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Nonlinear δf Method for Beam-Beam Simulation *

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Abstract

We have developed an efficacious algorithm for simulation of the beam-beam interaction in synchrotron colliders based on the nonlinear δf method, where δf is the much smaller deviation of the beam distribution from the slowly evolving main distribution f_0. In the presence of damping and quantum fluctuations of synchrotron radiation it has been shown that the slowly evolving part of the distribution function satisfies a Fokker-Planck equation. Its solution has been obtained in terms of a beam envelope function and an amplitude of the distribution, which satisfy a coupled system of ordinary differential equations. A numerical algorithm suited for direct code implementation of the evolving distributions for both δf and f_0 has been developed. Explicit expressions for the dynamical weights of macro-particles for δf as well as an expression for the slowly changing f_0 have been obtained.

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1 Introduction

The effects of the beam-beam interaction on particle dynamics in a synchrotron collider are the key element that determines the performance of the collider such as luminosity [1] - [3]. In order to accurately understand these effects, it is necessary to incorporate not only the overall collisional effects of the beam-beam interaction, but also the collective interaction among individual parts of the beam in each beam and its feedback on the beam distribution. The particle-in-cell (PIC) approach [4], [5] has been adopted to address such a study need [6], [7], [8].

Particle-in-cell codes typically use macro-particles to represent the entire distribution of particles. In the beam-beam interaction for the PEP-II [9] (for example), the beams consist of 10^{10} particles each. Simulating this many particles with the PIC technique is computationally prohibitive. With the conventional PIC code 10^{10} particles are represented by only $10^3 - 10^4$ macro-particles allowing simulation of the beam-beam interaction in a reasonable computation time. However, the statistical fluctuation level of various quantities such as the beam density ρ in the code is much higher than that of the real beam. The fluctuation level $\delta\rho$ goes as approximately

$$\frac{\delta\rho}{\rho} \approx \frac{\sqrt{N}}{N}, \tag{1.1}$$

where N is the number of particles. Therefore, the fluctuation level of the PIC code is about 10^3 times higher than that of the real beam. Although this probability is not significant for beam blowup near resonances, the higher fluctuation level has a large effect on more subtle phenomenon such as particle diffusion. The purpose of the δf algorithm is to facilitate the study of subtle effects and has been introduced in [10], [11], [12].

The δf method follows only the fluctuating part of the distribution instead of the entire distribution. This is essentially modeling the numerator of the right-hand side of equation (1.1). So the $10^3 - 10^4$ macro particles are used to represent $\sqrt{10^{10}}$ or 10^5 real fluctuation particles in PEP-II beams. This is only one or two orders of magnitude beyond the number of macro particles. Such a modest gap between the number of macro particles and the real fluctuating particles maybe ameliorated by the standard techniques of the PIC approach, such as the method of finite-sized macro-particles [4], [5].

PIC strong-strong codes use a finite number of particles to represent the Klimontovich equation for the microscopic phase space density (MPSD) [13]. In the particular case of one-dimensional beam-beam interaction,

$$\frac{\partial f}{\partial s} + p\frac{\partial f}{\partial x} - (K(s)x - F(x;s))\frac{\partial f}{\partial p} = 0, \tag{1.2}$$

where $K(s)x$ is the usual magnetic guiding force and $F(x;s)$ is the beam-beam force

$$F(x;s) = \frac{2eE_x(x)}{m\gamma v^2}\delta_p(s). \tag{1.3}$$

The electric field $E_x(x)$ is calculated from the distribution of the particles of the on-coming beam and $\delta_p(s)$ is the periodic δ-function with a periodicity of the accelerator circumference. The distribution function $f(x,p;s)$ is represented by a finite number of macro-particles by

$$f(x,p;s) = \frac{1}{N}\sum_{n=1}^{N} \delta(x - x_n(s))\delta(p - p_n(s)), \tag{1.4}$$

where N is the number of macro-particles.

The strategy of the δf method is that only the perturbative part of the distribution is followed. The total distribution function $f(x,p;s)$ is decomposed into

$$f(x,p;s) = f_0(x,p;s) + \delta f(x,p;s), \tag{1.5}$$

where $f_0(x,p;s)$ is the steady or slowly varying part of the distribution and $\delta f(x,p;s)$ is the perturbative part. The key to this method is finding a distribution $f_0(x,p;s)$ which is close to the total distribution $f(x,p;s)$. The perturbative part $\delta f(x,p;s)$ is then small, causes only small changes to the distribution, and thus represents only the fluctuation levels. If a distribution $f_0(x,p;s)$ close to the total distribution is not found or found poorly, then $\delta f(x,p;s)$ represents more than the fluctuation part of the total distribution; defeating the purpose of the method. The ideal situation is having an analytic solution for $f_0(x,p;s)$. In this case any numerical truncation errors which result from the necessary derivatives of this function are eliminated. If an analytic solution cannot be found, then a numerical solution needs to be found which is close to the total distribution $f(x,p;s)$ and is slowly varying. A frequent numerical update of $f_0(x,p;s)$ would also defeat the purpose of the δf method, since the PIC technique essentially does this also.

