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Quantum-mechanical Analysis of Optical Stochastic Cooling

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Abstract

Quantum theory of the optical stochastic cooling [1] is presented. Consideration follows the evolution of the density matrix of a bunch of particles interacting with radiation in the undulators and quantum amplifier.

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I. INTRODUCTION AND BASICS OF OSC

In the classical rf stochastic cooling [3], the signal from a pickup is amplified and send to a kicker where it affects the same particle changing its momentum from initial p_j to $\bar{p}_j$. With the proper choice of the time delay, this produces damping. The signal affects also other particles within the length $c/(2\Delta f)$ defined mostly by the (full) bandwidth of the amplifier Δf producing diffusion and emittance growth. The balance of two processes can be understood from the equation for the momentum of the j-th particle:

$$\bar{p}_j = p_j - \Lambda p_j - \Lambda \sum_{i \neq j} p_i, \tag{1}$$

where Λ is parameter proportional to the electronic gain of the amplifier. The rms value $\Delta^2 = (1/N_b) \sum_j [< p_j^2 > - < p_j >^2]$ for initially uncorrelated particles changes to $\bar{\Delta}^2 = \Delta^2 [1 - 2\Lambda + N_s \Lambda^2]$, where N_s is number of particles interacting through the amplifier (number of particles per "slice"). The maximum cooling rate $(\bar{\Delta}^2 - \Delta^2)/\Delta^2 = -\Lambda$, and is achieved for $\Lambda = 1/N_s$. The number of particles per slice $N_s = N_B c/(\sigma_B^0 \Delta f)$, where σ_B^0 is the bunch length in the laboratory frame, and N_B is number of particles per bunch.

To achieve fast cooling (for example, for muon collider) it is desirable to reduce N_s. Optical stochastic cooling was proposed recently [2],[1] and can be applied for short bunches. In the method, radiation is generated by a bunch in a (pickup) undulator and, after amplification in the optical amplifier, is send to another undulator (kicker). In the kicker, amplified wave of radiation interacts with the same bunch providing desirable cooling. The phase shift between bunch and the wave is controlled in a dispersion section between two undulators.

In the case of the optical stochastic cooling, the bandwidth $\Delta f \simeq \gamma_0^2 (c/L_u)$ where $L_u = N_u \lambda_u$ is the undulator length and γ_0 is relativistic factor of the beam in the laboratory frame. For large Δf cooling is fast but number of particles per slice becomes small and there may be concern that classical and quantum fluctuations could be dangerous. This is the primary motivation of the study we present here. Related problem might be amplification of the noise induced by interaction of particles in the undulator and by the noise of the amplifier.

The paper is the summary of three previous publications [7] [8] [9] where details of calculations may be found. In our consideration we follow evolution of the density matrix of the system (bunch plus radiated mode) through the undulators and quantum amplifier. Dynamics in the undulators is described as 1D dynamics in the rest frame of a bunch as it is outlined by Dattoli-Renieri [4], [5] where other references can be found. The formalism we use to describe interaction in the undulators reproduces results but is different from Becker and McIver [6] formalism. In this formalism as well as in the Becker-McIver's formalism, number of particles per bunch N_B can be arbitrary, but effect of bunching is neglected. In this sense the interaction of particles with radiation is weak. This assumption substantially simplifies consideration being quite adequate for describing optical stochastic cooling. The theory of evolution of the density matrix in quantum amplifier includes the non-diagonal components of the matrix.

II. DENSITY MATRIX OF A BUNCH

We assume that, at the entrance to the pickup, the bunch is a superposition of N_B relativistic particles, there is no initial z, p correlation, and correlations generated in one pass are wiped out in one turn. The bunch is described by the density matrix $\hat{\rho} = |p' > \rho(p', p) < p|$, $\mathrm{Tr}[\rho] = 1$, where $\rho(p', p) = \Pi_i \rho^0(p_i, z_i)$, $i = 1, 2..N_B$.

