

Lectures on Warped Compactifications and Stringy Brane Constructions

Shamit Kachru

Department of Physics and SLAC
Stanford University
Stanford, CA 94305/94309

In these lectures, two different aspects of brane world scenarios in 5d gravity or string theory are discussed. In the first two lectures, work on how warped compactifications of 5d gravity theories can change the guise of the gauge hierarchy problem and the cosmological constant problem is reviewed, and a discussion of several issues which remain unclear in this context is provided. In the next two lectures, microscopic constructions in string theory which involve D-branes wrapped on cycles of Calabi-Yau manifolds are described. The focus is on computing the superpotential in the brane worldvolume field theory. Such calculations may be a necessary step towards understanding e.g. supersymmetry breaking and moduli stabilization in stringy realizations of such scenarios, and are of intrinsic interest as probes of the quantum geometry of the Calabi-Yau space.

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1. Introduction

Scenarios for an underlying string theory description of nature have been considerably enriched following the duality revolution of the mid 1990s. Perhaps the most striking qualitative new feature is the emergence of scenarios in which standard model gauge fields are confined to some submanifold of a larger bulk spacetime, while of course gravity propagates in the bulk. For instance, such models are natural in the Horava-Witten extension of the $E_8 \times E_8$ heterotic string theory, where finite string coupling opens up an additional dimension with the geometry of an interval, and the $E_8 \times E_8$ gauge fields live on the boundaries [1]. More generally, after the realization of the important role played by D-branes in string duality [2], it was found that the world-volume quantum field theory on coincident D-branes enjoys a non-Abelian gauge symmetry [3]. This makes it natural to construct type II or type I string models where the standard model gauge fields are confined to stacks of D-branes (see e.g. [4] for a discussion of some such attempts).

String constructions of this sort have also motivated new ideas in long wavelength effective field theory for reformulating the gauge hierarchy problem [5,6] and the cosmological constant problem [7,8,9] in terms of brane world constructions. The translation of these problems to brane world language does not solve them, but certainly provides a different way of thinking about them, and opens up exciting new possibilities for phenomenology.

In the following lectures, we will first review some of the new ideas for reformulating the hierarchy problems of fundamental physics in the language appropriate to such “brane world” scenarios. We will then switch tracks and talk about the detailed investigation of one class of microscopic brane constructions that exist in string theory. These latter lectures will start with a telegraphic review of some aspects of closed string Calabi-Yau compactifications. They will then focus on superpotential computations in models with 4d $\mathcal{N} = 1$ supersymmetry, since these are quite relevant to issues of physical interest like supersymmetry breaking and stabilization of moduli.

2. Trapped Gravity and the Gauge Hierarchy

2.1. *Trapping Gravity*

Our world might actually be contained on a 3+1 dimensional defect, e.g. a domain wall, in a higher dimensional spacetime [10]. Why would we see 4d gravity in such a model?

Suppose the extra spatial dimension, parametrized by x_5 , is a circle of radius r , and we are localized around some point on this circle. The global 5d metric looks like the metric on $R^{3,1}$ times a circle. The 5d Einstein action is:

$$S_5 = \int d^5x \sqrt{-G} R M_5^3 \quad (2.1)$$

where M_5 is the 5d Planck scale. Integrating out the “extra” x_5 dimension gives rise to a 4d effective action with an effective Planck scale

$$M_4^2 \sim r M_5^3 \quad (2.2)$$

Hence, at length scales larger than r , gravity will appear to be four-dimensional with a Newton’s constant determined by (2.2). For sufficiently small r , this of course would reproduce everything we know about gravity from present day experiments.

However, there is a more general possibility. The metric can be *warped*. For instance, consider pure 5d gravity with a cosmological constant, and a source term for a domain wall located at $x_5 = 0$:

$$S = \int d^5x \sqrt{-G} (R - \Lambda) + \int d^4x \sqrt{-g} (-V_{brane}) \quad (2.3)$$

where $g_{\mu\nu} = \delta_\mu^M \delta_\nu^N G_{MN}(x_5 = 0)$, $\mu, \nu = 1, \dots, 4$, and $M, N = 1, \dots, 5$. Following Randall and Sundrum [11], we will find solutions of (2.3) which give rise to 4d gravity and in which non-trivial warping plays an essential role. If we want the 4d world to look flat, we should look for solutions of the equations of motion following from (2.3) which exhibit an $SO(3,1)$ symmetry (the Poincare invariance of our world). The most general such ansatz for the 5d metric is:

$$ds^2 = e^{2A(x_5)} \eta_{\mu\nu} dx^\mu dx^\nu + dx_5^2 \quad (2.4)$$

With this ansatz, Einstein’s equations just become equations for the warp factor A (primes denote differentiation with respect to x_5):

$$6(A')^2 + \frac{1}{2}\Lambda = 0 \quad (2.5)$$

$$3A'' + \frac{1}{2}V\delta(x_5) = 0 \quad (2.6)$$

Choosing $\Lambda < 0$, a negative 5d cosmological constant, one can quickly solve (2.5) to find

$$A = \pm kx_5, \quad k = \sqrt{-\frac{\Lambda}{12}} \quad (2.7)$$

Integrating (2.6) from $x_5 = -\epsilon$ to $x_5 = \epsilon$ to pick up the delta function contribution, one finds:

$$3\Delta(A') = -\frac{1}{2}V \quad (2.8)$$

where Δ denotes the discontinuity across $x_5 = 0$.

So to solve the equations with the ansatz (2.4), we must take $A = -kx_5$ for $x_5 > 0$, $A = kx_5$ for $x_5 < 0$. Furthermore, we must tune the brane tension V in terms of the bulk cosmological constant Λ so that

$$V = 12k = 12\sqrt{-\frac{\Lambda}{12}} \quad (2.9)$$

This yields a solution where

$$ds^2 = e^{-2k|x_5|}\eta_{\mu\nu}dx^\mu dx^\nu + dx_5^2 \quad (2.10)$$

The warp factor is sharply peaked at $x_5 = 0$, where the domain wall, which we will call the ‘‘Planck brane,’’ is located. This fact leads to the existence of localized gravity at the Planck brane [11]. Namely, doing the naive 5d to 4d reduction by simply integrating over the x_5 direction, one finds

$$M_4^2 = M_5^3 \int dx_5 e^{-2k|x_5|} < \infty \quad (2.11)$$

This is *finite* despite the existence of an infinite 5th dimension, so the 4d Newton’s constant on the Planck brane is finite, and an observer there would see effective four-dimensional gravity.

There is a natural concern that arises in this case, that does not arise in the case of a 5d theory compactified on a circle of radius r . In the latter case, the lightest 5d Kaluza-Klein (KK) modes have masses which go like $\frac{1}{r}$. For r small enough to avoid experimental detection, this leads to a gap in the KK spectrum, and the low energy theory is clearly just 4d general relativity coupled to the brane worldvolume fields.

In the warped case, however, the infinite extent of the x_5 dimension means that there is *no gap* in the spectrum of bulk modes! So, one should seriously worry that they will

appear as particles with a continuum of masses in 4d, and ruin 4d effective field theory. It has been argued in e.g. [11] and [12] that despite the gapless KK spectrum, a physicist on the Planck brane would still see effectively four-dimensional physics. This is because, although the KK spectrum is gapless, most of the bulk KK modes have wavefunctions with support far from $x_5 = 0$ where the brane fields are localized. Due to this very small overlap of wavefunctions in the x_5 direction, the brane fields and localized graviton couple only very weakly to the bulk continuum. So for instance the Newtonian form of the gravitational potential $V(r) \sim \frac{1}{r}$ receives only small $\frac{1}{r^3}$ corrections (curiously, as if there were two extra flat dimensions) [11,12].