The beam-beam interaction can lead to beam instabilities that disrupt or severely distort the beam or gradual beam spreading. The higher the beam current, and thus the beam-beam interaction, the stronger these effects become. Therefore, when one wants to maximize the luminosity of a collider, one needs to confront the beam-beam interaction effects. The operation of PEP-II, for example, is critically dependent on the beam-beam interaction and optimal parameters to minimize the related beam instabilities are under intense study.

The paper is organized as follows. In the next Section we present a brief formulation of the problem of beam-beam interaction in synchrotron colliders. In Section 3 we develop the nonlinear δf method for solving the equation for the microscopic phase space density in the presence of random external forces. The equation for the fluctuating part δf is being derived and its solution is found explicitly in terms of dynamical weight functions, prescribed to each macro-particle. In Section 4 we solve the Fokker-Planck equation for the averaged slowly evolving part of the distribution. We show that the solution is an exponential of a bilinear form in coordinates and momenta with coefficients that can be regarded as generalized Courant-Snyder parameters. In Section 5 we outline numerical algorithms to alternatively solve the Fokker-Planck equation and the macro particle distribution with dynamical weight. Finally, Section 6 is dedicated to our summary and conclusions.

2 Description of the beam-Beam Interaction

In order to describe the beam dynamics in an electron positron storage ring, we introduce the equations of motion in the following manner. The beam propagation in a reference frame attached to the particle orbit is usually described in terms of the canonical conjugate pairs

$$\widehat{u}^{(k)} = u^{(k)} - D_u^{(k)} \widehat{\eta}^{(k)} \qquad ; \qquad \widehat{p}_u^{(k)} = \frac{p_u^{(k)}}{p_0^{(k)}} - \widehat{\eta}^{(k)} \frac{dD_u^{(k)}}{ds}, \tag{2.1}$$

$$\widehat{\sigma}^{(k)} = \widetilde{\sigma}^{(k)} + \sum_{u=x,z} \left(u^{(k)} \frac{dD_u^{(k)}}{ds} - D_u^{(k)} \frac{p_u^{(k)}}{p_0^{(k)}} \right) \qquad ; \qquad \widehat{\eta}^{(k)} = \frac{1}{\beta_{k0}^2} \frac{E^{(k)} - E_{k0}}{E_{k0}}, \tag{2.2}$$

where $u = (x, z)$, s is the path length along the particle orbit, and the index k refers to either beam ($k = 1, 2$). In equations (2.1) and (2.2) the quantity $u^{(k)}$ is the actual particle displacement from the reference orbit in the plane transversal to the orbit, $p_u^{(k)}$ is the actual particle momentum, and $E^{(k)}$ is the particle energy. Furthermore, $p_0^{(k)}$ and E_{k0} are the total momentum and energy of the synchronous particle, respectively, and $D_u^{(k)}$ is the well-known dispersion function. The quantity

$$\widetilde{\sigma}^{(k)} = s - \omega_0^{(k)} R t \tag{2.3}$$

is the longitudinal coordinate of a particle from the k-th beam with respect to the synchronous particle, where $\omega_0^{(k)}$ is the angular frequency of the synchronous particle and R is the mean machine radius.

It is known that the dynamics of an individual particle is governed by the Langevin equations of motion:

$$\frac{d\widehat{u}^{(k)}}{ds} = \frac{\partial \widehat{H}^{(k)}}{\partial \widehat{p}_u^{(k)}} - D_u^{(k)} \widetilde{F}_\eta^{(k)} \qquad ; \qquad \frac{d\widehat{p}_u^{(k)}}{ds} = -\frac{\partial \widehat{H}^{(k)}}{\partial \widehat{u}^{(k)}} + \widetilde{F}_u^{(k)} - \widetilde{F}_\eta^{(k)} \frac{dD_u^{(k)}}{ds}, \tag{2.4}$$

$$\frac{d\widehat{\sigma}^{(k)}}{ds} = \frac{\partial \widehat{H}^{(k)}}{\partial \widehat{\eta}^{(k)}} - \sum_{u=x,z} D_u^{(k)} \widetilde{F}_u^{(k)} \qquad ; \qquad \frac{d\widehat{\eta}^{(k)}}{ds} = -\frac{\partial \widehat{H}^{(k)}}{\partial \widehat{\sigma}^{(k)}} + \widetilde{F}_\eta^{(k)}, \tag{2.5}$$

