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Weak Beam-Laser Interaction in Undulator

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Abstract

Evolution of the density matrix of a bunch of particles in an undulator is described in the quantum mechanical theory neglecting effect of beam bunching. The approach of the paper is the same but the formalism is different from Becker-McIver's operator formalism [1].

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I. INTRODUCTION

Interaction of a bunch of particles with the laser field in an undulator has been described by Becker and McIver [1] using operator formalism of quantum mechanics. This note reproduces the main results of this paper in a different formalism where we follow the time evolution of the density matrix of the system. This approach will be used in the following publication to study the optical stochastic cooling [2].

The number of particles within a bunch is arbitrary, but theory neglects retardation which would lead to bunching in the self-amplification regime. In this sense the interaction is weak.

II. KINEMATICS

Here we reproduces for completeness the main features of laser/beam kinematics and introduce the density matrix of the beam.

The circularly polarized undulator in the lab (L) frame is described by the field $\vec{B}_L(z) = B_u[\vec{y}\cos(k_u z_L) + \vec{x}\sin(k_u z)]$, or transverse vector potential $\vec{A}_{u,L} = (B_u/k_u)[\vec{x}\sin(k_u z_L) + \vec{y}\cos(k_u z)]$, where z_L is coordinate along the axes of the undulator.

The equilibrium particle is defined as a particle moving in the lab frame with relativistic factor γ_0 and velocity along the undulator axes $v_{0,z} = v_0\cos\theta$, where $\theta = K_0/\gamma_0$, K_0 is undulator parameter, $K_0 = eB_u/(m_0 c^2 k_u)$. The equilibrium particle with momentum p_0 in the lab frame is at rest in the frame moving with velocity $v/c = k_L/(k_L + k_u)$, $\gamma \simeq \gamma_0/\sqrt{1+K_0^2}$, where $k_L = \omega_L/c = 2k_u\gamma_0^2/(1+K_0^2)$, ω_L is frequency radiated by the equilibrium particle in the lab frame in the forward direction. The frequency $\omega = ck$ in the moving frame, $k = \omega/c = \gamma(v/c)k_u \simeq \gamma k_u$. A particle with offset momentum in the lab frame $p_L = p_0 + \Delta p_L$ and the longitudinal velocity $v_{z,L} = v_L\cos\theta$ has, in the moving frame, longitudinal momentum $p_z/(m_0 c) = (\Delta p_L/p_0)/\sqrt{1+K_0^2}$. This defines transform of the rms bunch distribution to the moving frame. In particular, particles can be considered as non-relativistic in the moving frame for small $\Delta p_L/p_0$.

In the moving frame with coordinates (z,t), the undulator field $\vec{B}(z) = \vec{y}\gamma B_u\cos[\gamma k_u(z+$

$vt)]$ can be considered as flux of equivalent photons with frequencies $\omega = \gamma v k_u$. This frequency is equal to the frequency of photons radiated by the equilibrium particle in the moving frame. The last is the definition of the Bambini-Renieri moving frame and shows that, for the equilibrium particle, radiation is just back-scattering of undulator photons.

In quantum mechanics, the vector potential operator of the EM field in the moving frame in Shrodinger picture is

$$\vec{A} = \sqrt{\frac{2\pi h c^2}{V\omega}}\vec{y}[\hat{a}e^{ik\hat{z}} + cc], \quad |\vec{y}| = 1, \tag{1}$$

where $\hat{z}$ is operator of longitudinal coordinate, and V is normalization volume. As usual, $\hat{a}$ is Shrodinger time-independent annihilation operator, $\hat{a}|n >= |n-1> \sqrt{n}$, where $|n>$ is the n-photon state. The undulator field can be described in the moving frame as an external time-dependent field, for example,

$$A_u(z) = \frac{B_u}{2ik_u}e^{i\gamma k_u(\hat{z}+vt)} + c.c. \tag{2}$$

Hamiltonian of the system $H = (p - \frac{e}{c}[\vec{A}_u + \vec{A}])^2/2m_0$ in 1D case does not contain the linear in A term [3] for longitudinal initial $p(0)$, because both operators $\hat{A}_u$ and $\hat{A}$ are transverse and the canonical momentum p is constant of motion.