2.2. Hierarchies from Multiple Branes

Consider now a case with two branes, located at $x_5 = 0$ and $x_5 = \pi$. We will take x_5 to live on the interval between 0 and π , so the space now has an extra dimension of finite extent. The total action looks like:

$$S = \int d^5x \sqrt{-G}(R - \Lambda) + S_{SM} + S_{Pl} \quad (2.12)$$

where S_{SM} is the action on the ‘‘Standard Model brane’’ located at $x_5 = \pi$ and S_{pl} is the action on the ‘‘Planck brane’’ located at $x_5 = 0$, i.e.

$$S_{SM} = \int d^4x \sqrt{-g_{SM}}(\mathcal{L}_{SM} - V_{SM}) \quad (2.13)$$

$$S_{pl} = \int d^4x \sqrt{-g_{pl}}(\mathcal{L}_{pl} - V_{pl}) \quad (2.14)$$

Following Randall and Sundrum [6], we will demonstrate solutions of the theory (2.12) which give rise to large hierarchies between scales in a somewhat natural way.

We again look for a warped metric which maintains the 4d Poincare invariance we desire:

$$ds^2 = e^{-2A(x_5)} \eta_{\mu\nu} dx^\mu dx^\nu + r^2 dx_5^2 \quad (2.15)$$

It follows from (2.15) that the size of the x_5 interval is πr .

The Einstein equations are now:

$$6 \frac{(A')^2}{r^2} + \frac{1}{2} \Lambda = 0 \quad (2.16)$$

$$3 \frac{A''}{r^2} + \frac{1}{2} \frac{V_{pl}}{r} \delta(x_5) + \frac{1}{2} \frac{V_{SM}}{r} \delta(x_5 - \pi) = 0 \quad (2.17)$$

As before, we can define $k = \sqrt{-\frac{\Lambda}{12}}$ (and take the bulk Λ to be negative). Then, (2.16) is solved by taking

$$A(x_5) = kr|x_5| \quad (2.18)$$

Notice that (2.18) is consistent with a Z_2 symmetry under which x_5 is reflected; our strategy will be to find a Z_2 symmetric solution where $-\pi < x_5 < \pi$, and then orbifold by the Z_2 to get the desired setup.

From (2.18) (and the periodicity of x_5 , where $x_5 = \pm\pi$ are identified), it follows that

$$A'' = 2kr(\delta(x_5) - \delta(x_5 - \pi)) \quad (2.19)$$

Comparing this to (2.17), we see that we should choose

$$V_{pl} = -V_{SM} = 12k \quad (2.20)$$

in order to find a Poincare invariant solution.

Taking this as our background gravity solution, what is the 4d effective field theory on the ‘‘Standard Model’’ brane that follows from it? First of all notice that it is natural to write the metric ansatz in terms of 4d fields as follows (here x runs over the dimensions other than x_5):

$$ds^2 = e^{-2kT(x)} (\eta_{\mu\nu} + h_{\mu\nu}(x)) dx^\mu dx^\nu + T(x)^2 dx_5^2 \quad (2.21)$$

There are two dynamical 4d fields in (2.21); the four-dimensional metric $\bar{g}_{\mu\nu}(x) = h_{\mu\nu}(x) + \eta_{\mu\nu}$, and the 4d scalar field $T(x)$ (the so-called ‘‘radion’’). We recover the desired vacuum solution by choosing the radion to have a constant VEV $\langle T(x) \rangle = r$.

One can easily compute the 4d effective action for the metric \bar{g} by starting from the 5d Einstein action; one finds a 4d Einstein term with

$$M_4^2 = M_5^3(1 - e^{-2kr\pi}) \quad (2.22)$$

for the 4d Planck scale. In particular, notice that for r of reasonable magnitude (in Planck units), M_4 depends only very weakly on r . This is intuitively because the 4d graviton is largely localized in the vicinity of the Planck brane, which is at $x_5 = 0$.

This leads to an interesting phenomenon. If we compute the metrics g_{pl} and g_{SM} which appear in the source terms for the Planck and Standard Model branes, it follows that

$$g_{\mu\nu}^{pl} = \bar{g}_{\mu\nu} \quad (2.23)$$

while

$$g_{\mu\nu}^{SM} = e^{-2kr\pi} \bar{g}_{\mu\nu} \quad (2.24)$$

This reflects the fact that an object with energy E at the Planck brane would be seen, at the Standard Model brane, as an object with energy $Ee^{-kr\pi}$; equivalently, length scales at the Standard Model brane are “redshifted” to be longer than the corresponding lengths at the Planck brane. This is a familiar manifestation of scale/radius duality in AdS/CFT [13]. However, here it has the interesting consequence that if one starts with dimensionful parameters in the Standard Model Lagrangian \mathcal{L}_{SM} (e.g. a Higgs mass) of order the 4d Planck scale, they can easily be “redshifted” down to energies which are hierarchically smaller, simply by the factors of the metric (2.24). Hence, to find TeV scale physics on the Standard Model brane, one simply needs to choose r to be of order ten times the fundamental scale. This sounds relatively natural, and provides a candidate solution to the hierarchy problem.

Finally, one should ask, how easy is it to accomplish the stabilization of the radion around the required value (that leads to an “explanation” of the hierarchy)? Goldberger and Wise have argued that the presence of a bulk scalar field, with fairly natural bulk and brane couplings, can do the job [14].

2.3. Some Remarks on Randall-Sundrum scenarios

In this section, we make some remarks which have relevance to both the naturalness of RS scenarios, and their possible embedding into a fully consistent microscopic theory of gravity (like string theory). There has been a great deal of research on this topic.

There are several different things to say about this. One is that, via the AdS/CFT correspondence [13], the Randall-Sundrum scenario is more or less a strong coupling version of an older idea for solving the hierarchy problem just within quantum field theory. It has long been realized that if one starts with some ultraviolet fixed point CFT around the UV scale (say, just below the Planck scale) and perturbs it by a marginally relevant operator (whose dimension is very close to 4, say $4 - \epsilon$) then one can naturally generate scales much lower than M_{pl} . Namely, the RG running of the couplings in the perturbed field theory is logarithmic, and therefore the relevant coupling will have significant dynamical effects only after a vast amount of RG running (in energy scale space). Roughly speaking, the scale at which the operator produces significant dynamical effects might be $M = e^{-1/\epsilon} M_{pl}$. A scenario of this sort was advocated recently by Frampton and Vafa [15].

Obviously, for ϵ finely tuned close enough to zero, one can achieve $M \ll M_{pl}$. However, it might be quite a challenge to find a 4d conformal field theory whose most relevant perturbation is of dimension $4 - \epsilon$ with ϵ small; and if one cannot find such a theory, then this mechanism becomes unnatural (because the other, more relevant operators will also be activated along the RG flow).

This concern has a direct translation into the Randall-Sundrum scenario, via the AdS/CFT dictionary [16].¹ Their scenario requires the existence (between the Planck and the Standard Model branes) of a region many AdS radii in size where the 5d metric is approximately that of AdS space. By introducing a Planck brane, they have also rendered normalizable those modes in AdS space which are normally non-normalizable (due to divergent behavior near the boundary of AdS). These modes will fluctuate. It is difficult to think of concrete scenarios where none of these fluctuating modes in the gravity theory is a “tachyon” (which maps, via the AdS/CFT duality, to a relevant operator). Such tachyons, when they fluctuate and condense, will destroy the AdS form of the metric between the Planck and Standard Model branes. The question of why tachyons (except perhaps those which are very close to being non-tachyonic) should be absent, is the same as the question raised above in the field theory picture. This is not surprising; by AdS/CFT duality, the Randall-Sundrum scenario is exactly the same as the previous one, except in the limit of strong field theory ’t Hooft coupling. Of course, such a limit was never tractable before, so interesting new features could emerge there.