where

$$\widetilde{F}_u^{(k)} = -p_0^{(k)} \mathcal{A}_k \left(\widehat{p}_u^{(k)} + \widehat{\eta}^{(k)} \frac{dD_u^{(k)}}{ds} \right), \tag{2.6}$$

$$\widetilde{F}_\eta^{(k)} = -p_0^{(k)} \mathcal{A}_k \left[1 + \left(3 - \beta_{k0}^2 + \alpha_M^{(k)} \right) \widehat{\eta}^{(k)} + \sum_{u=x,z} K_u^{(k)} \widehat{u}^{(k)} \right], \tag{2.7}$$

$$\mathcal{A}_k = \mathcal{C}_1 |\mathbf{B}_k|^2 + \sqrt{\mathcal{C}_2} |\mathbf{B}_k|^{3/2} \xi_k(s), \tag{2.8}$$

$$\mathcal{C}_1 = \frac{2 r_e e^2}{3 (m_e c)^3} \qquad ; \qquad \mathcal{C}_2 = \frac{55}{24\sqrt{3}} \frac{r_e \hbar e^3}{(m_e c)^6} \qquad ; \qquad r_e = \frac{e^2}{4\pi\epsilon_0 m_e c^2}. \tag{2.9}$$

Here $\alpha_M^{(k)}$ is the momentum compaction factor, $K_u^{(k)}(s)$ is the local curvature of the reference orbit, and $\mathbf{B}_k = \left(B_x^{(k)}, B_z^{(k)}, B_s^{(k)}\right)$ is the magnetic field. The variable $\xi_k(s)$ is a Gaussian random variable with formal properties:

$$\langle \xi_k(s) \rangle = 0 \qquad ; \qquad \langle \xi_k(s) \xi_k(s') \rangle = \delta(s - s'). \tag{2.10}$$

The hamiltonian part in equations (2.4) and (2.5) consists of three terms:

$$\widehat{H}^{(k)} = \widehat{H}_0^{(k)} + \widehat{H}_2^{(k)} + \widehat{H}_{BB}^{(k)}, \tag{2.11}$$

where

$$\widehat{H}_0^{(k)} = -\frac{\mathcal{K}^{(k)}}{2} \widehat{\eta}^{(k)2} + \frac{1}{2\pi\beta_{k0}^2} \frac{\Delta E_{k0}}{E_{k0}} \cos\left(\frac{h_k \widetilde{\sigma}_k}{R} + \Phi_{k0}\right), \tag{2.12}$$

$$\widehat{H}_2^{(k)} = \frac{1}{2}\left(\widehat{p}_x^{(k)2} + \widehat{p}_z^{(k)2}\right) + \frac{1}{2R^2}\left(G_x^{(k)} \widehat{x}^{(k)2} + G_z^{(k)} \widehat{z}^{(k)2}\right), \tag{2.13}$$

$$\widehat{H}_{BB}^{(k)} = \lambda_k \delta_p(s) V_k\left(x^{(k)}, z^{(k)}, \widetilde{\sigma}^{(k)}; s\right). \tag{2.14}$$

The parameter $\mathcal{K}^{(k)}$ is the so called slip phase coefficient, h_k is the harmonic number of the RF field and ΔE_{k0} is the energy gain per turn. The coefficients $G_{x,z}^{(k)}(s)$ represent the focusing strength of the linear machine lattice, $\delta_p(s)$ is the periodic delta-function, while λ_k and $V_k\left(x^{(k)}, z^{(k)}, \widetilde{\sigma}^{(k)}; s\right)$ are the beam-beam coupling coefficient and the beam-beam potential, respectively. The latter are given by the expressions:

$$\lambda_k = \frac{r_e N_{3-k}}{\gamma_{k0}} \frac{1 + \beta_{k0}\beta_{(3-k)0}}{\beta_{k0}^2}, \tag{2.15}$$

$$V_k\left(x^{(k)}, z^{(k)}, \widetilde{\sigma}^{(k)}; s\right) = \int dx dz d\widetilde{\sigma} \mathcal{G}_k\left(u^{(k)} - u, \widetilde{\sigma}^{(k)} - \widetilde{\sigma}; s\right) \rho_{3-k}(u, \widetilde{\sigma}; s), \tag{2.16}$$

where N_k is the number of particles in the k-th beam and the Green's function $\mathcal{G}_k(u, \widetilde{\sigma}; s)$ for the Poisson equation in the fully 3D case, in the ultra-relativistic 2D case and in the 1D case can be written respectively as:

$$\mathcal{G}_k\left(u^{(k)} - u, \widetilde{\sigma}^{(k)} - \widetilde{\sigma}; s\right) = \begin{cases} -\left[\left(x^{(k)} - x\right)^2 + \left(z^{(k)} - z\right)^2 + \left(\widetilde{\sigma}^{(k)} - \widetilde{\sigma} + 2s\right)^2\right]^{-1/2}, \\ \\ \delta\left(\widetilde{\sigma}^{(k)} - \widetilde{\sigma} + 2s\right) \ln\left[\left(x^{(k)} - x\right)^2 + \left(z^{(k)} - z\right)^2\right], \\ \\ 2\pi\delta\left(\widetilde{\sigma}^{(k)} - \widetilde{\sigma} + 2s\right)\delta\left(z^{(k)} - z\right)\left|x^{(k)} - x\right|. \end{cases} \tag{2.17}$$