The average $< a > \propto e^{-i\omega t}$. Therefore, neglecting the small corrections $2\pi r_0 c^2/(V\omega)$ to ω and $(eA_u/c)^2$ to m_0c^2, we get [1]:

$$H = \sum_{i=1}^{N_B}\frac{\hat{p}_i^2}{2m_0} + h\omega(a^+a + 1/2) - ihg[a_u e^{2ik\hat{z}+i\omega t} - cc]. \tag{3}$$

The sum here is over N_B particles in the bunch, parameter of interaction

$$g = \frac{r_0 B_u}{2k_u}\sqrt{\frac{2\pi c^2}{V h\omega}} = cK_0\sqrt{\frac{e^2}{hc}\frac{\pi}{2kV}}, \tag{4}$$

where $r_0 = \frac{e^2}{m_0c^2}$, $K_0 = eB_u/(m_0c^2k_u)$. The interaction time in the moving frame is $t = 2\pi N_u/(v\gamma k_u)$, where N_u is number of the undulator periods.

In classical mechanics, bunch of particles is described approximately as product of normalized particle distribution functions $\Pi_{i=1}^{N_B} f(\vec{P}_i, \vec{r}_i)$, $\int d\vec{P}dV f(\vec{P}, \vec{r}) = 1$. Respectively, in quantum mechanics, it can be described as product of one-particle density matrices

$\hat\rho = |\vec{p}' > \rho(\vec{p}',\vec{p}) < \vec{p}|$, where the single particle density matrix $\rho(\vec{p}',\vec{p})$ in momentum representation is related to $f(\vec{P},\vec{r})$ by the Wigner's transform:

$$f(\vec{P},\vec{r}) = \int \frac{Vd\vec{q}}{(2\pi h)^6} e^{i\vec{q}\vec{r}/h}\rho(\vec{P}+\vec{q}/2,\vec{P}-\vec{q}/2). \tag{5}$$

The density matrix $\rho(\vec{p}',\vec{p})$ is normalized by the condition

$$\sum_{\vec{p}} \rho(\vec{p},\vec{p}) = 1, \quad \sum_{\vec{p}} = \int \frac{Vd\vec{p}}{(2\pi h)^3}. \tag{6}$$

Inverse transform

$$\rho(\vec{p}'\,\vec{p}) = \int \frac{(2\pi h)^3 d\vec{r}}{V} e^{-i(\vec{p}'-\vec{p})\vec{r}/h} f(\frac{\vec{p}+\vec{p}'}{2},\vec{r}) \tag{7}$$

can be used to define density matrix corresponding to classical distribution function. Consider the Gaussian distribution function with (z,p) correlation specified by parameter η:

$$f(\vec{p},\vec{r}) = \frac{1}{2\pi\Delta\sigma} e^{-\frac{(p-p_0)^2}{2\Delta^2}-\frac{(z-z_0-\eta p)^2}{2\sigma^2}}\delta(\vec{r}_\perp)\delta(\vec{p}_\perp). \tag{8}$$

The corresponding density matrix $\rho(\vec{p}',\vec{p}) = \rho_\perp(\vec{p}'_\perp,\vec{p}_\perp)\rho_l(p'_z,p_z)$, where

$$\rho_\perp(\vec{p}'_\perp,\vec{p}_\perp) = \frac{(2\pi h)^2}{S}\delta[\frac{\vec{p}'_\perp+\vec{p}_\perp}{2}], \tag{9}$$

$$\rho^0_l(p',p) = \frac{h\sqrt{2\pi}}{L\Delta} e^{-(i/h)(p'-p)(z_0+\eta(p+p')/2)-(1/2)(\sigma/h)^2(p'-p)^2-(1/2)(1/\Delta)^2((p+p')/2-p_0)^2}. \tag{10}$$

Here $V = SL$. As usual, integration over p can be converted to the sum. This can be done using correspondence shown in Eq. (6) and replacing $(2\pi h/L)\delta(p-p')$ by Kroneker's symbol $\delta_{p,p'}$, $L\delta(z-z')$ by $\delta_{z,z'}$, and $\int dz/L$ by the sum over z, $\sum_z$. For example, transform from the momentum representation of an operator O to the coordinate representation is given by $< p'|O|p >= \sum_{z,z'} < p'|z' >< z'|O|z >< z|p >$, where $< z|p >= e^{ipz/h}$, and $\sum_z < p'|z >< z|p >= \delta_{p',p}$.

