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Quantum theory of Optical Stochastic Cooling *

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1 Abstract

Quantum theory of the optical stochastic cooling [1] is presented. Consideration follows the evolution of the density matrix of a bunch of particles interacting with radiation in the undulators and quantum amplifier.

2 Introduction

Optical stochastic cooling was proposed recently [1],[2]. In the method, radiation is generated by a particle in a (pickup) undulator and, after amplification in the optical amplifier, is send to another undulator (kicker). In the kicker, amplified wave of radiation interacts with the same particle providing desirable cooling. The phase shift of the off-momentum partilce in respect with the wave is controlled in a dispersion section between two undulators. Effect of radiation of a given particle on other particles in the beam leads to diffusion and limits the damping rate. In this respect, optical stochastic cooling is not different from the rf stochastic cooling. In the later [3], interaction of particles changes momentum of the j-th particle

$$\bar{p}_j = p_j - \Gamma p_j - \Gamma \sum_{i \neq j} p_i, \tag{1}$$

where Γ is parameter of interaction between particles proportional to the electronic gain of the amplifier. The rms value $\Delta^2 = (1/N_b)\sum_j [< (p_j)^2 > - < p_j >^2]$ for initially uncorrelated particles changes to $\bar{\Delta}^2 = \Delta^2[1 - 2\Gamma + N_s\Gamma^2]$, where N_s is number of particles interacting through the amplifier (number of particles per "slice"). The maximum cooling rate $(\bar{\Delta}^2 - \Delta^2)/\Delta^2 = -\Gamma$, and is achieved for $\Gamma = 1/N_s$. The number of particles per slice $N_s = N_B c/(\sigma_B^0 \Delta f)$, where σ_B^0 is the bunch length in the laboratory frame, N_B is number of particles per bunch, and Δf is the (full) bandwidth of the amplifier.

In the case of the optical stochastic cooling, the bandwidth $\Delta f \simeq \gamma_0^2(c/L_u)$ where $L_u = N_u\lambda_u$ is the undulator length and γ_0 is relativistic factor of the beam in the laboratory frame. Large Δf is advantage of the optical stochastic cooling allowing fast cooling. Parameters of the undulator has to be chosen to match the undulator mode to the central frequency and bandwidth Δf of the amplifier. For the typical solid state Ti:Sapphire amplifier ($\lambda = 0.8\mu$, $\Delta f/f \simeq 1/5$). Given bandwidth, the fast cooling (for example, for the muon collider) can be achieved reducing N_s. However, with small number of particles per slice, classical and quantum fluctuations could be dangerous. This is the primary motivation of the study we present here. Related problem might be amplification of the noise induced by interaction of particles in the undulator and by the noise of the amplifier.

In our consideration we follow evolution of the density matrix of the system (bunch plus radiated mode) through the undulators and quantum amplifier. Dynamics in the undulators is described in the next section as 1D dynamics in the rest frame of a bunch as it is outlined by Dattoli-Renieri [4], [5] where other references can be found. The formalism we use to describe radiation of the beam in the undulators reproduces results but is different from Becker and McIver [6] formalism and has been described elsewhere [7]. In this formalism as well as in the Becker-McIver's formalism, number of particles per bunch N_s can be arbitrary, but effect of bunching is neglected. In this sense the interaction of particles with radiation is weak. This assumption substantially simplifies consideration being quite adequate for describing optical stochastic cooling. Evolution of the density matrix in quantum amplifier follows our previous note [8]. The theory of quantum amplifier includes the non-diagonal components of the matrix. In the following sections we describe radiation in the kicker in the same way as it was done for the pickup, and then combine results of the previous sections to get moments of the final distribution function. All phase relations are retained through the whole system. In conclusion, we compare the final result for the rms energy spread with the classical theory.

3 Pickup

We assume that, at the entrance to the pickup, there are N_B relativistic particles, there is no initial z, p correlation, and correlations generated in one pass are wiped out in one turn. The pickup (and the kicker) undulators are helical with the undulator parameter K_0 and period $\lambda_u = 2\pi/k_u$. The bunch dynamics is considered in the Bambini-Renieri frame moving with the relativistic factor $\gamma = \gamma_0/\sqrt{1+K_0^2}$, where the bunch centroid initially has zero velocity, and the resonance frequency of the mode is $k = \gamma k_u$. At entrance to the pickup, each particle is described by the density matrix $\rho^0(p_i, z_i)$, $i = 1, 2..N_b$. ρ_0 is the wave packet localized at the point (z_i^0, p_i^0) in the space phase,

$$\rho^0(p', p) = \frac{h\sqrt{2\pi}}{L\Delta} e^{-\frac{i}{h}(p'-p)z_0 - \frac{1}{2}(\frac{\sigma}{h})^2(p'-p)^2 - \frac{1}{2}(\frac{1}{\Delta})^2(\frac{p+p'}{2}-p_0)^2}, \quad (2)$$

where σ and Δ are the rms values of the wave packet which may be small compared to the rms energy spread Δ_B and rms length σ_B of a bunch, and L is normalization length. The density matrix of the whole bunch $\hat{\rho} = \Pi_{i=1}^{N_B} |p' > \rho^0(p_i', p_i) < p|$.

In the moving frame, interaction of particles with the mode $k = \omega/c$ is described by the Hamiltonian

$$H = \sum_{i=1}^{N_B} \frac{\hat{p}_i^2}{2m} + h\omega(a^+a + 1/2) - ihg[ae^{2ik\hat{z}+i\omega t} - cc], \quad (3)$$

where $m = m_e\sqrt{1+K_0^2}$.

If the vector-potential of the radiation is normalized to one photon per volume V [6], [7]

$$\vec{A} = \sqrt{\frac{2\pi hc^2}{V\omega}}\vec{y}[\hat{a}e^{ik\hat{z}} + cc], \quad |\vec{y}| = 1, \tag{4}$$

then parameter of interaction $g_k = c\frac{K_0}{\sqrt{1+K_0^2}}\sqrt{\frac{e^2}{hc}\frac{2\pi}{kV}}$. However, we consider 1D model where beam interacts with a single radiated mode. In this case, operator a, a^+ are operators changing number of coherent photons in the mode, and the vector potential Eq. (4) has to be normalized to the phase volume Ω of the mode.