In what follows we focus on the two-dimensional case, entirely neglecting the longitudinal dynamics. Let us write down the Langevin equations of motion (2.4) and (2.5) once again in the following form:

$$\frac{d\mathbf{x}^{(k)}}{ds} = \mathbf{p}^{(k)}, \tag{2.18}$$

$$\frac{d\mathbf{p}^{(k)}}{ds} = \mathbf{F}_L^{(k)} + \mathbf{F}_B^{(k)} + \mathbf{F}_R^{(k)}, \tag{2.19}$$

$$\mathbf{x}^{(k)} = \left(\widehat{x}^{(k)}\ ,\ \widehat{z}^{(k)}\right) \qquad ; \qquad \mathbf{p}^{(k)} = \left(\widehat{p}_x^{(k)}\ ,\ \widehat{p}_z^{(k)}\right), \tag{2.20}$$

where

$$\mathbf{F}_L^{(k)} = \left(-\frac{G_x^{(k)}}{R^2}\widehat{x}^{(k)}\ ,\ -\frac{G_z^{(k)}}{R^2}\widehat{z}^{(k)}\right) \tag{2.21}$$

is the (external) force acting on particles from the k-th beam, that is due to the linear focusing properties of the corresponding confining lattice. Furthermore,

$$\mathbf{F}_B^{(k)} = \lambda_k \delta_p(s)\left(-\frac{\partial V_k}{\partial \widehat{x}^{(k)}}\ ,\ -\frac{\partial V_k}{\partial \widehat{z}^{(k)}}\right) \tag{2.22}$$

is the beam-beam force and

$$\mathbf{F}_R^{(k)} = -p_{k0}\mathcal{A}_k\left(\widehat{p}_x^{(k)} - \frac{dD_x^{(k)}}{ds}\ ,\ \widehat{p}_z^{(k)} - \frac{dD_z^{(k)}}{ds}\right) \tag{2.23}$$

is the synchrotron radiation friction force with a stochastic component due to the quantum fluctuations of synchrotron radiation [cf expression (2.8)].

3 The Nonlinear δf Method

It can be checked in a straightforward manner that the Klimontovich microscopic phase space density

$$f_k(\mathbf{x}, \mathbf{p}; s) = \frac{1}{N_k} \sum_{n=1}^{N_k} \delta\left[\mathbf{x} - \mathbf{x}_n^{(k)}(s)\right] \delta\left[\mathbf{p} - \mathbf{p}_n^{(k)}(s)\right] \tag{3.1}$$

satisfies the following evolution equation:

$$\frac{\partial f_k}{\partial s} + \mathbf{p} \cdot \nabla_x f_k + \left(\mathbf{F}_L^{(k)} + \mathbf{F}_B^{(k)}\right) \cdot \nabla_p f_k + \nabla_p \cdot \left(\mathbf{F}_R^{(k)} f_k\right) = 0, \tag{3.2}$$

where $\left\{\mathbf{x}_n^{(k)}(s)\ ,\ \mathbf{p}_n^{(k)}(s)\right\}$ is the trajectory of the n-th particle from the k-th beam. Next we split the MPSD f_k into two parts according to the relation:

$$f_k(\mathbf{x}, \mathbf{p}; s) = f_{k0}(\mathbf{x}, \mathbf{p}; s) + \delta f_k(\mathbf{x}, \mathbf{p}; s), \tag{3.3}$$

where f_{k0} is a solution to the equation

$$\frac{\partial f_{k0}}{\partial s} + \mathbf{p} \cdot \nabla_x f_{k0} + \left(\mathbf{F}_L^{(k)} + \mathbf{F}_{L0}^{(k)}\right) \cdot \nabla_p f_{k0} + \nabla_p \cdot \left(\mathbf{F}_R^{(k)} f_{k0}\right) = 0. \tag{3.4}$$

The quantity $\mathbf{F}_{L0}^{(k)}$ in Eq. (3.4) is the *linear* part of the beam-beam force $\mathbf{F}_B^{(k)}$. The beam-beam force should be calculated with the on-coming beam distribution $f_{(3-k)0}$. In what follows it will prove convenient to cast the beam-beam force into the form:

$$\mathbf{F}_B^{(k)} = \mathbf{F}_{L0}^{(k)} + \mathbf{F}_{N0}^{(k)} + \delta\mathbf{F}_B^{(k)}, \tag{3.5}$$

where $\mathbf{F}_{N0}^{(k)}$ is the nonlinear (in the transverse coordinates) contribution calculated with $f_{(3-k)0}$, while $\delta\mathbf{F}_B^{(k)}$ denotes the part of the beam-beam force due to δf_{3-k}.