Without addressing these concerns, one can still ask whether one can plausibly realize the mechanism of §2.2 in some class of string theory backgrounds. A good argument that this is possible has been provided by H. Verlinde [17]. He recalls that in certain compactifications of F-theory on a Calabi-Yau fourfold X_4 , one can introduce

$$N = \frac{\chi(X_4)}{24} \tag{2.25}$$

space filling D3 branes to satisfy tadpole cancellation conditions (at least if the sign of the Euler characteristic is correct). Since the known list of Calabi-Yau fourfolds includes some with $|\chi| \sim 200,000$, this can lead to the introduction of large numbers of D3 branes. Of course, with 4d $\mathcal{N} = 1$ supersymmetry, dynamics will undoubtedly dictate the positions of these D3 branes in the end (there will be a superpotential for the chiral fields which

¹ This clear relationship and the corresponding concerns were stressed to the author by Maldacena and Witten on separate occasions.

fixes their positions). But it is quite plausible that large numbers of D3 branes will be stacked on top of one another, generating an AdS throat which is “glued” into the CY fourfold asymptotically. Then, the D3 brane field theory will hopefully, in the infrared, manufacture some analogue of the RS Standard Model brane, while the gluing of the throat to the CY fourfold acts effectively as a Planck brane. More explicit models of what the TeV brane might look like have emerged in the recent papers [18].

3. Brane Worlds and the Cosmological Constant

3.1. The Problem

It is an old idea, going back at least to Rubakov and Shaposhnikov [19], that if the Standard Model were confined to a defect in a higher dimensional space (e.g. a domain wall), this defect might naturally like to be flat. Suitably interpreted, the flatness of the defect could then explain the extreme smallness of the cosmological constant in our 4d world.

In this section, we discuss the extent to which this idea seems realizable in the “wall world” scenarios which have become common today. To see that the idea doesn’t always fare well, let’s begin by reviewing the relevant portions of the Randall-Sundrum scenario. For simplicity, we discuss the scenario of §2.1, but that of §2.2 would differ in no essential way.

So, suppose we did live on a domain wall in a 5d gravity theory with bulk $\Lambda < 0$. The key point is to recall that, in searching for a Poincare invariant 4d world, we were forced by the Einstein equations to tune the tension of the brane V in terms of the bulk cosmological term, as expressed in equation (2.20):

$$V = 12\sqrt{-\frac{\Lambda}{12}}$$

Now, if we imagine the Standard Model degrees of freedom living on the wall at $x_5 = 0$, small changes in the Standard Model parameters (the electron mass, QCD scale, weak scale,...) will result in a renormalization of the brane tension $V \rightarrow V + \Delta V$. Equivalently, quantum loops of Standard Model fields enter in V . But under such a shift, the relation (2.20) will be violated, and hence one will no longer be able to find a flat solution!

This is the manifestation of the cosmological constant problem in such wall world scenarios. One must tune the brane tension V , which depends in a sensitive way on the Standard Model parameters, in terms of other microscopic parameters, or one cannot find a Poincare invariant 4d world.

3.2. Adding Scalars

In most microscopic theories which could be responsible for the 5d bulk action in a wall world scenario, there are degrees of freedom other than the 5d metric. For instance, in string theory generic compactifications result in massless scalar moduli. So, it is natural to consider a theory with additional 5d bulk scalars, and see if the situation of §3.1 improves. In fact, as discussed in [8,9], it does to some extent.

So, take now for the action:

$$S = \int d^5x \sqrt{-G} \left(R - \frac{4}{3} (\nabla\phi)^2 \right) + \int d^4x \sqrt{-g} (-f(\phi)) \quad (3.1)$$

In addition to the action for the 5d gravity and scalar field, we have a source term for a domain wall at $x_5 = 0$. In the presence of the scalar ϕ , it is natural to take the wall tension to be ϕ dependent. For instance, the tension of branes in string theory can depend on the string dilaton, or the moduli controlling the volumes of cycles which they are wrapping. Also, we have chosen to start with an action with no 5d cosmological term. Our philosophy throughout this section will be that the 5d bulk is supersymmetric, while the theory on the 4d domain wall breaks SUSY. Hence, it is natural (in a controlled expansion in small parameters, which we will discuss later) to choose the bulk to have vanishing cosmological term.

For simplicity, we will for now take

$$f(\phi) = V e^{b\phi} \quad (3.2)$$

Most of what we say generalizes to far more generic $f(\phi)$, as detailed in [8]. To look for Poincare invariant 4d worlds, we again choose the metric ansatz:

$$ds^2 = e^{2A(x_5)} \eta_{\mu\nu} dx^\mu dx^\nu + dx_5^2 \quad (3.3)$$

and we take the scalar $\phi = \phi(x_5)$.

The resulting equations are (where again $'$ denotes differentiation with respect to x_5):

$$\frac{8}{3} \phi'' + \frac{32}{3} A' \phi' = b V e^{b\phi} \delta(x_5) \quad (3.4)$$

$$6(A')^2 - \frac{2}{3} (\phi')^2 = 0 \quad (3.5)$$

$$3A'' + \frac{4}{3} (\phi')^2 = -\frac{1}{2} e^{b\phi} V \delta(x_5) \quad (3.6)$$

An important fact which is immediately evident from the equations above is that finding flat solutions will NOT require any fine tune of the coefficient V in (3.2) in terms of any microscopic parameters. This is obvious because the only non-derivative coupling of the scalar ϕ is through the brane tension term (in $f(\phi)$). So given a solution for one value of V , a shift of V to $V + \Delta V$ can be compensated by an appropriate shift in the zero mode of ϕ , leaving the equations of motion unchanged.

Why is this significant? The Standard Model physics at $x_5 = 0$ is purely reflected (in this approximation, where the theory is in its ground state) through the wall source term. Now, suppose the Standard Model gauge couplings are independent of ϕ . Then varying Standard Model parameters, or summing Standard Model radiative corrections, will shift V in a way that is ϕ independent. Hence, one can effectively absorb any cosmological constant generated by Standard Model physics, while still finding a Poincare invariant 4d world [8,9].

A picture where ϕ is, at leading order, unrelated to Standard Model couplings is not unreasonable. For instance if we are in string theory, we could let the brane at $x_5 = 0$ be a D-brane and ϕ be a geometrical modulus for a cycle the brane does not wrap. Alternatively, if we wish to treat ϕ as the dilaton, we could imagine the brane at $x_5 = 0$ is a wrapped NS brane whose effective gauge coupling is determined by some geometrical modulus, as in the examples of [20].

3.3. What about 4d gravity?

To proceed, let's write down the explicit solutions to the Einstein equations. Solving the bulk equations of motion, we find

$$\phi(x_5) = \frac{3}{4} \log \left| \frac{4}{3} x_5 + c \right| + d \quad (3.7)$$

$$A' = \frac{1}{3} \phi' \quad (3.8)$$

Notice that at $x_5 = -\frac{3}{4}c$, there is a singularity: the scale factor vanishes (A goes to $-\infty$), and the $|\text{curvature}| \rightarrow \infty$. If one momentarily views x_5 as a time-like direction, and the slices of constant x_5 as 4d spatial slices in a cosmology, then this singularity looks like a big bang or big crunch singularity where the spatial slices collapse to zero size.

Next, we need to include the wall source terms. For simplicity, we specialize to the case:

$$f(\phi) = V e^{-\frac{4}{3}\phi} \quad (3.9)$$

However, with one exception (to be mentioned later), basically all of our considerations carry over for much more generic $f(\phi)$ [8].

From the form of the bulk solutions, it is clear that the solution with the wall should have the general form:

$$\phi(x_5) = \frac{3}{4} \log \left| \frac{4}{3} x_5 + c_1 \right| + d_1, \quad x_5 < 0 \quad (3.10)$$

$$\phi(x_5) = \frac{3}{4} \log \left| \frac{4}{4} x_5 + c_2 \right| + d_2, \quad x_5 > 0 \quad (3.11)$$

and $A' = \frac{1}{3} \phi'$.