In the laboratory frame [11], $\Omega = (V/(2\pi)^3)(\pi k^3/N_u^2)$. The later is defined in the laboratory frame by the constrain $|2\pi N_u - \psi| <= \Pi$ on the phase slippage $\psi = |\omega t - k_z z|$ along the undulator, and requirement that the frequency spread $|(\omega - \omega_r)/\omega_r| < 1/(2N_u)$, where ω_r is resonance frequency of radiation at zero angle. Result in the moving frame follows from relativistic invariance of d^3k/ω.

The normalized vector potential is obtained by multiplying Eq.(4) by $\sqrt{\Omega}$. Parameter of interaction with the mode is then $g = g_k\sqrt{\Omega}$ and, using time of the interaction in the moving frame $t = 2\pi N_u/(ck)$, we get $gt = (K_0/\sqrt{1+K_0^2})\sqrt{\pi e^2/hc}$, i.e. $(gt)^2$ of the order of $\alpha_0 = e^2/hc$.

Interaction of particles with the mode described by Hamiltonina Eq. (3) is just back-scattering of equivalent photons. Initial state $|p_i, n>= |p_1, p_2, .., p_{N_B}, n>$ of the system with n-photons and particles with momentums p_i, $i = 1..N_B$ is transformed by the interaction with the mode $k = \omega/c = \gamma k_u$ to the vector

$$|\Psi(t)>= \sum_{l_i, p_i} |p_i - 2hkl_i, n + l_\Sigma> \sqrt{\frac{n!}{(n+l_\Sigma)!}}F_n(t, p_i, l_i)e^{-i\omega t(n+l_\Sigma)-(it/h)\sum_i E(p_i,l_i)}, \tag{5}$$

where $E(p_i, l_i) = (p_i - 2hkl_i)^2/(2m_e)$, and $l_\Sigma = \sum_i l_i$ is the total number of radiated photons. Function $F_n(t)$ is given [7] by

$$F_n(t,p,l) = \int_0^\infty d\lambda \frac{\lambda^n}{n!}e^{-\lambda}\hat{O}_{\lambda\kappa}\{\Pi_{i=1}^{N_B}(\frac{\lambda a_i}{\kappa a_i^*})^{l_i/2}J_{l_i}(2g|a_i|\sqrt{\lambda\kappa})\}|_{\kappa=1}, \tag{6}$$

where we neglected a small phase factor. Here J_l is Bessel function, operator $\hat{O}_{\lambda\kappa} = e^{-(1/2)\frac{\partial^2}{\partial\lambda\partial\kappa}}$, and

$$a_i(t) = \frac{\sin(\epsilon_i t/2)}{(\epsilon_i/2)}e^{-i\epsilon_i t/2}, \quad \dot{a}_i(t) = e^{-i\epsilon_i t}, \quad \epsilon_i = \frac{2kp_i}{m_e}. \tag{7}$$

As the main simplification [4] of the theory, terms of the order of hk^2/m_e in Eq. (4) are neglected. As a result, we loose effect of bunching due to radiation. However, this is sufficient for our purpose. For short undulators, $kpt/m_e << 1$, $\frac{\sin(\epsilon_i t/2)}{(\epsilon_i/2)} \simeq t$, and F_n depends on parameter gt, where t is time of flight in the undulator ($t = N_u\lambda_u/(c\gamma)$ in the moving frame), and g is parameter defining coupling of a particle to radiation.

We assume that at the entrance to the pickup there is no radiation, $n = 0$. In this case, initial density matrix $\hat{\rho} = \Pi_{i=1}^{N_B}|p' > \rho^0(p'_i, p_i) < p|$ is transformed according to Eqs. (5), (6) (cp. with Eq. (37) of the reference [7]) to $\hat{\rho}(t) = |q', l'_\Sigma > \rho(q', q, l_\Sigma, l'_\Sigma) < q, l_\Sigma$, where

$$\rho(q', q, l'_\Sigma, l_\Sigma) = \frac{1}{\sqrt{l_\Sigma! l'_\Sigma!}} \int \frac{d\psi d\psi'}{(2\pi)^2} e^{-i(l'_\Sigma\psi' - l_\Sigma\psi)} e^{i\omega t(l_\Sigma - l'_\Sigma)} \int d\lambda d\lambda' e^{-\lambda-\lambda'} \hat{O}_{\lambda\kappa}\hat{O}_{\lambda'\kappa'} F_{loc}(q', q). \tag{8}$$

Here $|q >$ stands for the set $|q_1..q_{N_B} >$, $F_{loc}(q', q) = \Pi_{i=1}^{N_B} F^i_{loc}$,

$$F^i_{loc}(q'_i, q_i) = \sum_{l,l'} f'_i f^*_i \rho^0(q'_i + 2hkl'_i, q_i + 2hkl_i) e^{-i\frac{((q'_i)^2 - q_i^2)t}{2m_e h}}, \tag{9}$$

where $f_i = f(q_i, l_i, \psi)$, $f'_i = f(q'_i, l'_i, \psi')$,

$$f(q, l, \psi) = (\frac{\lambda a}{\kappa a^*})^{l/2} J_l[2g|a(t)|\sqrt{\lambda\kappa}] e^{il\psi}. \tag{10}$$

Integration over ψ, ψ' is introduced in Eq. (8) to separate the global parameters l_Σ, l'_Σ of the radiation and particle parameters $\{q_i, l_i\}$. It is convenient to consider Fourier transform

$$F^i_{loc}(p, z) = \int \frac{Ldq}{2\pi h} e^{iqz/h} F^i_{loc}(p + q/2, p - q/2). \tag{11}$$