It is worthwhile to note here that the representation (3.3) is unique, embedding the basic idea of the δf method. However, one is completely free to fix the f_0 part, which usually describes those features of the evolution of the system one can solve easily (and preferably in explicit form). In the next Section we show that f_{k0}, averaged over the statistical realizations of the process $\xi_k(s)$ satisfies a Fokker-Planck equation and find its solution.

Subtract now the two equations (3.2) and (3.4) to obtain an equation for the δf_k

$$\frac{\partial \delta f_k}{\partial s} + \mathbf{p} \cdot \nabla_x \delta f_k + \left(\mathbf{F}_L^{(k)} + \mathbf{F}_B^{(k)}\right) \cdot \nabla_p \delta f_k + \nabla_p \cdot \left(\mathbf{F}_R^{(k)} \delta f_k\right) =$$

$$= -\left(\delta \mathbf{F}_B^{(k)} + \mathbf{F}_{N0}^{(k)}\right) \cdot \nabla_p f_{k0}. \tag{3.6}$$

The next step consists in defining the weight function that is relative to the total distribution as

$$W_k(\mathbf{x}, \mathbf{p}; s) = \frac{\delta f_k(\mathbf{x}, \mathbf{p}; s)}{f_k(\mathbf{x}, \mathbf{p}; s)}. \tag{3.7}$$

Substituting

$$\delta f_k = W_k f_k \qquad ; \qquad f_k = \frac{f_{k0}}{1 - W_k} \tag{3.8}$$

into (3.6) and taking into account (3.2) we finally arrive at the evolution equation for the weights:

$$\frac{\partial W_k}{\partial s} + \mathbf{p} \cdot \nabla_x W_k + \left(\mathbf{F}_L^{(k)} + \mathbf{F}_B^{(k)} + \mathbf{F}_R^{(k)}\right) \cdot \nabla_p W_k =$$

$$= -\frac{1}{f_k}\left(\delta \mathbf{F}_B^{(k)} + \mathbf{F}_{N0}^{(k)}\right) \cdot \nabla_p f_{k0} =$$

$$= \frac{W_k - 1}{f_{k0}}\left(\delta \mathbf{F}_B^{(k)} + \mathbf{F}_{N0}^{(k)}\right) \cdot \nabla_p f_{k0}. \tag{3.9}$$

Equation (3.9) can be solved formally by the method of characteristics. The first couple of equations for the characteristics are precisely the equations of motion (2.18) and (2.19). Suppose their solution (particle's trajectory in phase space) $\{\mathbf{x}(s)\ ,\ \mathbf{p}(s)\}$ is known, and let us write down the last one of the equations for the characteristics

$$\frac{1}{W_k - 1} \frac{dW_k}{ds} = \frac{1}{f_{k0}}\left(\delta \mathbf{F}_B^{(k)} + \mathbf{F}_{N0}^{(k)}\right) \cdot \nabla_p f_{k0}\bigg|_{\mathbf{x},\mathbf{p} \longrightarrow \mathrm{trajectory}}. \tag{3.10}$$

Note that its right-hand-side is a function of s only, provided $\mathbf{x}$ and $\mathbf{p}$ are replaced by particle's trajectory in phase space $\{\mathbf{x}(s)\ ,\ \mathbf{p}(s)\}$. Therefore equation (3.10) can be integrated readily to give:

$$W_k(s) = 1 + [W_k(s_0) - 1] \exp\left\{\int\limits_{s_0}^{s} \frac{d\sigma}{f_{k0}(\sigma)}\left[\delta \mathbf{F}_B^{(k)}(\sigma) + \mathbf{F}_{N0}^{(k)}(\sigma)\right] \cdot \nabla_p f_{k0}(\sigma)\bigg|_{\mathbf{x}(\sigma)\ ,\ \mathbf{p}(\sigma)}\right\}. \tag{3.11}$$