Imposing the matching conditions at the wall, we find a Z_2 invariant solution (symmetric under left/right exchange) if

$$c_1 = -c_2 = c, \quad d_1 = d_2 = d, \quad e^{-\frac{4}{3}d} = \frac{4}{V} \frac{c}{|c|} \quad (3.12)$$

with *arbitrary* c .

Now, suppose we choose c positive. Then, the solution (3.10) (3.11) has curvature singularities at $x_5 = \pm \frac{3}{4}c$. Let us suppose the space *ends* at the singularities, so that the x_5 dimension is effectively an interval (with the Standard Model brane in the middle). Then, a simple computation reveals:

$$M_4^2 \sim M_5^3 \int dx_5 e^{2A} < \infty \quad (3.13)$$

so there is indeed four dimensional gravity coupled to the brane field theory at $x_5 = 0$.

3.4. Discussion of Several Important Issues

There are several issues that need to be addressed about this framework for discussing the cosmological constant in wall world scenarios.

1) What about bulk quantum corrections?

In general, choosing a bulk action with vanishing 5d cosmological term and only kinetic terms for the bulk scalar ϕ , as in (3.1), is only sensible in an approximate sense. We are assuming the bulk is supersymmetric, with the brane breaking supersymmetry. Still, eventually the interaction of bulk and brane fields will transmit the SUSY breaking to the bulk, and there will be subleading results which correct the action (3.1) and lead to slight

curvature of the previously Poincare invariant slices. How do we estimate the size of these effects?

It follows from the matching conditions (3.12) that if one chooses the brane tension $Vf(\phi(0))$ to be roughly a TeV, and one fixes $M_4 \sim 10^{19} GeV$, then the 5d Planck scale is fixed to be $10^5 TeV$ (and the size of the x_5 interval is about a millimeter). Interactions of bulk and brane fields are suppressed by explicit powers of $\frac{1}{M_5}$; therefore, bulk corrections to the 4d effective field theory will arise in a power series in $\epsilon = (TeV/M_5)$. Hence, in this scenario one can arrange to cancel the leading Standard Model $(TeV)^4$ contribution to the effective 4d cosmological term, but there will be contributions at subleading orders in ϵ . While these are too large to be tolerated given the observed value of the cosmological constant (unless one can somehow cancel the first few terms in the power series), they are nevertheless hierarchically smaller than the expected answer. So, our philosophy should be, that we are looking for a system where the induced cosmological term is hierarchically smaller than what is expected (and we can postpone understanding how to get precisely the right magnitude of the suppression).

2) We have shown there are generically Poincare invariant solutions to the equations, independent of Standard Model parameters. However, are there also other curved solutions, which would be characteristic of a 4d effective field theory with nonzero cosmological term?

For generic $f(\phi)$, it turns out that curved solutions with de Sitter or anti de Sitter symmetry *do* exist [8]. For fine tuned $f(\phi)$, e.g. $f(\phi) \sim Ve^{\pm\frac{4}{3}\phi}$, there are no de Sitter or anti de Sitter solutions [9]. However, in systems with massless scalar fields, the solutions which arise when there is a nonzero cosmological term are often not dS or AdS solutions, but instead solutions where the scalar field is spatially varying. A prototypical example is string theory, where introducing a slight cosmological constant can lead to linear dilaton solutions instead of dS or AdS [21]. So it seems rather likely that in this case also, one can find solutions with 4d slices that are characteristic of a nonzero 4d cosmological term; however a definitive answer to this question is lacking, since the 4d effective field theory has not been written down.

Even given this fact, we find it quite interesting that one can find Poincare invariant solutions as well. It has been very hard to find *any* examples in string theory of Poincare invariant vacua without supersymmetry. Some candidates were proposed in [22], but have a very non-generic low energy effective field theory (with Bose-Fermi degeneracy) and have only been studied at low orders in perturbation theory. Indeed it has been advocated by

Banks [23] that perhaps M theory does not admit nonsupersymmetric, Poincare invariant solutions. We find it intriguing that these “wall worlds” do admit Poincare invariant solutions, and are quite similar to systems one can realize in string theory with wrapped branes.

3) What about the singularities?

Of course, the 5d effective field theory defined by (3.1) breaks down in regions of large curvature. However, it is often the case that string theory can regulate and provide a definition of singular geometries. So, it is an important problem to find a microphysical realization of these systems, which regulates the singularities or describes the physics occurring there.

There are obvious analogies between our x_5 interval and the intervals encountered in e.g. Horava-Witten theory or Type I' string theory. Instead of expanding on those here (see e.g. [8] for more discussion), we concentrate instead on the similarity to geometries arising from RG flows in AdS/CFT duality.

Polchinski and Strassler, for instance, have studied a class of RG flows from the deformed $\mathcal{N} = 4$ super Yang-Mills to confining gauge theories with less supersymmetry [24]. Because their geometries involve a 5d gravity theory with small curvature in the UV (near the “Standard Model” brane, in our language) and large curvature in the IR (which corresponds to our singularities), they are quite similar to our setup. As discussed by Bousso and Polchinski [25], we can then use the results of [24] to infer some important “facts” about the singularities we encounter here.

What Polchinski and Strassler find is that the RG flow results in a “discretuum” of possible IR branes – there are roughly $e^{\sqrt{N}}$ possibilities for the IR brane, where N is a large number in the (super)gravity limit. This translates immediately to the statement that, in our solutions, it is quite likely that the integration constants which arise in $\phi(x_5)$ cannot take arbitrary values, but are rather quantized to certain allowed values at the singularities. The question is then, do the allowed values allow for a solution which is closer to Poincare invariant than would be possible with the expected $(TeV)^4$ cosmological constant?

The answer seems to be yes. As argued in [25],² the AdS/CFT results strongly suggest that cosmological constants which are suppressed from TeV scale by powers of $e^{-\sqrt{N}}$ should be achievable. The question of why the allowed singularity with the smallest possible norm

² This was independently known and stated by several others including N. Arkani-Hamed, E. Silverstein and the author.

of the 4d cosmological constant would be cosmologically preferred is a difficult one. A possible scenario, following earlier work of Brown and Teitelboim [26], was presented in [25](in a different context, involving M theory compactifications with four-form fluxes). Several aspects of their work might generalize to the case under discussion. A related approach can be found in [27], and a critical discussion appears in [28].

Another method of dealing with the issue of singularities, is to attempt to find solutions that retain the “self-tuning” property (the existence of a flat solution independent of “Standard Model” parameters), but are either not singular or have singularities which are physically innocuous. One interesting approach of this sort has been detailed in [29], where they find a self-tuning model which has no naked singularities and is known to arise in exact string theory constructions. While the particular solution they find is not attractive for other reasons (there is a strongly time-dependent 4d Newton’s constant), the basic idea seems promising. Another recent self-tuning construction which is free of singularities appears in [30].

There has been some controversy in the literature about the various brane world approaches to studying the cosmological constant problem (regarding issues like physical admissibility of the singularities). While my point of view is well represented here, alternative viewpoints can be found in e.g. [31,32].

4. Calabi-Yau Compactifications and Closed String Mirror Symmetry

In the next two lectures, we will work up to the study of building blocks for microscopic “brane worlds” which clearly *are* realizable in string theory. These backgrounds involve D-branes in curved geometries, and the open string sectors which live on these branes. Such brane theories exhibit some interesting duality symmetries, which we will also briefly explore. To make the discussion self-contained, we must provide a brief description of the relevant closed string backgrounds first.