For a short undulator, parameter $\epsilon_i t << 1$. In this case, $a(t) \simeq te^{-i\epsilon_i t/2}$. Parameter $(gt)^2$ has meaning of the average number of photons radiated in the undulator per particle and is always small. This justifies expansion of f_i in series over gt. Neglecting terms of the order of $(gt)^3$, we write for the $i-$th particle $F^i_{loc}(p, z) = F^0_i(p, z)(1 + gtF^{(1)}_i + (gt)^2 F^{(2)}_i)$,

$$F^{(1)}_i = e^{-(1/2)(hk/\Delta)^2}\{-\kappa e^{\frac{hk(p-p^0)}{\Delta^2} + i\psi - 2ik(z - \frac{pt}{2m_e})} - \kappa' e^{\frac{hk(p-p^0)}{\Delta^2} - i\psi' + 2ik(z - \frac{pt}{2m_e})} \tag{12}$$

$$+\lambda e^{-\frac{hk(p-p^0)}{\Delta^2} - i\psi + 2ik(z - \frac{pt}{2m_e})} + \lambda' e^{-\frac{hk(p-p^0)}{\Delta^2} + i\psi' - 2ik(z - \frac{pt}{2m_e})}\}, \tag{13}$$

where

$$F^0_i(p, z) = \frac{h}{\sigma\Delta} e^{-\frac{(p-p_0)^2}{2\Delta^2} - \frac{(z - z^0 - pt/m_e)^2}{2\sigma^2}}. \tag{14}$$

$F^{(2)}$ has a similar structure.

With the same accuracy,

$$F_{loc}(p, z) = \{\Pi_{i=1}^{N_B} F^0_i(p_i, z_i)\} e^{gt\Sigma_i F^{(1)}_i + (gt)^2 \Sigma_i F^{corr}_i}, \tag{15}$$

where $F_i^{corr} = F_i^{(2)} - (1/2)[F_i^{(1)}]^2$. Eq. (15) takes into account all terms of the order of $N_b gt$ and $N_b(gt)^2$ neglecting terms $N_b(gt)^3$.

The sum $f_0 \equiv gt\sum_i F_i^0$ in Eq. (15) is defined by parameters

$$\sigma_\pm(p,z) = gt\sum_{i=1}^{N_B} e^{-2ik(z_i-\frac{p_i t}{2m_e})\pm\frac{hk(p_i-p_i^0)}{\Delta^2}} e^{-\frac{1}{2}(\frac{hk}{\Delta})^2}. \tag{16}$$

This expression has to be averaged over frequency spread in the mode around $\bar{k} = \gamma k_u$:

$$\sigma_\pm(p,z) = gt\sum_{i=1}^{N_B} e^{-2i\bar{k}(z_i-\frac{p_i t}{2m_e})\pm\frac{h\bar{k}(p_i-p_i^0)}{\Delta^2}} e^{-\frac{1}{2}(\frac{h\bar{k}}{\Delta})^2} s_i. \tag{17}$$

where

$$s_i = \int \frac{dk}{\pi} e^{-2i(k-\bar{k})(z_i-p_i t/2m_e)} \frac{\sin^2(\pi N_u(k-\bar{k})/\bar{k})}{(\pi N_u/\bar{k})(k-\bar{k})^2}. \tag{18}$$

Factor s_i restricts summation over particles within the length (length of a "slice") $\propto 2\pi N_u/(2\bar{k})$ or, in the laboratory system, within $l_s = N_u\lambda_{lab}$. Parameter $N_s =<< \sigma_-\sigma_-^* >> /(gt)^2$ is the fundamental parameter of the theory defining number of interacting particles within the bandwidth of the mode (number of particles per slice). Here double averaging means averaging with the density matrix of the wave packet Eq.(15) and within the Gaussian bunch $\rho_B(z_0,p_0) = (1/2\pi\sigma_B\Delta_B)e^{-p_0^2/2\Delta_B^2 - z_0^2/2\sigma_B^2}$ over z^0, p^0. If the width of the packet σ is of the order of the length of a slice and $N_u >> 1$, then $\bar{k}\sigma >> 1$, and

$$N_s = \sum_i \int \frac{dx}{\pi}\frac{\sin^2 x}{x^2}\frac{dy}{\pi}\frac{\sin^2 y}{y^2} << e^{-\frac{2ik}{\pi N_u}(x-y)(z_i-\frac{p_i t}{2m_e})} >> . \tag{19}$$

Neglecting terms of the order of h, we get

$$N_s = N_b\frac{N_u\sqrt{2\pi}}{3k\sigma_B}, \tag{20}$$

where σ_B is rms bunch length in the moving frame and we use $\int(dx/\pi)(\sin x/x)^4 = 0.6666$. In terms of the wave length of the mode and the bunch length in the laboratory frame, $N_s = N_B(\frac{N_u\lambda_L}{3\sqrt{2\pi}\sigma_B^0})$.

In terms of averaged $\sigma_\pm$, Eq. (17),

$$f_0(\psi,\psi') = -\kappa\sigma_+e^{i\psi} - \kappa'\sigma_+^*e^{-i\psi'} + \lambda\sigma_-^*e^{-i\psi} + \lambda'\sigma_-e^{i\psi'}. \tag{21}$$

The second terms $(gt)^2\Sigma_i F_i^{corr}$ in the exponent of Eq. (15) can be expanded over h. Expansion starts with the term proportional to h^2. It can be split in two parts: one,

$$f_{cor}^{(1)} = -N_s(gt)^2(\frac{hk}{\Delta})^2(\kappa e^{i\psi} + \lambda' e^{i\psi'})(\kappa' e^{-i\psi'} + \lambda e^{-i\psi}), \tag{22}$$

which is proportional to the number of particles N_s, and $f^{(2)}_{cor}$, proportional to the sum over oscillating factors. Introducing $r_\pm = \Sigma_i e^{\pm 4ik(z_i - p_i t/2m_e)}$, we can write

$$f^{(2)}_{cor} = -\frac{(gt)^2}{2}(\frac{hk}{\Delta})^2[(\kappa e^{i\psi} + \lambda' e^{i\psi'})^2 r_- + (\kappa' e^{-i\psi'} + \lambda e^{-i\psi})^2 r_+]. \qquad (23)$$