III. BEAM DYNAMICS

Initial state of the system with n-photons and a bunch of particles with moments p_i, $i = 1..N_B$ is transformed by the interaction with the mode k to the vector

$$|\Psi(t) >= \sum_{l_i,p_i} |p_i - 2hkl_i, n + l_\Sigma > \sqrt{\frac{n!}{(n+l_\Sigma)!}} F_n(t,p_i,l_i) e^{-(it/h)\sum_i E(p_i,l_i) - i\omega t(n+l_\Sigma)}, \tag{11}$$

where $E(p_i, l_i) = (p_i - 2hkl_i)^2/(2m_0)$ and $l_\Sigma = \sum_i l_i$.

Equation for the amplitudes $F(t, [p_i, l_i])$ follows from the Shrodinger equation:

$$\dot{F}_n(t, p_j, l_j) = g\sum_j[(n+l_\Sigma)F_n(t, p_j, l_j-1)e^{-2i(k/m_0)t(p-2hkl_j)} - F_n(t, p_j, l_j+1)e^{2i(k/m_0)t(p_j-2hkl_j)}]. \tag{12}$$

Here, we use $< p_j - 2hkl_j|e^{\pm 2ik\hat{z}}|p'_j - 2hkl'_j >= (2\pi h/L)\delta[p'_j - p_j + 2hk(l_j - l'_j \pm 1)]$, and write explicitly only quantum numbers which are changed by the interaction, $F_n(t, p_j, l_j \pm 1) = F_n(t, (p_1, l_1).., (p_j, l_j \pm 1), .., p_{N_B}, l_{N_B})$.

Neglecting terms $hk^2/2m_0$ in the exponent (i.e., in the laboratory frame, terms of the order of $hk_uk_L/m_0 << 1$), we can solve Eq. (12) by the Fourier transform

$$F_n(t, p_j, l_j) = \{\Pi_{i=1}^{N_B}\int_0^{2\pi}\frac{d\phi_i}{2\pi}e^{il_i\phi_i}\}F(t, p_1..p_{N_B}, \phi_1, .., \phi_{N_B}). \tag{13}$$

Function $F(t, \phi) = F(t, p_1, ..p_{N_B}, \phi_1..\phi_{N_B})$ is given by

$$\dot{F}(t, \phi_) = -gv_0^*F(t, \phi) + g(n+1)v_0F + igv_0\sum_i\frac{\partial F}{\partial\phi_i}, \tag{14}$$

where

$$v_0(t, \phi) = \sum_i e^{-i\epsilon_it - i\phi_i}, \quad \epsilon_i = \frac{2kp_i}{m_0}. \tag{15}$$

If initial condition is the n-photon state $|\Psi(0) >= |p_i, n >$, then $F_n(0, p_i, l_i) = \Pi_i\delta_{l_i,0}$ and $F(0, \phi) = 1$.

Characteristics $\phi(t)$ of the Eq. (14) are defined by $\dot{\phi}_i(t) = -igv_0(t, \phi)$, with solutions $\phi_i(t) = \phi_i^0 + V(t)$, where $V(0) = 0$, $\phi_i^0 = \phi_i(0)$ are constants, and $V(t)$ is the same for all $\phi_i(t)$.

The function $V(t)$ satisfies $\dot{V}(t) = -igv_0$. Substitution of $\phi(t)$ in Eq. (15) gives

$$v_0(t, \phi) = u_0(t, \phi^0)e^{-iV(t)}, \quad u_0(t, \phi^0) = \sum_i e^{-i\epsilon_it - i\phi_i^0}. \tag{16}$$

Hence, $(\partial/\partial t)e^{iV(t)} = gu_0(t, \phi^0)$,

$$e^{iV(t)} = 1 + g\sum a_j(t)e^{-i\phi_j^0}, \tag{17}$$

where

$$a_i(t) = \frac{\sin(\epsilon_it/2)}{(\epsilon_i/2)}e^{-i\epsilon_it/2}, \quad \dot{a}_i(t) = e^{-i\epsilon_it}. \tag{18}$$