$\rho_0(p_i, z_i)$ is the wave packet of individual particles related by the Wigner's transform to the classical distribution function $f(p, z)$,

$$f(p, z) = \frac{1}{2\pi\Delta\sigma} e^{-\frac{(p-p_0)^2}{2\Delta^2} - \frac{(z-z_0-\eta p)^2}{2\sigma^2}}, \quad \int dpdz f(p, z) = 1. \tag{2}$$

The $p - z$ correlation defined by parameter η may be introduced in the dispersion section with momentum compaction α_{MC} and length L_{ds}. Then parameter $\eta = \gamma_0 \alpha_{MC} L_{ds}/m_e c$. The density matrix corresponding to $f(p, z)$ in the case $\eta = 0$ is

$$\rho^0(p', p) = \frac{h\sqrt{2\pi}}{L\Delta} e^{-\frac{i}{h}(p'-p)z_0 - (\frac{1}{2}(\frac{\sigma}{h})^2(p'-p)^2 - \frac{1}{2}(\frac{1}{\Delta})^2(\frac{p+p'}{2} - p_0)^2}, \tag{3}$$

The rms values of the wave packet σ and Δ may be small compared to the rms energy spread Δ_B and rms length σ_B of a bunch.

With the correlation, the corresponding density matrix is different from Eq. (3) by the factor $e^{-(i/h)\eta[q'^2 - q^2]/2}$.

III. PICKUP

We consider helical pickup and kicker undulators with the undulator parameter K_0, period $\lambda_u = 2\pi/k_u$, and the length $L_u = N_u\lambda_u$. The bunch dynamics is considered in the Bambini-Renieri frame moving with the relativistic factor $\gamma = \gamma_0/\sqrt{1+K_0^2}$, where the bunch centroid initially has zero velocity, and the resonance frequency of the mode is $k = \gamma k_u$. The momentum spread $(\delta p/p)_L$ in the lab frame corresponds to the momentum spread $\Delta p/m_e c = (\delta p/p)_L\sqrt{1+K_0^2}$ in the moving frame.

In the moving frame, interaction of particles with the mode $k = \omega/c$ is described by the Hamiltonian

$$H = \sum_{i=1}^{N_B} \frac{\hat{p}_i^2}{2m} + h\omega(a^+a + 1/2) - ihg[ae^{2ik\hat{z}+i\omega t} - cc], \tag{4}$$

where $m = m_e\sqrt{1+K_0^2}$, parameter of interaction

$$g = c\frac{K_0}{\sqrt{1+K_0^2}}\sqrt{2\pi\frac{e^2}{hc}\frac{\Omega}{kV}}, \tag{5}$$

V is normalization volume and $\Omega = (V/(2\pi)^3)(\pi k^3/N_u^2)$ is the phase volume of the mode [12]. The interaction time (time of light in the undulator) in the moving frame is $t = 2\pi N_u/(ck)$. Note that $(gt)^2 \simeq e^2/(hc)$.

Hamiltonian Eq. (4) describes back-scattering of equivalent photons. The state $|p_i, n> = |p_1, p_2, .., p_{N_B}, n>$ of the system with n-photons and particles with momentums p_i, $i = 1..N_B$ is transformed by the interaction with the mode $k = \omega/c = \gamma k_u$ to the state [8]

$$|\Psi(t)> = \sum_{l_i,p_i} |p_i - 2hkl_i, n + l_\Sigma> \sqrt{\frac{n!}{(n+l_\Sigma)!}} F_n(t, p_i, l_i) e^{-i\omega t(n+l_\Sigma)-(it/h)\sum_i E(p_i,l_i)}, \tag{6}$$

where $E(p_i, l_i) = (p_i - 2hkl_i)^2/(2m_e)$, and $l_\Sigma = \sum_i l_i$ is the total number of radiated photons. Function $F_n(t)$ is given [7] by

$$F_n(t,p,l) = \int_0^\infty d\lambda \frac{\lambda^n}{n!} e^{-\lambda} \hat{O}_{\lambda\kappa}\{\Pi_{i=1}^{N_B}(\frac{\lambda a_i}{\kappa a_i^*})^{l_i/2} J_{l_i}(2g|a_i|\sqrt{\lambda\kappa})\}|_{\kappa=1}, \tag{7}$$

where we neglected a small phase factor. Here J_l is Bessel function, operator $\hat{O}_{\lambda\kappa} = e^{-(1/2)\frac{\partial^2}{\partial\lambda\partial\kappa}}$, and

$$a_i(t) = \frac{\sin(\epsilon_i t/2)}{(\epsilon_i/2)} e^{-i\epsilon_i t/2}, \quad \dot{a}_i(t) = e^{-i\epsilon_i t}, \quad \epsilon_i = \frac{2kp_i}{m_e}. \tag{8}$$