In these notations,

$$F_{loc}(p, z) = \{\Pi_{i=1}^{N_B} F_i^0(p_i, z_i)\} e^{f_0(\psi,\psi') + f^{(1)}_{cor} + f^{(2)}_{cor}}. \qquad (24)$$

The first factor is the product of unperturbed single particle distribution functions while exponent describes particle interaction. The last term, $f^{(2)}_{cor}$ is small. Eq. (15) can be simplified writing $e^{f^{(2)}_{cor}} = (1 + f^{(2)}_{cor})$ and replacing $-gt\kappa' e^{-i\psi'}$, $gt\lambda e^{-i\psi}$, $-gt\kappa e^{i\psi}$, and $gt\lambda' e^{i\psi'}$ by the derivatives over σ_+^*, σ_-^*, σ_+, and σ_-, respectively. The result is differential operator $\hat{P}(\sigma_\pm)$. The factor $e^{f^{(1)}_{cor}}$ can be written as

$$e^{f^{(1)}_{cor}} = \hat{O}_{\mu,\nu} e^{-\nu(\kappa e^{i\psi} + \lambda' e^{i\psi'}) - \mu(\kappa' e^{-i\psi'} + \lambda e^{-i\psi})}|_{\mu=\nu=0}, \qquad (25)$$

where $\hat{O}_{\mu,\nu} = e^{-\zeta^2 \frac{\partial^2}{\partial\mu\partial\nu}}$, and $\zeta^2 = N_s (gt)^2 (\frac{hk}{\Delta})^2$. Then,

$$F_{loc}(p, z) = \{\Pi_{i=1}^{N_B} F_i^0(p_i, z_i)\}(1+\hat{P})\hat{O}_{\mu,\nu} e^{-\kappa(\sigma_+ + \nu)e^{i\psi} - \kappa'(\sigma_+^* + \mu)e^{-i\psi'} + \lambda(\sigma_-^* - \mu)e^{-i\psi} + \lambda'(\sigma_- - \nu)e^{i\psi'}}. \qquad (26)$$

Now it is easy to calculate

$$\hat{O}_{\kappa,\lambda}\hat{O}_{\kappa',\lambda'} e^{-\kappa(\sigma_+ + \nu)e^{i\psi} + \lambda(\sigma_-^* - \mu)e^{-i\psi} - \kappa'(\sigma_+^* + \mu)e^{-i\psi'} + \lambda'(\sigma_- - \nu)e^{i\psi'}}|_{\kappa=\kappa'=1} \qquad (27)$$

$$= e^{(1/2)(\sigma_+ + \nu)(\sigma_-^* - \mu) + (1/2)(\sigma_+^* + \mu)(\sigma_- - \nu)} e^{-(\sigma_+ + \nu)e^{i\psi} + \lambda(\sigma_-^* - \mu)e^{-i\psi} - (\sigma_+^* + \mu)e^{-i\psi'} + \lambda'(\sigma_- - \nu)e^{i\psi'}}. \qquad (28)$$

Integration over ψ and ψ' can be carried out using

$$\int \frac{d\psi}{2\pi} e^{il\psi} e^{\lambda e^{-i\psi} - \kappa e^{i\psi}} = (\frac{\lambda}{\kappa})^{l/2} J_l(2\sqrt{\lambda\kappa}). \qquad (29)$$

After that, integrals over λ and λ' are known [9]

$$\int_0^\infty d\lambda e^{-\lambda} \lambda^{l/2} J_l(2\sqrt{\lambda a}) = a^{l/2} e^{-a}. \qquad (30)$$

The distribution function at the end of the pickup

$$\rho(p, z, l'_\Sigma, l_\Sigma) = \int \frac{L dq}{2\pi h} e^{iqz/h} \rho(p + q/2, p - q/2, l'_\Sigma, l_\Sigma), \qquad (31)$$

takes form

$$\rho(p, z, l'_\Sigma, l_\Sigma) = \frac{1}{\sqrt{l_\Sigma! l'_\Sigma!}} e^{i\omega t(l_\Sigma - l'_\Sigma)} \{\Pi_{i=1}^{N_B} F_i^0(p_i, z_i)\}(1 + \hat{P}) R(p, z), \qquad (32)$$

where

$$R(p,z) = \hat{O}_{\mu,\nu}(\sigma_-^* - \mu)^{l_\Sigma}(\sigma_- - \nu)^{l'_\Sigma} e^{-(1/2)(\sigma_-^* - \mu)(\sigma_+ + \nu) - (1/2)(\sigma_- - \nu)(\sigma_+^* + \mu)}. \quad (33)$$

For small $\zeta^2 << 1$, and $N = (l_\Sigma + l'_\Sigma)/2$, $\mu = (l_\Sigma - l'_\Sigma)/2$,

$$R(p,z) = e^{-(1/2)(\sigma_-^* \sigma_+ + c.c.)}(\frac{\sigma_-}{\sigma_-^*})^\mu |\sigma_-|^{2N}. \quad (34)$$

Correction $\zeta^2|\sigma_-|^2$ is of the order of $(N_s(gt)^2\frac{hk}{\Delta})^2$ and always negligible.

