999.
- [6] T. Curtright and C. Zachos, U. Miami report TH/1/98, ANL report ANL-HEP-PR-98-132 (1998).
- [7] Z.R. Huang, P. Chen, and R. Ruth, Phys. Rev. Lett. 74, 1759 (1995).
- [8] M. Venturini, this conference.