Eq. (17) defines characteristics $\phi_i(t)$

$$e^{i\phi_i} = e^{i\phi_i^0}[1 + g\sum a_j(t)e^{-i\phi_j^0}], \quad e^{-i\phi_i} = [e^{i\phi_i}]^{-1}. \tag{19}$$

Eq. (19) can be reversed to get constants of motion ϕ_i^0 in terms of ϕ_i. Define $\Lambda(\phi_1, ..., \phi_{N_B})$ as $e^{i\phi_i^0} = \Lambda e^{i\phi_i(t)}$. Substitute this in the right-hand-side of Eq. (19) to get

$$\Lambda = 1 - g\sum_j a_j(t)e^{-i\phi_j}. \tag{20}$$

Hence,

$$e^{i\phi_i^0} = e^{i\phi_i}[1 - g\sum a_j(t)e^{-i\phi_j}], \quad e^{-i\phi_i^0} = [e^{i\phi_i^0}]^{-1}. \tag{21}$$

The general solution of the Eq. (14) can be found as $F(t, \phi) = \Phi(t, e^{i\phi_i^0})$, where Φ is arbitrary function of the arguments $\xi_i(t, \phi) \equiv e^{i\phi_i^0}$ given by Eq. (21). Eq. (14) in terms of t, ξ takes form

$$\frac{\partial\Phi(t, \xi)}{\partial t} = -gv_0^*\Phi(t, \xi) + g(n + 1)v_0\Phi, \tag{22}$$

where

$$v_0(t) = \frac{u_0}{1 + g\sum_i a_i(t)/\xi_i}, \quad v_0^*(t) = u_0^*[1 + g\sum_i a_i(t)/\xi_i]. \tag{23}$$

Integrating Eq. (22) over t and substituting ξ_i from Eq. (21) gives

$$F(t, \phi) = \Phi_0(\xi)\frac{e^{-g\sum_i a_i^* e^{i\phi} + (g^2/2)|\sum_i a_i^* e^{i\phi}|^2 - (g^2/2)\sum_{i,j} e^{i(\phi_i - \phi_j)}\int_0^t d\tau[a_j\dot{a}_i^* - a_i^*\dot{a}_j]}}{[1 - g\sum_i a_i e^{-i\phi_i}]^{n+1}}. \tag{24}$$

Note, $a(0) = 0$. Hence, initial condition $F(0, \phi) = 1$ allows us to choose $\Phi_0(\xi) = 1$.

Parameter ϵt is of the order of ϵt $4\pi 2N_u(\delta p/p)_L/\sqrt{1 + K_0^2}$ where N_u is number of periods the undulator and $(\delta p/p)_L$ is rms energy spread in the lab system. We assume that $\epsilon t <<$ 1 what means that the rms energy spread of the bunch is small compared to the width $\Delta\omega/\omega \simeq 1/N_u$ of the mode. In this case, the integral term in the numerator

$$\int_0^t d\tau[a_j\dot{a}_i^* - a_i^*\dot{a}_j] \simeq \frac{it^3}{6}(\epsilon_i + \epsilon_j) - \frac{t^4}{12}(\epsilon_i^2 - \epsilon_j^2), \tag{25}$$

i.e. ϵt times smaller than other terms in the exponent (which are of the order of gt and $(gt)^2$) and can be neglected. We retain only diagonal terms $i = j$ for comparison with a single particle theory. Remaining terms can be written introducing additional integration over λ in terms of $b = g\sum_i a_i(t)e^{-i\phi_i}$:

$$F(t,\phi) = \int_0^\infty d\lambda \frac{\lambda^n}{n!} e^{-\lambda} e^{-b^* + \lambda b + (1/2)bb^*}. \qquad (26)$$

Let us factorize this expression using identity

$$e^{-b^* + \lambda b + (1/2)bb^*} = \hat{O}_{\lambda\kappa} e^{\lambda b - \kappa b^*}|_{\kappa=1}, \qquad (27)$$

where operator $\hat{O}_{\lambda\kappa} = e^{-(1/2)\frac{\partial^2}{\partial\lambda\partial\kappa}}$.