4 Optical Amplifier and Dispersion Section

For small ζ, the density matrix at the end of the pickup takes form

$$\rho(p,z,l'_\Sigma,l_\Sigma) = \frac{1}{\sqrt{l_\Sigma! l'_\Sigma!}}\{\Pi_{i=1}^{N_B} F_i^0(p_i,z_i)\}(1+\hat{P})R(p,z,N,\mu)e^{i\omega t(l_\Sigma - l'_\Sigma)}, \quad (35)$$

where $R = e^{-(1/2)(\sigma_-^* \sigma_+ + c.c.)}\tilde{R}(p,z,N,\mu)$, and

$$\tilde{R}(p,z,N,\mu) = (\frac{\sigma_-^*}{\sigma_-})^\mu |\sigma_-|^{2N}, \quad N = \frac{l_\Sigma + l'_\Sigma}{2}, \quad \mu = \frac{l_\Sigma - l'_\Sigma}{2}. \quad (36)$$

Now let us transform the density matrix $\rho(p,z,l'_\Sigma,l_\Sigma)$ back to the momentum representation, $\rho(p+q/2, p-q/2) = \{\Pi_i \int (dz_i/L) e^{-i(q'-q)z_i}\}\rho(\frac{q'_i+q_i}{2,z_i})$. The result is

$$\hat{\rho} = |q', l'_\Sigma > \rho(q'q) < q, l_\Sigma|, \quad (37)$$

where

$$\rho(q',q) = \frac{1}{\sqrt{l_\Sigma! l'_\Sigma!}}\{\Pi_i \int \frac{dz_i}{L} F^i(q',q,z)\}(1+\hat{P})R(\frac{q'+q}{2},z,N,\mu)e^{2i\mu\omega t}, \quad (38)$$

and

$$F^i(q',q,z) = \frac{h}{\sigma\Delta} e^{-i(q'-q)z/h - \frac{1}{2\Delta^2}(\frac{q'+q}{2})^2 - \frac{1}{2\sigma^2}(z - \frac{q'+q}{2m_e}t)^2}. \quad (39)$$

Note that $\sigma_\pm$ are functions of the set of coordinates $(z_i, \frac{q'_i+q_i}{2})$ of all particles.

The density matrix Eqs. (35), (36) at the exit of the pickup undulator is the superposition of coherent states. Transformation of such a state in the optical amplifier can be obtained following the recipe formulated in our previous note [8]. The Mellin transform $\tilde{R}_M(N,\mu)$ of $\tilde{R}(\frac{q'+q}{2},z,N,\mu)$,

$$\tilde{R}_M(\zeta,\mu) = \int_{-i\infty}^{i\infty} \frac{dN}{2\pi i}\zeta^{-N}\tilde{R}(\frac{q'+q}{2},z,N,\mu), \quad (40)$$

is proportional to $\delta(\zeta-\zeta_0)$,

$$\tilde{R}_M(\zeta,\mu)=\zeta_0[\frac{\sigma_-^*}{\sigma_-}]^\mu\delta(\zeta-|\sigma_-|^2). \tag{41}$$

Let us for simplicity consider two-level fully inverted amplifier. In this case, after the amplifier, $\tilde{R}(\frac{q'+q}{2},z,N,\mu)$ should be replaced after the amplifier by (see [8], Eq. (22)) by F_{ampl},

$$F_{ampl}(N,\mu)=(N-|\mu|)!\frac{1}{G}[\frac{\sigma_-^*}{\sigma_-}]^\mu[\frac{\sigma_-^*\sigma_-}{G-1}]^{|\mu|} \tag{42}$$

$$(\frac{G-1}{G})^N L_{N-|\mu|}^{2|\mu|}(-\frac{|\sigma_-|^2}{G-1}). \tag{43}$$

Here G is power gain of the amplifier, L_N^m are Laguerre polynomials, and $N=(l_\Sigma+l'_\Sigma)/2$, $\mu=(l_\Sigma-l'_\Sigma)/2$.

So far we considered transform of the main term in Eq. (35). Calculation of the derivatives in the correction term, $\hat{P}R(p,z,N,\mu)$ where $\hat{P}$ is differential operator of the second order in $\sigma_\pm$, gives polynomial of the second order in N multiplied by $R(p,z,N,\mu)$. The result can be written as $\hat{P}(y\frac{\partial}{\partial y})x^\mu y^N$ where $\hat{P}$ is now a differential operator of the second order in y independent of N, and $y=|\sigma_-|^2$, $x=\sigma_-^*/\sigma_-$. It can be transformed in the amplifier in the same way as the main term above.

Dispersion section with momentum compaction α_{MC} and length L_{ds}, introduces (z,p) correlation for each particle by changing the path length in the lab frame by $\Delta z=\alpha_{MC}L_{ds}(p-p^0)/q_0$. In the moving frame, this corresponds to the classical distribution function

$$f(p,z)=\frac{1}{2\pi\Delta\sigma}e^{-\frac{(p-p_0)^2}{2\Delta^2}-\frac{(z-z_0-\eta p)^2}{2\sigma^2}}, \tag{44}$$

where parameter $\eta=\gamma_0\alpha_{MC}L_{ds}/m_ec$. The corresponding density matrix is different from Eq. (2) by the factor $e^{-(i/h)\eta[q'^2-q^2]/2}$.

Hence, the dispersion section modifies $F^i(q',q,z)$ in Eq. (39) which have to be replaced by

$$F^i(q',q,z)e^{-(i/h)\eta[q'^2-q^2]/2}e^{i\theta}. \tag{45}$$

Here, a phase slip θ of a bunch centroid is added and should be controlled in the experiment.

5 Kicker

We obtain the density matrix at the entrance to the kicker combining Eqs. (38), (42), and (45)

$$\hat{\rho}_{in}(t)=|q',l'_\Sigma>\frac{F_{in}}{\sqrt{l_\Sigma!l'_\Sigma!}}<q,l_\Sigma|, \tag{46}$$

where $F_{in} = F_{ds}(q', q)(1 + \hat{P})F_{ampl}(N, \mu)e^{2i\mu\omega t}e^{-\frac{1}{2}[\sigma_-^* \sigma_+ + c.c.]}$, and

$$F_{ds} = \Pi_i \int \frac{dz}{L} F^i(q', q, z) e^{-i\frac{\eta}{2h}[(q')^2 - q^2]}. \quad (47)$$

The transform of the density matrix at the end of the kicker is given by Eq. (5) where n has to be replaced by the number of photons l_Σ. We will use notation m_i for the number of photons radiated by the i-th electron in the kicker and $m_\Sigma = \sum_i m_i$ for the total number of photons. We also assume that parameters of both undulators are the same.