4.1. Type II Calabi-Yau Compactifications

Suppose one wants to find a supersymmetry-preserving compactification of the type IIA or type IIB theory, by compactifying on a smooth manifold M of complex dimension d .³ One can argue that a necessary condition is [33]

$$\text{Holonomy of } M \subset \text{SU}(d) \tag{4.1}$$

³ We want to achieve Poincare supersymmetry in the remaining $10 - 2d$ dimensional theory, so we will not have to worry about e.g. the Freund-Rubin ansatz and AdS solutions.

The possible choices of M become more and more plentiful as d is increased. For $d = 1$, the only choice is the two-torus T^2 , and the resulting 8d theory has 32 supercharges. For $d = 2$, one can choose either T^4 or $K3$, which preserve 32 and 16 supercharges respectively. Finally, in the case we will utilize later on, $d = 3$, there are (at least) thousands of choices (for the earliest large compendium of such spaces that I am aware of, see [34]). The generic choice of such a complex threefold preserves 8 supercharges, corresponding to 4d $\mathcal{N} = 2$ supersymmetry. The study of such compactifications has been a rich and beautiful subject about which we will necessarily be very brief here: for a much more comprehensive review, see [35].

These so-called *Calabi–Yau* manifolds are Ricci flat and Kähler. The Ricci-flat Kähler metrics on a Calabi-Yau space M come in a family of dimension $h^{1,1}(M) + 2h^{2,1}(M)$, where $h^{1,1}$ parametrizes the choice of a Kähler form and $2h^{2,1}$ is the dimension of the space of inequivalent complex structures on M .

A simple example of such a space is the quintic Calabi-Yau threefold in CP^4 . CP^4 is defined by taking 5 homogeneous coordinates (z_1, \dots, z_5) , subject to the identification $(z_1, \dots, z_5) \sim (\lambda z_1, \dots, \lambda z_5)$ where λ is a nonzero complex number (and with the origin deleted). The quintic is defined by a homogeneous equation of degree 5 in this space, for instance

$$P = \sum_{i=1}^5 z_i^5 = 0 \quad (4.2)$$

The complex structure deformations of this manifold are parametrized simply by monomial deformations of the equation (4.2), modulo linear changes of variables $z_i \rightarrow A_{ij} z_j$. In the end, this leads to a 101 possible (complex) deformations of the equation (4.2). The Kähler deformations are, in this case, simply inherited from those of P^4 – there is a single real Kähler parameter, parametrizing the overall volume.

4.2. Spectrum of IIA or IIB String Theory on M

Compactifying either type II string theory on a Calabi-Yau threefold M results in a 4d, $\mathcal{N} = 2$ supersymmetric theory in the remaining $R^{3,1}$. Such a theory admits two kinds of light supermultiplets:

- The vector multiplet, which consists of a complex scalar field, a vector field, and fermions, all in the adjoint representation of the gauge group \mathcal{G} .

- The hypermultiplet, which consists of 2 complex scalars and fermi superpartners, in any representation of the gauge group \mathcal{G} .

The moduli of the Ricci-flat metric on M show up as scalars in such multiplets in the compactified IIA/B theory. However, the correspondence between the geometry of M and the type of multiplet is different for the two theories.

Type IIA on M :

The IIA theory in ten dimensions has a metric, an NS $B_{\mu\nu}$ field, a dilaton ϕ , and 1 and 3 form RR gauge fields. On dimensional reduction on M , this gives rise to $h^{1,1}(M)$ abelian vector multiplets (where the scalars come from the real Kähler moduli of the metric plus the $B_{\mu\nu}$ field, and the vector comes from $C_{\mu\nu\rho}$). On the other hand, the complex structure moduli of the metric together with the scalars coming from absorbing 3-forms on M with $C_{\mu\nu\rho}$ give rise to (the scalar components of) $h^{2,1}(M)$ hypermultiplets. It turns out that the dilaton in the IIA theory is also part of a hyper, yielding a total of $h^{2,1}(M) + 1$ hypers.

Type IIB on M :

In the IIB theory, in addition to the metric, NS B field and dilaton, there are 0,2 and 4 form RR gauge fields. These give rise to $h^{2,1}(M)$ abelian vector multiplets in the low energy theory (with the scalars coming from complex structure moduli, and the vectors coming from $C_{\mu\nu\rho\lambda}$). On the other hand, the Kähler moduli, the $B_{\mu\nu}$ and $C_{\mu\nu}$ fields (NS and RR two forms), and the RR 4-form give rise to $h^{1,1}(M)$ hyper multiplets coming from the (1,1) forms on M . Including the dilaton, which again transforms as part of a hyper, this yields a total of $h^{1,1}(M) + 1$ hypers.

By $\mathcal{N} = 2$ supersymmetry, there are several simplifications in the low energy effective action for these theories. First of all, with this much supersymmetry, there is no potential generated for “flat directions” which are present in the tree-level theory. Hence, there are moduli spaces \mathcal{M} of exactly degenerate supersymmetric vacua (the physical reflection of the moduli space of Ricci-flat metrics on the Calabi-Yau M , if you will). Furthermore, because of the extended supersymmetry, the moduli space \mathcal{M} takes the form of a product of vector and hypermultiplet moduli spaces:

$$\mathcal{M} = \mathcal{M}_v \times \mathcal{M}_h \tag{4.3}$$

where the metric on $\mathcal{M}_{v,h}$ is independent of VEVs of scalars in the “other” kind of multiplet.

4.3. Quantum Corrections

String theory on M comes with two natural perturbative expansions: an expansion in string loops, controlled by the dilaton $g_s = e^{-\phi}$, and an expansion in sigma model perturbation theory (or curvatures), which is roughly controlled by $\frac{R^2}{\alpha'}$ where R is some characteristic “size” of M (controlled by the Kähler moduli). Large g_s corresponds to strong string coupling, while large sigma model coupling means that classical geometry is not necessarily a good approximation, and the “stringy” phenomena characteristic of quantum geometry can occur [35].

One can then consider corrections to the tree-level picture in both of these expansions. To be concrete, let’s consider the geometry (metric) of the vector multiplet moduli space \mathcal{M}_v . It is controlled, as is familiar from Seiberg-Witten theory [36], by a holomorphic prepotential $F(\phi_i)$ where ϕ_i are the scalar moduli in the vector multiplets (F also determines the kinetic terms of the gauge fields, the so-called “gauge coupling functions”).

- In the IIB theory, \mathcal{M}_v is independent of the Kähler moduli and g_s , because both of them are in hypermultiplets and the geometry of the vector moduli space is independent of the VEVs of hypers. Therefore, it is exactly determined at both string and sigma model tree level: it is computable in terms of classical geometry. To be slightly more precise, each Calabi-Yau manifold M is characterized by a holomorphic (3,0) form Ω , which is unique up to scale. If we let i, j, k index directions in the moduli space of complex structures, then

$$\frac{\partial^3 F}{\partial \phi_i \partial \phi_j \partial \phi_k} \sim \int_M \Omega \wedge \partial_i \partial_j \partial_k \Omega \quad (4.4)$$

A detailed explanation of this formula can be found, for instance, in [37].

- In the IIA theory, there is a more complicated story. Because the Kähler moduli are in *vector* multiplets now, there *are* quantum corrections to the prepotential F controlled by the sigma model coupling. However the dilaton is still in a hypermultiplet, so there are no g_s corrections – F is computable at string tree level. Considerations of holomorphy dictate that the form of F is such that $\partial^3 F$ will contain contributions which are either tree-level or non-perturbative in $\frac{R^2}{\alpha'}$ (i.e. going like $e^{-\frac{R^2}{\alpha'}}$). This is because the Kähler parameter R is real, and its scalar partner (which arises from the dimensional reduction of the NS B field and complexifies it) is an axion a [38]. Although the continuous shift symmetry for the axion can be broken non-perturbatively, there is a discrete symmetry under which a

shifts by 2π (in a natural normalization). This forbids any corrections to the prepotential in perturbation theory, but is consistent with nonperturbative corrections.