4 The Fokker-Planck Equation

To derive the desired equation let us define the distribution function $\mathcal{F}_{k0}(\mathbf{x}, \mathbf{p}; s)$ and the fluctuation $\delta f_{k0}(\mathbf{x}, \mathbf{p}; s)$ according to the relations:

$$\mathcal{F}_{k0}(\mathbf{x}, \mathbf{p}; s) = \langle f_{k0}(\mathbf{x}, \mathbf{p}; s) \rangle \qquad ; \qquad \delta f_{k0}(\mathbf{x}, \mathbf{p}; s) = f_{k0}(\mathbf{x}, \mathbf{p}; s) - \mathcal{F}_{k0}(\mathbf{x}, \mathbf{p}; s), \tag{4.1}$$

where $\langle \cdots \rangle$ implies statistical average. Neglecting second order terms and correlators in δf_{k0} and $\delta f_{(3-k)0}$ that generally give rise to collision integrals, we write down the equations for $\mathcal{F}_{k0}$ and δf_{k0}

$$\frac{\partial \mathcal{F}_{k0}}{\partial s} + \mathbf{p} \cdot \nabla_x \mathcal{F}_{k0} + \left(\mathbf{F}_L^{(k)} + \mathbf{F}_{L0}^{(k)} \right) \cdot \nabla_p \mathcal{F}_{k0} + \nabla_p \cdot \left(\bar{\mathbf{F}}_R^{(k)} \mathcal{F}_{k0} \right) =$$

$$= -\nabla_p \cdot \left\langle \widetilde{\mathbf{F}}_R^{(k)} \xi_k(s) \delta f_{k0} \right\rangle, \tag{4.2}$$

$$\frac{\partial \delta f_{k0}}{\partial s} = -\nabla_p \cdot \left(\widetilde{\mathbf{F}}_R^{(k)} \xi_k(s) \mathcal{F}_{k0} \right) + O(\delta f_{k0}), \tag{4.3}$$

where $\bar{\mathbf{F}}_R^{(k)}$ and $\widetilde{\mathbf{F}}_R^{(k)}$ denote the deterministic and the stochastic parts of the radiation friction force $\mathbf{F}_R^{(k)}$ respectively. Moreover, the force $\mathbf{F}_{L0}^{(k)}$ should be calculated now with the distribution function $\mathcal{F}_{k0}$. Equation (4.3) has a trivial solution

$$\delta f_{k0}(s) = -\nabla_p \cdot \int_0^\infty d\sigma \widetilde{\mathbf{F}}_R^{(k)}(s - \sigma) \xi_k(s - \sigma) \mathcal{F}_{k0}(s - \sigma), \tag{4.4}$$

which is substituted into equation (4.2) yielding the Fokker-Planck equation:

$$\frac{\partial \mathcal{F}_{k0}}{\partial s} + \mathbf{p} \cdot \nabla_x \mathcal{F}_{k0} + \left(\mathbf{F}_L^{(k)} + \mathbf{F}_{L0}^{(k)} \right) \cdot \nabla_p \mathcal{F}_{k0} + \nabla_p \cdot \left(\bar{\mathbf{F}}_R^{(k)} \mathcal{F}_{k0} \right) =$$

$$= \nabla_p \cdot \left[\widetilde{\mathbf{F}}_R^{(k)} \nabla_p \cdot \left(\widetilde{\mathbf{F}}_R^{(k)} \mathcal{F}_{k0} \right) \right]. \tag{4.5}$$

In order to carry out the δf method effectively, it is important to find an equilibrium solution of f_0 (or very slowly varying solution) so that the evolution of δf is separate in time scale from that of f_0. In the following we discuss the equation and the solution of the f_0 distribution.

For the sake of simplicity, in what follows bellow in this Section, we consider one dimension only (say x), since the results can be easily generalized to the multidimensional case, provided the x-z coupling is neglected. Let us write down the Fokker-Planck equation (4.5) in the simplified form:

$$\frac{\partial \mathcal{F}_{k0}}{\partial s} + p \frac{\partial \mathcal{F}_{k0}}{\partial x} - F_k(s) x \frac{\partial \mathcal{F}_{k0}}{\partial p} = \Gamma_k \frac{\partial}{\partial p} (p \mathcal{F}_{k0}) + \mathcal{D}_k \frac{\partial^2 \mathcal{F}_{k0}}{\partial p^2}, \tag{4.6}$$

where

$$\Gamma_k = \frac{p_{k0} \mathcal{C}_1}{2\pi R} \int\limits_0^{2\pi R} ds |\mathbf{B}_k(s)|^2 \qquad ; \qquad \mathcal{D}_k = \frac{p_{k0}^2 \mathcal{C}_2}{4\pi R} \int\limits_0^{2\pi R} ds |\mathbf{B}_k(s)|^3 \left\langle p_{kx}^2(s) \right\rangle, \tag{4.7}$$

$$F_k(s) = \frac{G_x^{(k)}(s)}{R^2} + \lambda_k \delta_p(s) A_x^{(k)}(s) \qquad ; \qquad A_x^{(k)}(s) x = \left. \frac{\partial V_k}{\partial x} \right|_{\text{linear part}}. \tag{4.8}$$