As the main simplification [4] of the theory, terms of the order of hk^2/m_e in Eq. (7) are neglected. As a result, we loose effect of bunching due to radiation. However, this is sufficient for our purpose. For short undulators, $kpt/m_e << 1$, $\frac{\sin(\epsilon_i t/2)}{(\epsilon_i/2)} \simeq t$, and F_n depends on parameter gt, where t is time of flight in the undulator ($t = N_u\lambda_u/(c\gamma)$ in the moving frame). We assume that at the entrance to the pickup there is no radiation, $n = 0$. In this case, initial density matrix $\hat{\rho} = \Pi_{i=1}^{N_B}|p' > \rho^0(p'_i, p_i) < p|$ is transformed according to Eqs. (6), (7) to $\hat{\rho}(t) = |q', l'_\Sigma > \rho(q', q, l_\Sigma, l'_\Sigma) < q, l_\Sigma$, where

$$\rho(q', q, l'_\Sigma, l_\Sigma) = \int \frac{d\psi d\psi'}{(2\pi)^2} e^{-i(l'_\Sigma\psi' - l_\Sigma\psi)} e^{i\omega t(l_\Sigma - l'_\Sigma)} \int \frac{d\lambda d\lambda'}{\sqrt{l_\Sigma! l'_\Sigma!}} e^{-\lambda-\lambda'} \hat{O}\hat{O}' F_{loc}. \quad (9)$$

Here $|q >$ stands for the set $|q_1..q_{N_B} >$, $F_{loc}(q', q) = \Pi_{i=1}^{N_B} F^i_{loc}$,

$$F^i_{loc}(q'_i, q_i) = \sum_{l,l'} f'_i f^*_i \rho^0(q'_i + 2hkl'_i, q_i + 2hkl_i) e^{-i\frac{((q'_i)^2 - q_i^2)t}{2m_e h}}, \quad (10)$$

$f_i = f(q_i, l_i, \psi)$, $f'_i = f(q'_i, l'_i, \psi')$, and

$$f(q, l, \psi) = (\frac{\lambda a}{\kappa a^*})^{l/2} J_l[2g|a(t)|\sqrt{\lambda\kappa}] e^{il\psi}. \quad (11)$$

Integration over ψ, ψ' is introduced in Eq. (9) to separate the global parameters l_Σ, l'_Σ of the radiation and particle parameters $\{q_i, l_i\}$.

Some results can be obtained already from Eqs.(7), (9). In particular, the average $< p >= \mathrm{Tr}[\hat{p}\hat{\rho}] = -2hk(gt)^2$ gives the energy loss of a particle in the undulator. For a single particle,

$$F_n(t, p, l) = (ga)^l e^{-(1/2)(ga)^2} L^l_n(g^2|a|^2), \quad (12)$$

where L^l_n is Laguerre polynomials, what reproduces Dattoli-Renieri result [4] (here small phase factor is omitted).

The density matrix of radiation at the exit of the undulator can be obtained averaging Eq.(9) over the state of the bunch, $\hat{\rho}_{rad} = \rho_{rad}(l'_\Sigma, l_\Sigma)|l'_\Sigma >< l_\Sigma|$, where [8]

$$\rho_{rad}(l'_\Sigma, l_\Sigma) = \delta_{l_\Sigma, l'_\Sigma} \frac{(2s)^{l_\Sigma}}{(1+2s)^{l_\Sigma+1}}, \quad (13)$$

and $s = (1/2)N_B(gt)^2$. The average number of radiated photons $< a^+a >= Tr[\hat{\rho}_{rad}a^+a] = \sum_{l_\Sigma} l_\Sigma \rho_{rad}(l_\Sigma, l_\Sigma)$ is $< a^+a >= 2s = N_B(gt)^2$. Hence, parameter of expansion $(gt)^2$ in Eq. (44) is the number of photons radiated in the undulator per particle. Eq. (53) reproduces the thermal statistics of radiation [11].

IV. OPTICAL AMPLIFIER AND DISPERSION SECTION

Amplification by a linear phase-independent quantum amplifier can be modeled as interaction of radiation with two-level atoms being in an equilibrium with the thermal bath. Equation describing variation of the density matrix ρ in time is well known [13],

$$\dot{\rho} = -gN_+[aa^+\rho + \rho aa^+ - 2a^+\rho a] - gN_-[\rho a^+ a + a^+ a\rho - 2a\rho a^+]. \tag{14}$$

Here g is parameter of interaction of the atoms with radiation and $N_\pm$ are population of upper/lower levels. The model of the amplifier implies fixed in time inverse population of two levels $N_+ > N_-$. Solution for the diagonal components of the density matrix is well known in literature. This is, for example, sufficient for the estimate of the signal to noise ratio.