Now integration in Eq. (13) over ϕ_i can be easily carried out for each particle using

$$\int \frac{d\phi}{2\pi} e^{il\phi - \kappa a^* e^{i\phi} + \lambda a e^{-i\phi}} = (\frac{\lambda a}{\kappa a^*})^{l/2} J_l(2\sqrt{\lambda\kappa a a^*}). \qquad (28)$$

Replacement $k \to -k$ in the left-side of Eq. (28) changes the RHS by $J_l \to I_l$ with the rest intact.

Thus,

$$F_n(t,p,l) = \int_0^\infty d\lambda \frac{\lambda^n}{n!} e^{-\lambda} \hat{O}_{\lambda\kappa} \{\Pi_{i=1}^{N_B} (\frac{\lambda a_i}{\kappa a_i^*})^{l_i/2} J_{l_i}(2g|a_i|\sqrt{\lambda\kappa}) e^{-(g^2/2)\int_0^t d\tau [a_i \dot{a}_i^* - c.c]}\}|_{\kappa=1}, \qquad (29)$$

where J_l are Bessel functions.

For a single particle, $N_B = 1$, identity

$$\hat{O}_{\lambda\kappa}(\frac{\lambda}{\kappa})^{l/2} J_l(2g|a|\sqrt{\lambda\kappa})|_{\kappa=1} = \lambda^{l/2} e^{(1/2)g^2|a|^2} J_l[2g|a|\sqrt{\lambda}], \qquad (30)$$

which can be verified using series for the Bessel function, allows us to write

$$F_n(t,p,l) = \int_0^\infty d\lambda \frac{\lambda^{n+l/2}}{n!} e^{-\lambda} (\frac{a_i}{a_i^*})^{l/2} J_l(2g|a|\sqrt{\lambda}) e^{(1/2)g^2|a|^2} e^{-(g^2/2)\int_0^t d\tau [a_i \dot{a}_i^* - c.c]}. \qquad (31)$$

The phase factor $exp[-(g^2/2)\int_0^t d\tau (a\dot{a}^* - c.c.)] = exp\{-i(g/\epsilon)^2[\epsilon t - \sin(\epsilon t)]\}$.

Integration over λ gives [5] result in terms of Laguerre polynomials L_n^l:

$$F_n(t,p,l) = (ga)^l e^{-(1/2)g^2|a|^2} L_n^l(g^2|a|^2) e^{-(g^2/2)\int_0^t d\tau [a_i \dot{a}_i^* - c.c]}. \qquad (32)$$

Eq. (32) reproduces Dattoli-Renieri [4] result. Eq. (29) defines evolution of the initial state of the system for a bunch with N_b particles.

It is easy to see that Eq. (12) conserves the norm. By definition, function $F(t,p,l) = F_n(t, p_1, l_1, ..., p_{N_B}, l_{N_b})$ is normalized

$$\sum_{l_1,...,l_{N_B}} \frac{n!}{(n+l_\Sigma)!} |F_n(t,p,l)|^2 = 1, \quad l_\Sigma = \sum_i l_i. \qquad (33)$$

IV. TIME EVOLUTION OF THE DENSITY MATRIX

Let us consider time evolution of the density matrix $\hat{\rho}$ describing bunch of N_B particles with momentum $p_1, .., p_{N_B}$. At the entrance to the undulator, there is no radiation,

$$\hat{\rho}(0) = |p', p'_\perp, 0 > \rho^0_\perp \rho^0_l < p, p_\perp, 0|, \tag{34}$$

where $\rho^0_\perp$ and ρ^0_l are defined by Eqs. (9) and (10). $\hat{\rho}(0)$ is normalized by the condition $Tr[\hat{\rho}(0)] = 1$, or

$$\int \frac{V d\vec{p}}{(2\pi h)^3} \rho^0_\perp \rho^0_l = 1. \tag{35}$$

In the 1D case, $\rho_\perp$ does not change and it is suffice to consider evolution of the longitudinal part of $\hat{\rho}(t)$.