What is the source of the non-perturbative corrections to sigma model perturbation theory? At tree level in the g_s expansion, the string worldsheet is a sphere, and “instanton” corrections suppressed by $e^{\frac{-R^2}{\alpha'}}$ can arise when the worldsheet wraps a holomorphic sphere (of radius R) embedded in the Calabi-Yau space M [39]. Denote by H_i a basis for the homology 4-cycles in M , and by b_i a dual set of 2-forms ($i = 1, \dots, b_2(M)$). The H_i are in 1-1 correspondence with the scalars ϕ_i in the $\mathcal{N} = 2$ vector multiplets. At large radius, when non-perturbative corrections to the sigma model are irrelevant, there is an elegant formula expressing the prepotential F in terms of the intersection numbers of M (see e.g. [37])

$$\frac{\partial^3 F}{\partial \phi_i \partial \phi_j \partial \phi_k} \sim \int_M b_i \wedge b_j \wedge b_k \quad (4.5)$$

As argued in [39], this is corrected by instantons to an expression of the form

$$\frac{\partial^3 F}{\partial \phi_i \partial \phi_j \partial \phi_k} \sim \int_M b_i \wedge b_j \wedge b_k + \sum_C \int_C b_i \int_C b_j \int_C b_k e^{\frac{-Area(C)}{\alpha'}} \quad (4.6)$$

where the sum runs over holomorphic spheres C passing through all three of the cycles $H_{i,j,k}$.

4.4. Mirror Symmetry

Mirror symmetry basically is the statement that Calabi-Yau manifolds naturally come in pairs M and W such that the type IIA theory on M is *exactly equivalent* to the type IIB theory on W (see [40] and references therein for the development of this idea).

A moment’s thought reflects that this is an extremely nontrivial statement about the geometry of Calabi-Yau spaces, and a powerful computational tool for physics. For instance, the roles of the Kähler and complex structure moduli of M and W will be interchanged by the symmetry, since their physical role in the low energy effective $\mathcal{N} = 2$ gauge theory that arises from the string compactification is interchanged in the IIA and IIB theories. For a quick indication of the mathematical power of this statement, recall that the intricate prepotential (4.6) in the IIA theory will be computable in tree-level of both g_s and α' perturbation theory in the IIB theory, by a formula of the form (4.4). Equating the two makes highly nontrivial predictions about e.g. the multiplicity of holomorphic curves in M ; this has been exploited to great effect beginning with the work [41].

A heuristic proof of this statement (for some classes of Calabi-Yau manifolds) was provided by Strominger, Yau and Zaslow [42]. Their reasoning goes roughly as follows. Suppose the IIA theory on M is really equivalent to the IIB theory on W . Then the full theories, including all BPS states and their detailed properties, should match.

Now, what are the SUSY brane configurations allowed by Calabi-Yau geometry, that will give rise to BPS states in the two cases? There are basically two sets of possibilities for CY threefolds [43]:

- One can wrap D2, D4 or D6 branes on holomorphic 2,4 or 6 cycles.
- One can wrap D3 branes on “special Lagrangian” three-cycles; by definition, a special Lagrangian three-cycle Σ is a submanifold of M such that the Kähler form ω restricts to zero on Σ , and $Im(\Omega)|_{\Sigma} = 0$ as well.

Since the IIA theory only has supersymmetric Dp branes for even p , while the IIB theory only has supersymmetric Dp branes for odd p , the holomorphic cycles are relevant for IIA while the special Lagrangian cycles are relevant for IIB (as long as one is focusing only on point particles in the transverse $R^{3,1}$). It is a common abuse of terminology to simply call both special Lagrangian cycles and holomorphic cycles “supersymmetric cycles,” for obvious reasons.

So, let's start by considering the simplest possible case, the D0 brane in type IIA on M . The worldvolume theory is a supersymmetric quantum mechanics with 4 supercharges, and its moduli space is intuitively just given by the manifold M itself. Hence, if IIB on W is exactly equivalent to IIA on M , it must contain a “mirror” supersymmetric brane whose moduli space is also M !

As discussed above, it must be a D3 brane wrapping a SUSY 3-cycle $\Sigma \subset W$. What are the properties of Σ ? In particular, one needs the complex dimension of the moduli space of the wrapped D3 brane to be 3. By McLean’s theorem [44], Σ itself has $b_1(\Sigma)$ moduli as a supersymmetric cycle in W . In addition, Wilson lines of the $U(1)$ gauge field on the wrapped D3 brane provide another $b_1(\Sigma)$ moduli. Thus, we learn that we must have $b_1(\Sigma) = 3$ to match the expected dimension of the moduli space.

Furthermore, if we fix a point in the moduli space of the special Lagrangian cycle and simply look at the Wilson lines, they give rise to a T^3 factor in the moduli space of the wrapped brane. Hence, we learn that in some sense, M must be a T^3 fibration!

Now obviously, switching the role of M and W would yield an argument that W must also be a T^3 fibration. Hence, an elegant conjecture is that both M and W are fibered by special Lagrangian T^3 s, and in particular the mirror of the D0 brane on M is a D3 brane

wrapping the supersymmetric T^3 on W . This is intuitively sensible: T-dualizing on the 3 circles of the T^3 would turn the IIB theory into the IIA theory, and change the D3 brane into a D0 brane.

This chain of arguments indicates that all Calabi-Yau manifolds with mirrors are T^3 fibrations; it is furthermore a constructive argument, since one can in principle explicitly construct the mirror manifold by T-dualizing the supersymmetric T^3 s. In practice this is of course very difficult. Indeed, simply demonstrating the supersymmetric T^3 fibration is out of reach except in very special cases; weaker results about Lagrangian fibrations do exist. For a recent review, see [45].

4.5. An Example

Since the previous subsection was fairly abstract, we close this section with a (trivial) example. Consider IIA on a T^2 which is a product of two circles, $T^2 = S^1_{R_1} \times S^1_{R_2}$ where $R_{1,2}$ are the radii. We can clearly view this as a T^1 (i.e., S^1) fibration over S^1 . From standard results in elementary geometry, the complex structure of this torus is parametrized by

$$\tau \sim i \frac{R_2}{R_1} \quad (4.7)$$

while its Kähler structure (or volume) goes like

$$\rho \sim i R_1 R_2 \quad (4.8)$$

The i appears in (4.8) because the string theory modulus ρ satisfies $\rho = B + iJ$, where B is the NS B field and J is the geometrical Kähler form.

T-dualizing along the $S^1_{R_1}$ circle has the following effect. Define

$$R'_1 = \frac{1}{R_1} \quad (4.9)$$

(we are setting the string scale to unity for simplicity in this subsection). Then

$$\tau_{new} = i \frac{R_2}{R'_1} = i R_2 R_1 = \rho_{old} \quad (4.10)$$

and

$$\rho_{new} = i R'_1 R_2 = i \frac{R_2}{R_1} = \tau_{old} \quad (4.11)$$

And of course, T-dualizing along one circle exchanges the IIA and IIB theories.

We have succeeded in taking IIA on a torus with (complex, Kähler) moduli (τ, ρ) to IIB on a torus with moduli (ρ, τ) . This is mirror symmetry for T^2 , and it has precisely arisen here as T-duality on the S^1 “fibration.” One can do the slightly less trivial case of $K3$ with as much success, by viewing $K3$ as a T^2 fibration [42].

5. Open Strings and Mirror Symmetry

In the previous lecture, we saw that Calabi-Yau threefolds come in pairs M, W such that the IIA theory on M is equivalent to the IIB theory on W . This yields a powerful tool for the study of 4d $\mathcal{N} = 2$ supersymmetric string vacua.

By a slightly more elaborate construction, we can also manufacture 4d $\mathcal{N} = 1$ models starting with type II strings on Calabi-Yau spaces. Namely, we should compactify the type II theory on a Calabi-Yau, and then introduce additional (space-filling) $D(p+3)$ branes wrapping supersymmetric p cycles.⁴ It is natural to ask: what does mirror symmetry do for us in this context?