For our problem, however, the explicit form of the non-diagonal components of the density matrix is required. The solution for the density matrix $\rho(t) = |n' > \rho(n', n, t) < n|$ in the n-photon basis $|n >$, $a|n >= |n-1 > \sqrt{n}$, can be found in the form

$$\rho(n', n, t) = \frac{F_A(N, m, t)}{\sqrt{n!\, n'!}}, \tag{15}$$

where $N = (n + n')/2$ and $m = (n - n')/2$. We consider components with $m >= 0$. Components $m < 0$ can be obtained from the components with $m > 0$ by complex conjugation, $F_A(N, -m, t) = F_A^*(N, m, t)$. For simplicity, we consider the fully inverted system, $N_-/N_+ \to 0$. The general case is described in [7]. Equation for the function $F_A(N, m, t)$ follows from Eq.(14),

$$\dot{F}(N, m, t) = -[(N+1)F_A(N, m, t) - (N^2 - m^2)F_A(N-1, m, t)], \tag{16}$$

where dot means derivative over dimensionless time $\tau = 2gN_+t$.

All terms in Eq. (16) have the same dependence on m which is, therefore, a constant of motion. To solve Eq. (16) we use Mellin transform in z and Laplace transform in time. The solution is [7]

$$F_A(N, m, t) = \int_0^\infty dz' G_m(N, z', \tau) f_0(z', m), \tag{17}$$

where the kernel $G(N, z', t)$ is given in terms of the Laguerre's polynomials:

$$G_m(N, z, \tau) = (N-m)!\frac{\xi}{z}(1-\xi)^N b^m L^{2m}_{N-m}(-b), \tag{18}$$

and $b = \frac{\xi z}{(1-\xi)}$, $\xi = e^{-2N+gt}$. The function $f_0(z, m)$ is the Mellin transform of the initial condition $F_A(N, m, 0)$,

$$f_0(z, m) = \int_{-i\infty}^{i\infty} \frac{dN}{2\pi i} z^{-N} F_A(N, m, 0). \tag{19}$$

For the initial coherent state,

$$\rho(n', n, 0) = \frac{\alpha^{n'}(\alpha^*)^n}{\sqrt{n!n'!}} e^{-|\alpha|^2}, \quad F_A(N, m, 0) = (\frac{\alpha^*}{\alpha})^m |\alpha^*\alpha|^N e^{-|\alpha|^2}. \tag{20}$$

Eq. (19) gives

$$f_0(z, m) = (\frac{\alpha^*}{\alpha})^m |\alpha|^2 e^{-|\alpha|^2} \delta(z - |\alpha|^2). \tag{21}$$

Then, Eq. (18) defines $\rho(n', n, t) = F_A(N, m, t)/\sqrt{n!n'!}$,

$$F_A(N, m, t) = (N-m)!(\frac{\alpha^*}{\alpha})^m |\alpha|^{2m} e^{-|\alpha|^2} \xi^{m+1} (1-\xi)^{N-m} L^{2m}_{N-m}(-\frac{\xi|\alpha|^2}{1-\xi}). \tag{22}$$

In the limit $t->0$, $L^{2m}_{N-m}(x)->(-x)^{N-m}/(N-m)!$, and the initial condition is satisfied.

Let us, for example, find the lowest moment of the distribution. The amplitude

$$< a(t) >= \sum_{n,n'} \rho(n', n, t) < n|a|n' >= \sum_n \sqrt{n}\rho(n, n-1, t), \tag{23}$$

Initial amplitude $< a(0) >= \alpha$. Using

$$\sum_n z^n L^m_n(-x) = (1-z)^{-m-1} e^{\frac{xz}{1-z}}, \tag{24}$$

it is easy to find

$$< a(t) >= \frac{\alpha}{\sqrt{\xi}}. \tag{25}$$

Hence, $G_A = 1/\xi$ is power gain (amplification factor) of the amplifier. In the same way it is easy to get higher order moments. In particular, $< a(t)^2 >= \alpha^2 G$, $< a^+ a >= G(1 + |\alpha|^2)$, and the signal/noise ratio is independent of the gain of the amplifier, the well known result which can be obtained directly from Eq. (14).