In the undulator, the state $|p, 0 >$ evolves to the state $|\Psi >$ according to Eqs. (11), (29). The density matrix at the end of the undulator can be written separating variables of the individual particles and global parameters of the radiation l_Σ, l'_Σ. It can be done introducing additional integration:

$$\sum_{l_1,...,l_{N_B}} = \sum_{l_1,...,l_{N_B}} \sum_{l_\Sigma} \delta_{l_\Sigma, l_1+...+l_{N_B}} = \int \frac{d\psi}{2\pi} e^{-il_\Sigma \psi} \Pi_{i=1}^{N_B} \sum_{l_i} e^{il_i\psi}. \tag{36}$$

The density matrix takes form

$$\hat{\rho}(t) = \rho_\perp(\vec{q}'_\perp, \vec{q}_\perp) \rho(q', q, l_\Sigma, l'_\Sigma) |q', \vec{q}'_\perp, l'_\Sigma >< q, \vec{q}_\perp, l_\Sigma|, \tag{37}$$

$$\rho(q', q, l'_\Sigma, l_\Sigma) = \frac{1}{\sqrt{l_\Sigma! l'_\Sigma!}} \int \frac{d\psi d\psi'}{(2\pi)^2} e^{-i(l'_\Sigma \psi' - l_\Sigma \psi)} e^{i\omega t(l_\Sigma - l'_\Sigma)} \int d\lambda d\lambda' e^{-\lambda - \lambda'} \hat{O}_{\lambda\kappa} \hat{O}_{\lambda'\kappa'} F_{loc}(q', q). \tag{38}$$

Here $|q >$ stands for the set $|q_1..q_{N_B} >$, and $F_{loc}(q', q) = \Pi_{i=1}^{N_B} f_i(q'_i, q_i)$, where

$$f_i(q', q) = \sum_{l', l} (\frac{\lambda a}{\kappa a^*})^{l/2} J_l[2g|a(t)|\sqrt{\lambda\kappa}] (\frac{\lambda'[a']^*}{\kappa' a'})^{l'/2} J_{l'}[2g|a'(t)|\sqrt{\lambda'\kappa'}] e^{\frac{it}{2m_0 h}(q'^2 - q^2)} e^{i(l'\psi' - l\psi)} \rho^0_l, \tag{39}$$

$a = a(q, t)$, $a' = a(q', t)$, and $\rho^0_l = \rho^0_l[q' + 2hkl', q + 2hkl]$ is defined in Eq. (10).

Eq. (38) is generalized to the case of initial n photon state replacing factor $1/\sqrt{l_\Sigma! l'_\Sigma!}$ by $\sqrt{n!n'!/[(n + l_\Sigma)!(n' + l'_\Sigma)!]} \lambda^n (\lambda')^{n'}$.

Let us consider initial density matrix, Eq. (10), with no z, p correlation ($\eta = 0$). It describes probability to find a particle with moment p at location z. For a particle within a bunch with rms σ_B and Δ_B centered at $z_0 = p_0 = 0$,

$$\rho_l^0(q' + 2hkl', q + 2hkl) = \frac{h\sqrt{2\pi}}{L\Delta_B} e^{-\frac{1}{2}(\frac{\sigma_B}{h})^2[q'-q+2hk(l'-l)]^2-(1/2)(1/\Delta_B)^2[(q+q')/2+hk(l'+l)]^2}. \quad (40)$$

If the wave length of radiation is small compared to the rms bunch length, $k\sigma_B >> 1$, the non-diagonal terms $l' \neq l$ are exponentially small. We also can approximate $a_i(t) \simeq te^{-(i/2)\epsilon_i t}$. Then Eq. (39) is simplified

$$f_i(q_i', q_i) = \sum_l (\frac{\lambda}{\kappa})^{l/2} J_l[2gt\sqrt{\lambda\kappa}] (\frac{\lambda'}{\kappa'})^{l/2} J_l[2gt|\sqrt{\lambda'\kappa'}] e^{il(\epsilon'-\epsilon)t} e^{\frac{it}{2m_0 h}(q'^2-q^2)} e^{il(\psi'-\psi)} \rho_l^0, \quad (41)$$

where $\rho_l^0(q', q) = \frac{h\sqrt{2\pi}}{L\Delta_B} e^{-\frac{1}{2}(\frac{\sigma_B}{h})^2(q'-q)^2-(1/2)(1/\Delta_B)^2[(q+q')/2+2hkl]^2}$.