Let's begin the discussion in type IIA string theory. If we wish to make a “brane world” in type IIA string theory by compactifying on a Calabi-Yau M and then wrapping $D(p+3)$ branes on p cycles in M , and we also want to preserve 4d $\mathcal{N} = 1$ supersymmetry, then the only possibility is to wrap $D6$ brane(s) on supersymmetric (special Lagrangian) three-cycles. Recall that a 3-cycle $\Sigma \subset M$ is called special Lagrangian iff

- $\omega|_{\Sigma} = 0$
- $Im(\Omega)|_{\Sigma} = 0$

where ω is the Kähler form of M , and Ω is the holomorphic (3,0) form. Such cycles are volume minimizing in their homology class.

5.1. How to produce examples of Σ

Although quite generally it is difficult to produce examples of special Lagrangian 3-cycles in compact Calabi-Yau manifolds, there is a rather special construction that can be used to give a simple class of examples. Suppose we have local complex coordinates $z_{1,2,3}$ on M , chosen so that:

$$\omega \sim \sum_i dz_i \wedge d\bar{z}_i \tag{5.1}$$

$$\Omega \sim dz_1 \wedge dz_2 \wedge dz_3 \tag{5.2}$$

Furthermore, suppose that M comes equipped with a so-called *real involution* \mathcal{I} , which acts at

$$\mathcal{I} : z_i \rightarrow \bar{z}_i \tag{5.3}$$

⁴ In the full construction, one will also have to introduce orientifolds to cancel the RR tadpoles.

Consider now the fixed point locus of \mathcal{I} in M , i.e. the locus of points where $z_i = \bar{z}_i$. Let us call this $\Sigma_{\mathcal{I}}$. It is clear from (5.1) and (5.2) that \mathcal{I} acts on the Kähler form and the holomorphic three-form as

$$\mathcal{I} : \omega \rightarrow -\omega, \quad \Omega \rightarrow \bar{\Omega} \quad (5.4)$$

So in particular, we read off from (5.4) that on $\Sigma_{\mathcal{I}}$:

$$\omega|_{\Sigma_{\mathcal{I}}} = 0, \quad \text{Im}(\Omega)|_{\Sigma_{\mathcal{I}}} = 0 \quad (5.5)$$

Hence, the fixed point locus $\Sigma_{\mathcal{I}}$ of a real involution \mathcal{I} acting on M is always a special Lagrangian cycle.

Let's be very concrete by working out an example. Consider the Calabi-Yau hypersurface in $WP_{1,1,2,2,2}^4$ defined by the equation:

$$p = z_1^8 + z_2^8 + z_3^4 + z_4^4 + z_5^4 - 2(1 + \epsilon)z_1^4 z_2^4 = 0 \quad (5.6)$$

Notice that $p = dp = 0$ is soluble when $\epsilon \rightarrow 0$, indicating that the hypersurface (5.6) becomes singular at that point in moduli space. We will consider the region of small positive ϵ .

Now, consider the real involution $\mathcal{I} : z_i \rightarrow \bar{z}_i$ acting on (5.6). The fixed point locus is obviously the locus where all of the z_i are real. What is its topology? Let us define $u = z_1^4$, and work (without loss of generality) in the $z_2 = 1$ patch. Then $p = 0$ implies

$$u^2 - 2(1 + \epsilon)u + 1 + Q = 0 \quad (5.7)$$

where

$$Q \equiv z_3^4 + z_4^4 + z_5^4 \quad (5.8)$$

Solving (5.7) we find

$$u_{\pm} = 1 + \epsilon \pm \sqrt{\epsilon^2 + 2\epsilon - Q} \quad (5.9)$$

What is the point of this? The solutions (5.9) go imaginary for large Q , so Q is bounded to lie in some domain of size basically 2ϵ (for small ϵ). The locus $Q < 2\epsilon$ intersects the fixed point locus of \mathcal{I} in a 3-ball B_3 , and has boundary $Q = 2\epsilon$ which is (up to finite covering) an S^2 . The two different branches of solutions for u in (5.9) are glued together along this boundary S^2 ; so altogether $\Sigma_{\mathcal{I}}$ consists of two B_3 s glued together on an S^2 . But of course this is nothing but an S^3 .

It follows from these manipulations that the size of the S^3 goes to zero as $\epsilon \rightarrow 0$; the singular point in moduli space is related to the existence of this collapsing 3-cycle.

5.2. D6 Branes wrapping special Lagrangian cycles

Now that we have gotten some feeling for very simple examples of special Lagrangian cycles, let's start to consider the physical theory living on a $D6$ brane which wraps such a cycle Σ . Since the brane breaks half of the supersymmetry, the low energy theory on the brane (living in the noncompact $R^{3,1}$) will be a 4d $\mathcal{N} = 1$ field theory. The $D6$ brane gauge field will descend to yield a $U(1)$ gauge supermultiplet in 4d. The other kind of light multiplet in $\mathcal{N} = 1$ theories is the chiral multiplet; how many of these will be present?

It follows from the work of McLean [44] that the “geometrical” moduli space of Σ has (unobstructed) real dimension $b_1(\Sigma)$. String theory complexifies this with Wilson lines of the $U(1)$ gauge field, yielding a moduli space of vacua with $b_1(\Sigma)$ complex dimensions for the brane worldvolume field theory.

What are good coordinates on this moduli space? Choose a basis γ_j for $H_1(\Sigma)$, and choose discs $D_j \subset M$ such that $\partial D_j = \gamma_j$. Define

$$\omega_j = \int_{D_j} \omega \tag{5.10}$$

which is the area that a holomorphic disc in D_j s relative homology class would have (if it existed). To complexify this, consider also the $b_1(\Sigma)$ Wilson lines

$$a_j = \int_{\gamma_j} A \tag{5.11}$$

where A is the $U(1)$ gauge field on the $D6$ brane. Together, (5.10) and (5.11) yield the scalar components of $b_1(\Sigma)$ chiral multiplets ϕ_j which live in the 4d $\mathcal{N} = 1$ theory on the wrapped brane [46,47]:

$$\phi_j = \frac{\omega_j}{\alpha'} + ia_j + \dots \tag{5.12}$$

Now, in any $\mathcal{N} = 1$ supersymmetric field theory, a quantity of great interest which governs the vacuum structure and tends to be exactly computable is the superpotential $W(\phi)$ as a function of the chiral fields ϕ . How do we compute $W(\phi_j)$ in the theories at hand?

First of all, there is no superpotential for the ϕ_j to all orders in α' . The proof of this statement is quite analogous to the one used in discussions of heterotic string compactifications [39]. Because the Wilson line a_j has a shift symmetry under large gauge transformations, $W(\phi_j)$ must not have any polynomial dependence on a_j . But since a_j appears in the chiral field ϕ_j as in (5.12), and W is a holomorphic function of the chiral

fields, this implies that there can be no polynomial terms in $W(\phi_j)$ at all. This is consistent with McLean's result in pure mathematics, which roughly speaking sees α' perturbation theory.

On the other hand, terms of the form

$$e^{-(w_j + ia_j)} = e^{-\phi_j} \quad (5.13)$$

are consistent with shifts $a_j \rightarrow a_j + 2\pi$ which occur under large gauge transformations, and hence such terms in the superpotential cannot be ruled out. What would the source of such terms be? Just as closed string theories have worldsheet instantons, D brane theories on Calabi-Yau spaces can have "disc instantons." At tree level, the open string worldsheet is a disc D . One can consider holomorphic maps $D \rightarrow M$ such that $\partial D = \gamma_j \subset \Sigma$, and such that the normal derivative to the map at the boundary is in the pullback of the normal bundle to Σ in the Calabi-Yau. The claim is then that the superpotential in these D6 brane theories is entirely generated by such disc instanton effects.