Then, the density matrix at the exit of the kicker

$$\hat{\rho}_{out}(t) = |q' - 2hkm', l'_\Sigma + m'_\Sigma > \Phi_{loc}(q, q', \psi, \psi') F^*_{out}(q, l_\Sigma, m_\Sigma) \quad (48)$$

$$F^*_{out}(q', l'_{\Sigma'}, m'_\Sigma)(1 + \hat{P})F_{ampl}(N, \mu)e^{2i\mu\omega t}e^{-\frac{1}{2}[\sigma_-^* \sigma_+ + c.c.]} < q - 2hkm, l_\Sigma + m_\Sigma|. \quad (49)$$

Here $l_\Sigma = N + \mu$, $l'_\Sigma = N - \mu$, $m_\Sigma = M - \mu$, $m'_\Sigma = M + \mu$. Because l_Σ and l'_Σ are positive, the range of summation is $0 < N < \infty$, $-N < M < \infty$, and $|\mu| < N$. Functions $\sigma_\pm$ in F_{ampl} depend on coordinates of individual particles $\frac{q'_i + q_i}{2}$, z_i. The operator F_{out} is

$$F_{out}(q, l_\Sigma, m_\Sigma) = \frac{1}{\sqrt{(l_\Sigma + m_\Sigma)!}} \int \frac{d\psi}{2\pi} e^{-im_\Sigma \psi} e^{i\omega t(l_\Sigma + m_\Sigma)} \int d\lambda \frac{\lambda^{l_\Sigma}}{l_\Sigma!} e^{-\lambda} \hat{O}_{\lambda\kappa}, \quad (50)$$

where

$$\Phi_{loc}(q, q', \psi, \psi') = \Pi_i \frac{dz_i}{L} F^{(i)}(q', q, z) e^{-\frac{i\eta}{2h}[(q'_i)^2 - (q_i)^2]} S^*_{m_i}(q, \lambda, \kappa) S_i(q', \lambda', \kappa'), \quad (51)$$

and

$$S_{m_i}(q, \lambda, \kappa) = (\frac{\lambda a}{\kappa a^*})^{m_i/2} J_{m_i}[2g|a_i(t)|\sqrt{\lambda\kappa}] e^{im_i\psi} e^{\frac{-i[(q - 2hkm)^2]t}{2m_e h}}. \quad (52)$$

To describe stochastic cooling, it is suffice to calculate the momentum of a particle at the end of the kicker. The average moments for the j-th particle after a bunch passed through the system are $< p_j^k >= Tr[\hat{p}_j^k \hat{\rho}_{out}(t)]$, $k = 0, 1, ..,$ where $\hat{p}$ is momentum operator and brackets $< .. >$ mean averaging over the wave packet. In the momentum representation, only the diagonal components, $q'_i - 2hkm'_i = q_i - 2hkm_i$, $i = 1, 2..N_b$ and $l'_\Sigma + m'_\Sigma = l_\Sigma + m_\Sigma$, contribute in $< p_j^k >$. We can utilize the fact that $\sigma_\pm$ are functions only of the sum $q' + q$ and introduce P, $q'_i = P_i + hk(m'_i - m_i)$, $q_i = P_i - hk(m'_i - m_i)$. This allows us to write

$$< p^n >= [P_j - hk(m_j + m'_j)]^n \Phi_{loc}(1 + \hat{P}) F^*_{out}(q, l_\Sigma, m_\Sigma) F_{out}(q', l'_\Sigma, m'_\Sigma) F_{ampl}(N, \mu), \quad (53)$$

where

$$\Phi_{loc} = \Pi_i \frac{dz_i dP_i}{2\pi\sigma\Delta} \rho_0(P_i, z_i) \sum_{m'_i, m_i} S_{m_i}(\lambda, \kappa) S_{m'_i}(\lambda', \kappa') e^{-2ik(z_i + \eta P_i)(m'_i - m_i)}, \quad (54)$$

and

$$\rho_0(P_i, z_i) = e^{-(P_i-p_i^0)^2/2\Delta^2-(z_i-z_i^0-P_it/m0)^2}. \tag{55}$$

Note, that F_{ampl} depends on $\sigma_\pm$ which are given now by Eq.(17) where p_i are replaced by P_i.

Similarly to what was done for the pickup, we expand $S(\lambda,\kappa)$ in series over gt neglecting terms $o(gt)^3$. We skip over details of calculations and give the final result:

$$< p_j^n >= \sum_{i\neq j} \hat{K}_n(1+\hat{P})F_{out}(q,l_\Sigma,m_\Sigma)F_{out}(q',l'_\Sigma,m'_\Sigma)Q(b_1,b_2)F_{ampl}(P,z)|_{b_2->b_1}. \tag{56}$$

Here the sum stands for integrals $\Pi_i \frac{dz_i dP_i}{2\pi\sigma\Delta}\rho_0(P_i,z_i)$ over all particles in a bunch, and

$$Q(b_1,b_2) = e^{\lambda' b_1 e^{i\psi'}-\kappa' b_2^* e^{-i\psi'}+\lambda b_1^* e^{-i\psi}-\kappa b_2 e^{i\psi}}, \tag{57}$$

where $b_1 = gt\sum_i e^{-i\phi_i}$, and phase $\phi_j = 2k[z_j + p_j\eta]$. Operators $\hat{K}_n$ for different $n = 0,1,2$ are: $\hat{K}_0 = 1$, $\hat{K}_1 = q_j - hkgt(a_\lambda - a_\kappa)$, $\hat{K}_2 = \hat{K}_1^2 + (hk)^2gt(a_\lambda + a_\kappa)$, where

$$a_\kappa = e^{-i\phi_j}\frac{\partial}{\partial b_2} + +e^{i\phi_j}\frac{\partial}{\partial b_2^*}, \quad a_\lambda = e^{i\phi_j}\frac{\partial}{\partial b_1^*} + e^{-i\phi_j}\frac{\partial}{\partial b_1}. \tag{58}$$