Let us seek for a solution of the Fokker-Planck equation (4.6) in the general form:

$$\mathcal{F}_{k0}(x, p; s) = a_k(s) \exp \left[- \frac{\widetilde{\gamma}_k(s) x^2 + 2\widetilde{\alpha}_k(s) x p + \widetilde{\beta}_k(s) p^2}{2\epsilon_{x0}^{(k)}} \right], \tag{4.9}$$

where $\epsilon_{x0}^{(k)}$ is a scaling factor with dimensionality and meaning of emittance. Direct substitution of (4.9) into (4.6) and equating similar powers (up to second order) in x and p yield the following equations for the unknown coefficients:

$$\frac{da_k}{ds} = \Gamma_k a_k \left(1 - \frac{\widetilde{\beta}_k}{\beta_k^{(eq)}} \right), \tag{4.10}$$

$$\frac{d\widetilde{\alpha}_k}{ds} = F_k \widetilde{\beta}_k - \widetilde{\gamma}_k + \Gamma_k \widetilde{\alpha}_k \left(1 - \frac{2\widetilde{\beta}_k}{\beta_k^{(eq)}} \right), \tag{4.11}$$

$$\frac{d\widetilde{\beta}_k}{ds} = -2\widetilde{\alpha}_k + 2\Gamma_k \widetilde{\beta}_k \left(1 - \frac{\widetilde{\beta}_k}{\beta_k^{(eq)}} \right), \tag{4.12}$$

$$\frac{d\widetilde{\gamma}_k}{ds} = 2F_k \widetilde{\alpha}_k - 2\Gamma_k \frac{\widetilde{\alpha}_k^2}{\beta_k^{(eq)}}, \tag{4.13}$$

where

$$\beta_k^{(eq)} = \frac{\Gamma_k \epsilon_{x0}^{(k)}}{\mathcal{D}_k} \tag{4.14}$$

is the *equilibrium* β-function.

It is important to note that when the damping vanishes ($\Gamma_k = 0$) the above equations are exactly the same as the well-known differential equations for the Courant-Snyder parameters. In this sense the functions $\widetilde{\alpha}_k$, $\widetilde{\beta}_k$ and $\widetilde{\gamma}_k$ can be regarded as a generalization of the Courant-Snyder parameters in the case when radiation damping and quantum excitation are present. The well-known quantity

$$\widetilde{\mathcal{I}}_k = \det \begin{pmatrix} \widetilde{\gamma}_k & \widetilde{\alpha}_k \\ \widetilde{\alpha}_k & \widetilde{\beta}_k \end{pmatrix} = \widetilde{\beta}_k \widetilde{\gamma}_k - \widetilde{\alpha}_k^2 \tag{4.15}$$

is no longer invariant. It is easy to check that its dynamics is governed by the equation

$$\frac{d\widetilde{\mathcal{I}}_k}{ds} = 2\Gamma_k \widetilde{\mathcal{I}}_k \left(1 - \frac{\widetilde{\beta}_k}{\beta_k^{(eq)}} \right). \tag{4.16}$$

Comparison between equations (4.10) and (4.16) shows that

$$a_k(s) = C_{k0} \sqrt{\widetilde{\mathcal{I}}_k(s)} \tag{4.17}$$

with C_{k0} an arbitrary constant as it should be. Therefore the solution (4.9) takes its final form

$$\mathcal{F}_{k0}(x, p; s) = \frac{\sqrt{\widetilde{\mathcal{I}}_k(s)}}{2\pi \epsilon_{x0}^{(k)}} \exp \left[- \frac{\widetilde{\gamma}_k(s) x^2 + 2\widetilde{\alpha}_k(s) xp + \widetilde{\beta}_k(s) p^2}{2\epsilon_{x0}^{(k)}} \right], \tag{4.18}$$

Let us define now the dimensionless *envelope* function σ_k according to the relations

$$\sigma_k = \frac{\sqrt{\beta_{ke}}}{a_k} \qquad ; \qquad \beta_{ke} = \frac{\widetilde{\beta}_k}{\beta_k^{(eq)}}. \tag{4.19}$$

Manipulating equations (4.11), (4.12) and (4.13) for the generalized Courant-Snyder parameters one can eliminate $\widetilde{\alpha}_k$ and $\widetilde{\gamma}_k$ and obtain a single equation for the envelope σ_k, which combined with equation (4.10) comprises a complete set:

$$\frac{d^2\sigma_k}{ds^2} + \Gamma_k \frac{d\sigma_k}{ds} + F_k \sigma_k = \frac{1}{\beta_k^{(eq)2} a_k^2 \sigma_k^3}, \tag{4.20}$$

$$\frac{da_k}{ds} = \Gamma_k a_k \Big(1 - a_k^2 \sigma_k^2\Big). \tag{4.21}$$

By solving equations (4.20) and (4.21) one can obtain a complete information about the evolution of the $\mathcal{F}_{k0}$ part of the distribution function. However, solving the above system of equations for the beam envelopes and amplitudes of the distributions is not an easy task. For that purpose we develop in the next Section a numerical scheme which is more suited for direct code implementation.