V. BEAM DYNAMICS FOR OPTICAL STOCHASTIC COOLING

For small $N_s(gt)^2(hk/\Delta)^2$ and short undulators, $k\Delta t/m_e << 1$, $F_n(t,p,l)$ depends on the small parameter $(gt)^2 \simeq \alpha_0$, the average number of photons radiated in the undulator per particle. This justifies expansion of f_i in series over gt, $f_i = f_0 + gtf_i^{(1)} + (gt)^2 f_i^{(2)}$. Then, with the same accuracy,

$$\Pi_i f_i = \{\Pi_i f_0\} e^{gt\sum_i f_i^{(1)} + (gt)^2[f_i^{(2)} - (f_i^{(1)})^2]}. \tag{26}$$

In this way we take into account all powers of $(gt)N_s$ neglecting terms with the extra power of $gt << 1$. For small $(gt)^2 N_s$, this can be simplified to $\Pi_i f_i = \{\Pi_i f_0\} e^{gt\sum_i f_i^{(1)}}\{1+(gt)^2[f_i^{(2)} - (f_i^{(1)})^2]\}$.

Then, the density matrix at the end of the pickup takes form [9] $\hat{\rho} = |q', l'_\Sigma > \rho(q'q) < q, l_\Sigma|$, where

$$\rho(q',q) = \frac{1}{\sqrt{l_\Sigma! l'_\Sigma!}}\{\Pi_i \int \frac{dz_i}{L} F^i(q',q,z)\}(1+\hat{P})R(\frac{q'+q}{2}, z, N, \mu)e^{2i\mu\omega t}, \tag{27}$$

$$F^i(q',q,z) = \frac{h}{\sigma\Delta} e^{-i(q'-q)z/h - \frac{1}{2\Delta^2}(\frac{q'+q}{2})^2 - \frac{1}{2\sigma^2}(z - \frac{q'+q}{2m_e}t)^2}, \tag{28}$$

$$R(p,z,N,\mu) = e^{-(1/2)(\sigma_-^*\sigma_+ + c.c.)}(\frac{\sigma_-}{\sigma_-^*})^\mu |\sigma_-|^{2N}, \tag{29}$$

and $N = (l_\Sigma + l'_\Sigma)/2$, $\mu = (l_\Sigma - l'_\Sigma)/2$. The operator $\hat{P}$ is differential operator of the second order in yd/dy, where $y = |\sigma_-|^2$ defined in [8].

The function $R(p,z,N,\mu)$ is written in terms of parameters

$$\sigma_\pm(p,z) = gt\sum_{i=1}^{N_B} e^{-2ik(z_i - \frac{p_i t}{2m_e}) \pm \frac{hk(p_i - p_i^0)}{\Delta^2}} e^{-\frac{1}{2}(\frac{hk}{\Delta})^2} s_i. \tag{30}$$

Note that $\sigma_\pm$ in Eq. (30) are functions of coordinates $z_i, (q_i + q'_i)/2$ of all particles.

The factor s_i,

$$s_i = \int \frac{dk'}{\pi} e^{-2i(k'-k)(z_i - p_i t/2m_e)} \frac{\sin^2(\pi N_u(k'-k)/k')}{(\pi N_u/k)(k'-k)^2}, \tag{31}$$

is result of averaging over the frequency spread of the mode. It restricts summation over particles within the length (length of a "slice") $\propto 2\pi N_u/(2k)$ or, in the laboratory system, within $l_s = N_u \lambda_{lab}$.

Parameter $N_s =<< \sigma_- \sigma_-^* >> /(gt)^2$ is the fundamental parameter of the theory defining number of interacting particles within the bandwidth of the mode (number of particles per slice). Here double averaging means averaging with the density matrix of the wave packet Eq.(3) and over z^0, p^0 with the Gaussian bunch $\rho_B(z_0, p_0) = (1/2\pi\sigma_B\Delta_B)e^{-p_0^2/2\Delta_B^2 - z_0^2/2\sigma_B^2}$. If the width of the packet σ is of the order of the length of a slice and $N_u >> 1$, then $k\sigma >> 1$ and

$$N_s = N_b \frac{N_u}{3\sqrt{2\pi}k\sigma_B}, \tag{32}$$

where σ_B is rms bunch length in the moving frame. In terms of the wave length of the mode and the bunch length in the laboratory frame, $N_s = N_B(\frac{N_u\lambda_L}{3\sqrt{2\pi}\sigma_B^0})$.