Eq. (56) after some calculations, see Appendix, can be written as:

$$< p_j^n >= \sum_{i\neq j}\hat{K}_n\sum_\mu(1+\hat{P})(\frac{\sigma_-^*}{\sigma_-})^\mu I_{2\mu}[2\sqrt{G|b_2-b_1|^2\,|\sigma_- - \sigma_+|^2}] \tag{59}$$

$$(\frac{b_1-b_2}{b_1^*-b_2^*})^\mu e^{(G-1)|b_2-b_1|^2+(1/2)[b_2(b_2^*-b_1^*)+c.c.]}e^{(1/2)[\sigma_-(\sigma_-^*-\sigma_+)+c.c]}|_{b_2=b_1}. \tag{60}$$

The operators $\hat{K}_n$ are not more than the second order differential operators in b_2, b_1 and the function depends on $b_{2,1}$ only through powers of $b_2 - b_1$. Therefore, it is suffice to take into account only terms $\mu = 0$, $\mu = \pm 1/2$ and $\mu = \pm 1$ in the sum over μ. Additionally, we can expand the answer in series over gt and neglect terms $o(h^3)$.

To check the result, we calculated the average $< p_j^n >$ for $n = 0$. This quantity is just the norm of the distribution function and has to be equal one. Indeed, the answer is different from one by the term of the order of $N_s^2(gt)^4(hk/\Delta)^4$.

The result for the moments $n = 1$ and $n = 2$ were obtained with MATHEMATICA. As it will be shown below, the power gain G has to be of the order of $\Delta_b/(hk)$. Hence, $G >> 1$ and we can neglect terms which are independent of G. In this approximation, momentum $\tilde{p}_j$ of the j-th particle at the end of the kicker is

$$\tilde{p}_j = p_j - 2(gt)^2hk\sqrt{G}[\sigma_0^* e^{-2ik(z_j+\eta_{eff}p_j)+i\theta} + c.c.], \tag{61}$$

where $\eta_{eff} = \eta + t/2m_e$, and θ is phase slip of the bunch centroid. Calculation of $\tilde{p}_j^2$ at the end of the kicker gives

$$\tilde{p}_j^2 = p_j^2 - 4G(gt)^2(hk)p_j(\sigma_0^* e^{-2ik(z_j+\eta_{eff}p_j)+i\theta} + c.c.) + 8G(gt)^2(hk)^2(1+(gt)^2\sigma_0\sigma_0^* + c.c.) \tag{62}$$

$$+4\sqrt{G}(gt)^4(hk)^2(b_1\sigma_0^* e^{i\theta} + c.c.). \tag{63}$$

Here $\sigma_0 = \sigma_\pm|_{h\to 0}$. Double averaging over the wave packet $\rho_0(p_j, z_j)$ and over Gaussian distribution of particles in the bunch gives the rms $\Delta^2 =<< p^2 >> - << p >>^2$ at the end of the kicker:

$$\frac{\tilde{\Delta}^2 - \Delta^2}{\Delta^2} = -16\sqrt{G}(gt)^2 \frac{hk}{\Delta_B} \Lambda \sin\theta + 8G(gt)^2 (\frac{hk}{\Delta_B})^2 [1 + N_s(gt)^2] \tag{64}$$

$$+8\sqrt{G}(gt)^4 (\frac{hk}{\Delta_B})^2 N_s \cos\theta e^{-2(k\Delta_B)^2\eta_{eff}^2}. \tag{65}$$

Here $\Lambda = k\Delta_B \eta_{eff} e^{-2(k\Delta_B\eta_{eff})^2}$.

To get damping, we have to choose $\sin\theta = 1$. The damping is maximum if the power gain G of the amplifier is equal to

$$\sqrt{G} = \frac{\Lambda}{(hk/\Delta_B)[1 + N_s(gt)^2]}. \tag{66}$$

Parameter Λ as function of $x = k\Delta_B\eta_{eff}$ has maximum value $\Lambda_{max} \simeq 0.3$ at $x \simeq 2$. This defines the optimum parameter η of the dispersion section.

The optimized reduction of the rms in one pass through the system is

$$\frac{\tilde{\Delta}^2 - \Delta^2}{\Delta^2} = -\frac{8\Lambda_{max}^2}{(gt)^{-2} + N_s}. \tag{67}$$

6 Conclusion

The one pass reduction of the energy spread rms is derived following the evolution of the density matrix through all components of the system. The consideration is fully quantum-mechanical both for the beam and radiation but bunching effect is neglected and length of a slice of the order of $N_u\lambda_{lab}$ is assumed to be small compared to the bunch length in the laboratory frame σ_B^0. The final result Eq. (67) for large $N_s >> 1/(gt)^2$ corresponds to classical theory of stochastic cooling: the damping rate is given by the number of particles N_s per slice. However, for small N_s the damping rate goes to a constant proportional to $1/(gt)^2$, where $(gt)^2 \propto (K_0^2/(1+K_0^2))\alpha_0$. As a result, the minimum number of turns for cooling is of the order of $1/\alpha_0$. The term $1/(gt)^2$ is equivalent to the noise induced by $1/\alpha_0$ particles and is related to the quantum limit of the input noise of the amplifier equal to one photon in a mode. The other quantum mechanical corrections are small, of the order of $(hk)/\Delta_B$ (i.e. $hk_L/\Delta p_L$ in the laboratory frame) and can be noticeable only for very cold beams with energy spread comparable with the photon energy. The cooling is the result of interference of the amplified mode with the mode radiated in the kicker.