For instance, one can formally compute couplings like the $F_i \phi_j \phi_k$ coupling that would arise between two scalars and the auxiliary field F in chiral multiplets if there is a nontrivial superpotential. This is discussed at length in [47] (and is very closely related to the discussion in [46]). The upshot is that one can give a formula for this three-point function on the string worldsheet in terms of an infinite sum over disc instantons. If we call the vertex operators for the spacetime fields appearing in this coupling $V_F^i, V_\phi^j, V_\phi^k$, then the three-point function $\langle V_F^i V_\phi^j V_\phi^k \rangle$ has the following expression. Denote by $d_{\{m_l\}}^{\{n_a\}}(i, j, k)$ the number of holomorphic maps from the disc D to M with the following properties:

- i) $[\partial D] = \sum_l m_l \gamma_l$
- ii) $V^{i,j,k}$ are mapped in cyclic order to the intersection of ∂D with the 2-cycles in Σ dual to $\gamma_{i,j,k}$.
- iii) $[\partial D - \sum_l m_l D_l]$, which by i) is a closed 2-cycle in M , satisfies

$$[\partial D - \sum_l m_l D_l] = \sum_a n_a K_a \quad (5.14)$$

where the K_a are a basis for $H_2(M)$.

Then one can derive the statement:

$$\langle V_F^i V_\phi^j V_\phi^k \rangle \sim \sum_{m_l, n_a \geq 0} \int_{\partial D} \theta^i \int_{\partial D} \theta^j \int_{\partial D} \theta^k d_{\{m_l\}}^{\{n_a\}}(i, j, k) \prod_{l=1}^{b_1(\Sigma)} e^{-m_l \phi_l} \prod_{a=1}^{h^{1,1}(M)} e^{-n_a t_a} \quad (5.15)$$

where θ^i is the harmonic one-form associated to γ_i , and $t^a = \int_{K_a} \omega$ is the area of K_a . This formula is, in some natural sense, the open string analogue of the instanton sum formula (4.6) for the prepotential in closed string Calabi-Yau compactifications. In some very special examples, such disc instanton sums have proven directly computable [48].

Closed String Parameters

How do the closed string moduli of the Calabi-Yau M play a role in the brane theory? From (5.15) above, it is clear that the superpotential W of the brane theory really depends on the closed string Kähler moduli; they enter as *parameters* t_a , so we should really denote W as $W(\phi_j; t_a)$ to indicate the relevance of the closed string background. In fact, it was argued rather generally in [49] that in this class of theories (so-called “A-type” branes), the Kähler (complex structure) moduli of M will only enter in the superpotential (FI D-terms) of the wrapped brane theory. This is in accord with (5.15), where the Kähler dependence is manifest and there is no explicit complex structure dependence. The dependence of the FI D-terms on the complex structure of M has been explored, for instance, in [50,51].

5.3. Type IIB “Mirror” Brane Worlds

Thus far, we have been focusing our attention on the brane worlds we can construct in the IIA theory, but of course it is possible to make analogous type IIB constructions by wrapping 5,7 or 9 branes on holomorphic 2,4 or 6 cycles (or indeed, by having D3 branes transverse to the Calabi-Yau).

In §4.4, we saw that mirror symmetry was of great use in “solving” the $\mathcal{N} = 2$ theories that come out of string theory, by making the prepotential, which receives an infinite series of quantum corrections in the IIA picture (4.6), explicitly computable at tree level in the IIB picture via (4.4). Mirror symmetry should be a similarly powerful tool in studying brane worlds of the type under discussion. Computations of superpotentials, which by (5.15) are dauntingly difficult in the IIA picture, are much simpler in the IIB picture.

In fact, it was argued in [49] that in the case of $D(p+3)$ branes wrapping holomorphic p cycles, the superpotential is exact at tree level (receives no α' corrections whatsoever). Hence, in the IIB theory, superpotentials are effectively as computable as e.g. prepotentials in the closed string case. The challenge, then, is to find a mirror IIB brane configuration for a IIA configuration consisting of a $D6$ brane wrapping a special Lagrangian three-cycle $\Sigma \subset M$. Clearly, the IIB theory will be compactified on the mirror manifold W ; the question is, what is the mirror brane setup?

To find concrete examples it is most convenient to focus on cases where the mirror IIB setup turns out to be a $D5$ brane wrapping a rational curve $C \subset W$. The generalities of this kind of correspondence were discussed in [47] and very concrete examples, where a disc instanton generated superpotential in the IIA theory maps to a tree level superpotential in the IIB theory, were presented in [52].

So, what is the physics of a IIB $D5$ brane wrapping C ? As always, there is a $U(1)$ gauge supermultiplet. The number of massless chiral multiplets is given by the number of small deformations of the curve, parametrized by $H^0(C, N_C)$, where N_C is the normal bundle of $C \subset M$.⁵ However, by classical deformation theory, deformations of $C \subset W$ can be *obstructed*; this corresponds precisely to a massless chiral multiplet which has a nontrivial higher order superpotential! If $h^0(C, N_C) = 1$, and we call the chiral multiplet $\tilde{\phi}$, then an N th order obstruction is reflected in a superpotential on the brane which looks like

$$\tilde{W} \sim \tilde{\phi}^{N+2} \tag{5.16}$$

As in the IIA $D6$ brane theory, *closed* string moduli enter in the IIB $D5$ brane theory as parameters in the Lagrangian. From [49], the superpotential depends only on the complex structure of W , while the FI D-terms for the $U(1)$ gauge field can depend on the Kähler structure of W . In concrete examples, the moduli space of C can exhibit very intricate behavior as one varies the complex structure parameters ψ_a of W . So we should write the superpotential as $\tilde{W}(\tilde{\phi}_i; \psi_a)$ where a runs over the complex structure deformations of W , and the $\tilde{\phi}_i$ parametrize the deformations of C .

Therefore, if one finds a mirror pair consisting of a $D6$ brane wrapping a special Lagrangian cycle $\Sigma \subset M$ and a $D5$ brane wrapping $C \subset W$, then the IIA disc instanton sum (5.15) should basically map to *purely classical* geometrical data in the IIB theory. The superpotential $W(\phi_i; t_a)$ in IIA will encode the same data as $\tilde{W}(\tilde{\phi}_i; \psi_a)$. The map between the parameters t_a and ψ_a will of course be the mirror map between the closed string moduli spaces. On the other hand, working out the map between the *open* string fields ϕ and $\tilde{\phi}$ is a complicated problem about which little is known at this point [52].

In practice, how does one go about constructing such mirror pairs? The strategy followed in [52] was to wrap $D5$ branes on curves $C \subset W$ which collapse to zero volume at some particular point in the Kähler moduli space of W . Then the 3-cycle Σ that the mirror

⁵ If one were to wrap a higher genus curve, there would of course also be Wilson lines; but flat bundles on P^1 do not lead to any additional degrees of freedom.

$D6$ brane wraps must collapse at the mirror point in the complex structure moduli space of M .⁶ In examples where $b_1(\Sigma) > 0$ but the curve C has less than $b_1(\Sigma)$ unobstructed deformations, the $D6$ theory must “lose” some its tree-level moduli to an instanton generated potential. Examples of this sort were produced in [52], which is strong evidence for the presence of the disc instanton effects (5.15). It would be extremely interesting to actually find an open string analogue of the mirror map, which lets one directly map the IIA superpotential to the IIB superpotential. As a byproduct, one might obtain nice counting formulas for holomorphic discs with boundaries in a special Lagrangian cycle [46].

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⁶ A simple argument for this is that in the closed string theory, there will be light non-perturbative states associated with the collapsing curve; to replicate this phenomenon, there must also be a vanishing cycle on the mirror manifold.

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