5 Numerical Algorithm

In the previous Sections, we have established the theoretical foundation of the nonlinear δf method for the beam-beam interaction. In this Section we will apply those results to outline numerical algorithms suitable for computer simulation.

Starting with Eq. (3.4), because the forces in the equation both from lattice and the on-coming beam are linear, its solution is well known Gaussian distribution (for example as shown in the previous Section in the one-dimensional case)

$$\mathcal{F}_{k0}(\mathbf{z}; s) = \frac{1}{\left[2\pi \det\left(\widehat{\Sigma}_k\right)\right]^{\frac{3}{2}}} \exp\left(-\frac{1}{2}\mathbf{z}^T \cdot \widehat{\Sigma}_k^{-1} \cdot \mathbf{z}\right), \tag{5.1}$$

where $\widehat{\Sigma}_k$ is the matrix of the second moments for the distribution and $\mathbf{z}$ is a vector in the six-dimensional phase space. Based on the method of the beam-envelope [14], the propagation of $\mathcal{F}_{k0}$ can be represented as the iteration of the $\widehat{\Sigma}_k$ matrix,

$$\widehat{\Sigma}_k^{(i+1)} = \widehat{\mathcal{M}}_k \cdot \widehat{\Sigma}_k^{(i)} \cdot \widehat{\mathcal{M}}_k^T + \widehat{D}_k, \tag{5.2}$$

where $\widehat{\mathcal{M}}_k$ is the one-turn matrix including the linear beam-beam force of the on-coming beam, and the radiation damping and $\widehat{D}_k$ is the one-turn quantum diffusion matrix. Both $\widehat{\mathcal{M}}_k$ and $\widehat{D}_k$ can be extracted from the lattice using for example the LEGO code [15], [16]. However, there is a difference compared to the situation of a single storage ring, namely, we have to simultaneously iterate the Gaussian distribution for both beams, since the linear map $\widehat{\mathcal{M}}_k$ depends on the beam size of the other beam.

Combining Eqs. (3.1) and (3.7), the perturbative part of the beam distribution δf_k has a representation in terms of macro-particles

$$\delta f_k(\mathbf{x}, \mathbf{p}; s) = \frac{1}{N_k} \sum_{n=1}^{N_k} W_k^{(n)}(s) \delta\Big[\mathbf{x} - \mathbf{x}_n^{(k)}(s)\Big] \delta\Big[\mathbf{p} - \mathbf{p}_n^{(k)}(s)\Big], \tag{5.3}$$

where $W_k^{(n)}(s)$ is the dynamical weight of the n-th particle from the k-th beam.

As a part of the solution for Eq. (3.9), the propagation of the particle coordinates in phase space is the same as the conventional PIC code [8] provided that the beam-beam force is the sum of the two parts from both $\mathcal{F}_{k0}$ and δf_k.

For the $\mathcal{F}_{k0}$ part, we can apply the well known Erskine-Bassetti formula [17] for a Gaussian beam. The force due to the δf_k is obtained by solving the two-dimensional Poisson equation. In addition to the change of the coordinate, the weight of the particle should be propagated according to Eq. (3.11). The weight should be updated after the change of the coordinate since the change of the weight depends on the trajectory of the particle.

6 Summary

We have developed an efficacious algorithm for simulating the beam-beam interaction in a synchrotron collider with (or without) synchrotron radiation. The nonlinear δf method has been introduced into the evolutionary description of subtle changes of the counter streaming distribution of the colliding beams over many revolutions. The overall equation that describes this evolution is the Fokker-Planck equation (with the radiative process and quantum fluctuations). In order to isolate the δf distribution from the average distribution, we analyze the solution of the Fokker-Planck equation. Obtained is a form of solution in which the time dependence is parameterized through a slow evolution (slow compared with the changes in the δf distribution due to the individual beam-beam interaction) in the Courant-Snyder parameters and the emittance of the beam. This algorithm will enhance the analysis capability to scrutinize greater details and subtle effects in the beam-beam interaction than the PIC version which has been widely deployed [8].

The current algorithm as well as the previous one [8] have been developed with an immediate application to the PEP-II B-factory collider. The code [8] has already been applied to describe the beam-beam interaction in the PEP-II with unprecedented accuracy and reproduction faithfulness, and will be sufficient to study the overall dynamics such as the analysis of resonance instabilities and associated luminosity functions. It is anticipated, however, that the numerical noise associated with the PIC will require either an inordinate amount of macro-particle deployment or a level of noise high enough to mask some minute phase space structure that may manifest in subtle but important long-time evolution of the beam such as particle diffusion. It is here that the current algorithm will cope with the problem.

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