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8 Appendix

Eq. (56) can be simplified, first, integrating over ψ and ψ' and then by λ and λ' using formula:

$$\int d\lambda \frac{\lambda^l}{l!} e^{-\lambda} \hat{O}_{\lambda\kappa} (\frac{\lambda}{\kappa})^{m/2} J_m(2\sqrt{\lambda\kappa b})|_{\kappa=1} = b^{m/2} e^{-b/2} L_l^m(b), \tag{68}$$

It can be obtained expanding Bessel function and gives result in terms of Laguerre polynomials L_n^m. Eq. (71) is valid both for $m > 0$ and $m < 0$, where $L_l^{-|m|}(b)$ has to be understood as

$$L_l^{-|m|}(b) = (-1)^m \frac{(l-|m|)!}{l!} b^{|m|} L_{l-|m|}^{|m|}(b). \tag{69}$$

In this way we obtain

$$F_{out}(l_\Sigma, M_\Sigma) F_{out}(l'_\Sigma, M'_\Sigma) Q(b_1, b_2) = (b_1^*)^{M-\mu} (b_1)^{M+\mu} e^{-\frac{1}{2}(b_2 b_1^* + c.c.)} L_{N+\mu}^{M-\mu}(b_2 b_1^*) L_{N-\mu}^{M+\mu}(b_2^* b_1). \tag{70}$$

where $b_1 = gt \sum e^{-2ik(z_j+\eta p_j)}$. The average, $< p_j^k >$ is proportional to the sum

$$S(\mu) = \sum_{M=-\infty}^{\infty} x_0^M \sum_{N=max(-M,N)}^{\infty} \frac{(N-\mu)!}{(N+M)!} (\frac{G-1}{G})^N L_{N+\mu}^{M-\mu}[x] L_{N-\mu}^{M+\mu}[x^*] L_{N-\mu}^{2\mu}[-\frac{y}{G-1}], \tag{71}$$

where $y = |\sigma_-|^2$, $x = b_2 b_1^*$, and $x_0 = |b_1|^2$. Terms $\mu < 0$ can be obtained by complex conjugation.

The sum $S(\mu)$ can be split in two parts: one, for $-\mu < M < \infty$, $\mu < N < \infty$, and another one for $-\infty < M < -\mu$, $-M < N < \infty$. In the first sum we may start summation from $N = -\mu$ because the maximum power of z in $L_{N+\mu}^{M-\mu}(z)$ is $N+\mu$ and, therefore, derivatives over z give zero if $N < \mu$. After this, the sum can be calculated, first, expressing $L_{N-\mu}^{2\mu}[-y]$ in terms of the confluent hypergeometric factor and using integral representation for the last one,

$$L_{N-\mu}^{2\mu}[-y] = \frac{(N+\mu)!}{(N-\mu)!} y^{-2\mu} e^{-y} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} e^{sy} \frac{s^{N-\mu}}{(s-1)^{N+\mu+1}}. \tag{72}$$

Secondly, we write $L_{N-\mu}^{M+\mu}[x^*] = (-\frac{\partial}{\partial z})^{2\mu} L_{N+\mu}^{M-\mu}[z]|_{z=x^*}$, and use [9]

$$\sum_{N=-M}^{\infty} \frac{(N+\mu)!}{(N+M)!} \xi^{N+\mu} L_{N+\mu}^{M-\mu}(x) L_{N+\mu}^{M-\mu}(z) = \frac{(\xi xz)^{-(M-\mu)/2}}{1-\xi} e^{-\xi(x+z)/(1-\xi)} I_{|M-\mu|}(\frac{2\sqrt{\xi xz}}{1-\xi}), \tag{73}$$

where $\xi = (\frac{s(G-1)}{(s-1)G})$. In this form, the answer is valid also for the second part of the sum, $-\infty < M < -\mu$, $-M < N < \infty$.

The sum over M,

$$S(\mu) = (-\frac{\partial}{\partial z})^{2\mu} \sum_{M=-\infty}^{\infty} \frac{x_0^M}{y^{2\mu}} e^{-y} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{(\xi s)^{-\mu} e^{sy}}{(s-1)^{\mu+1}} \frac{(\xi xz)^{-(M-\mu)/2}}{1-\xi} e^{-\frac{\xi(x+z)}{(1-\xi)}} I_{|M-\mu|}(\frac{2\sqrt{\xi xz}}{1-\xi}), \tag{74}$$

can be calculated using

$$\sum_{k=-\infty}^{\infty} \alpha^{k/2} I_{|k|}(2\beta) = e^{\frac{\beta}{\sqrt{\alpha}} + \beta\sqrt{\alpha}}. \tag{75}$$

After that, each derivative over z gives factor $(\frac{\xi}{1-\xi})(1-\frac{x}{x_0})$. S takes form

$$S(\mu) = y^{-2\mu} x_0^{\mu} e^{-y} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} e^{sy} e^{[\xi x^*(x/x_0-1)+x_0-\xi x]/(1-\xi)} (1-\frac{x}{x_0})^{2\mu} \frac{G^{\mu+1}(G-1)^{\mu}}{(s-G)^{2\mu+1}}. \tag{76}$$

The integral is given by the residues of the poles at $s = G$,

$$S(\mu) = G(G-1)^{\mu} (\frac{x_0}{yA})^{\mu} (1-\frac{x}{x_0})^{2\mu} I_{2\mu}(2\sqrt{GAy}) e^{x_0+y+(G-1)A}, \tag{77}$$

where $A = |b_2 - b_1|^2$. Finally,

$$< p_j^n >= \sum_{i \neq j} \hat{K}_n \sum_{\mu} (1+\hat{P}) (\frac{\sigma_-^*}{\sigma_-})^{\mu} I_{2\mu}[2\sqrt{G|b_2-b_1|^2\,|\sigma_- - \sigma_+|^2}] \tag{78}$$

$$(\frac{b_1-b_2}{b_1^*-b_2^*})^{\mu} e^{(G-1)|b_2-b_1|^2+(1/2)[b_2(b_2^*-b_1^*)+c.c.]} e^{(1/2)[\sigma_-(\sigma_-^*-\sigma_+)+c.c]}|_{b_2=b_1}. \tag{79}$$

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