The density matrix Eqs. (27), (29) at the exit of the pickup undulator is superposition of coherent states. Transformation of such a state in the optical amplifier is described above. In this case, $R(\frac{q'+q}{2}, z, N, \mu)$ in Eq. (27) should be replaced by F_{ampl},

$$F_{ampl}(N, \mu) = (N - |\mu|)! \frac{1}{G} [\frac{\sigma_-^*}{\sigma_-}]^{\mu} [\frac{\sigma_-^* \sigma_-}{G-1}]^{|\mu|} \tag{33}$$

$$(\frac{G-1}{G})^N L_{N-|\mu|}^{2|\mu|}(-\frac{|\sigma_-|^2}{G-1}) e^{-(1/2)(\sigma_-^*\sigma_+ + c.c.)}. \tag{34}$$

Here G is power gain of the amplifier, $N = (l_\Sigma + l'_\Sigma)/2$, $\mu = (l_\Sigma - l'_\Sigma)/2$.

Dispersion section modifies $F^i(q', q, z)$ in Eq. (28) adding factor $e^{-(i/h)\eta[q'^2 - q^2]/2}$.

The beam dynamics in the kicker can be described in the same way as it is done above for the pickup but the initial number of photons L_Σ is not zero.

VI. COOLING

We skip some cumbersome details and give the final result for the average moments $n = 0, 1, 2$ of the momentum of the j-th particle. The average for $n = 0$ is just the norm of the distribution function and has to be equal one. Indeed, the answer we obtained is different from one by the term of the order of $N_s^2 (gt)^4 (hk/\Delta)^4$.

The result for the moments $n = 1$ and $n = 2$ were obtained with MATHEMATICA. As it will be shown below, the power gain G has to be of the order of $\Delta_b/(hk)$. Hence, $G >> 1$ and we can neglect terms which are independent of G. In this approximation, the double averaging over the wave packet $\rho_0(p_j, z_j)$ and over the Gaussian distribution of particles in the bunch gives the rms $\Delta^2 =<< p^2 >> - << p >>^2$ at the end of the kicker:

$$\frac{\tilde{\Delta^2} - \Delta^2}{\Delta^2} = -16\sqrt{G}(gt)^2\frac{hk}{\Delta_B}\Lambda\sin\theta + 8G(gt)^2(\frac{hk}{\Delta_B})^2[1 + vbbbbbbbbbbbbcN_s(gt)^2] \tag{35}$$

$$+8\sqrt{G}(gt)^4(\frac{hk}{\Delta_B})^2 N_s\cos\theta e^{-2(k\Delta_B)^2\eta_{eff}^2}. \tag{36}$$

Here $\Lambda = k\Delta_B\eta_{eff}e^{-2(k\Delta_B\eta_{eff})^2}$.

To get damping, we have to choose $\sin\theta = 1$. The damping is maximum if the power gain G of the amplifier is equal to

$$\sqrt{G} = \frac{\Lambda}{(hk/\Delta_B)[1 + N_s(gt)^2]}. \tag{37}$$

Parameter Λ as function of $x = k\Delta_B\eta_{eff}$ has maximum value $\Lambda_{max} \simeq 0.3$ at $x \simeq 2$. This defines the optimum parameter η of the dispersion section.

The optimized reduction of the rms in one pass through the system is

$$\frac{\tilde{\Delta^2} - \Delta^2}{\Delta^2} = -\frac{8(gt)^2\Lambda_{max}^2}{1 + N_s(gt)^2}. \tag{38}$$

VII. CONCLUSION

The one pass reduction of the energy spread is derived following the evolution of the density matrix through all components of the system. The consideration is fully quantum-mechanical both for the beam and radiation but bunching effect is neglected and length of a slice, of the order of $N_u\lambda_{lab}$, is assumed to be small compared to the bunch length in the laboratory frame σ_B^0. The final result Eq. (38) for large $N_s >> 1/(gt)^2 \simeq (e^2/hc)$ corresponds to classical theory of stochastic cooling: the damping rate is given by the number of particles N_s per slice. However, for small N_s the damping rate goes to a constant proportional to $(gt)^2 \propto \alpha_0$. This is the result of classical fluctuations of small number of particles in a slice. The quantum mechanical corrections per se are small, of the order of

$(hk)/\Delta_B$ (i.e. $hk_L/\Delta p_L$ in the laboratory frame) and can be noticeable only for very cold beams with energy spread comparable with the photon energy.

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