A The norm of the density matrix

First let us show that the density matrix Eqs. (38), (41) is normalized $Tr[\hat{\rho}(t)] = 1$. This also allows us to check approximations we are going to make. Trace

$$Tr[\hat{\rho}(t)] = \sum_{l_\Sigma} \frac{1}{l_\Sigma!} \int \frac{d\psi d\psi'}{(2\pi)^2} e^{-il_\Sigma(\psi'-\psi)} \int d\lambda e^{-\lambda} \hat{O}_{\lambda\kappa} \int d\lambda' e^{-\lambda'} \hat{O}_{\lambda'\kappa'} \Pi_{i=1}^{N_B} [\sum_{q_i} f_i(q_i, q_i)]. \quad (42)$$

Integration over dq_i gives

$$\sum_{q_i} f_i(q_i, q_i) = \sum_{l_i=-\infty}^{\infty} (\frac{\lambda}{\kappa})^{l_i/2} J_{l_i}[2gt\sqrt{\lambda\kappa}] (\frac{\lambda'}{\kappa'})^{l_i/2} J_{l_i}[2gt|\sqrt{\lambda'\kappa'}] e^{i(\psi'-\psi)l_i}. \quad (43)$$

The sum here includes $l_i < 0$ even for $l_\Sigma >= 0$ due to absorption of photons radiated by other electrons in a bunch. For $gt << 1$, we can use approximation $J_l[x] \simeq (x^l/l!)[1 - x^2/(l+1)]$ for $l > 0$, and $J_l[x] = (-1)^l J_{|l|}(x)$ for $l < 0$. Neglecting terms of the order of $o(gt)^4$, we get

$$\sum_{q_i} f_i(q_i, q_i) = 1 - g^2t^2(\lambda\kappa + \lambda'\kappa') + g^2t^2[\lambda\lambda' e^{i(\psi'-\psi)} + \kappa\kappa' e^{-i(\psi'-\psi)}]. \quad (44)$$

With the same accuracy,

$$\Pi_{i=1}^{N_B} [\sum_{q_i} f_i(q_i, q_i)] = e^{-N_B g^2 t^2(\lambda\kappa+\lambda'\kappa')+N_B g^2 t^2[\lambda\lambda' e^{i(\psi'-\psi)}+\kappa\kappa' e^{-i(\psi'-\psi)}]}. \quad (45)$$

This takes correctly into account terms of arbitrary power of $N_B g^2 t^2$ neglecting terms with additional factor $g^2 t^2$.

Calculation of $Tr[\hat{\rho}(t)]$ can be simplified, first, integrating by parts

$$\int d\lambda e^{-\lambda} \hat{O}_{\lambda\kappa} f(\lambda,\kappa) = \int d\lambda e^{-\lambda} e^{-(1/2)\frac{\partial}{\partial\kappa}} f(\lambda,\kappa) \tag{46}$$

and then by the identity $e^{-(1/2)\frac{\partial}{\partial\kappa}} f(\lambda,\kappa)|_{\kappa=1} = f(\lambda,\kappa-1/2)|_{\kappa=1} = f(\lambda,1/2)$ for an arbitrary $f(\lambda,\kappa)$. This gives

$$Tr[\hat{\rho}(t)] = \sum_{l_\Sigma} \frac{1}{l_\Sigma!} \int \frac{d\psi d\psi'}{(2\pi)^2} e^{-il_\Sigma(\psi'-\psi)} \int d\lambda \int d\lambda' e^{-(1+p)(\lambda+\lambda')} e^{2p[\lambda\lambda' e^{i(\psi'-\psi)} + (1/4)e^{-i(\psi'-\psi)}]}, \tag{47}$$

where $p = (1/2) N_B g^2 t^2$. Integration over ψ, ψ' now can be carried out using Eq. (28):

$$Tr[\hat{\rho}(t)] = \sum_{l_\Sigma} \frac{1}{l_\Sigma!} \int d\lambda e^{-\lambda} \int d\lambda' e^{-\lambda'} e^{-p(\lambda+\lambda')} (4\lambda\lambda')^{l_\Sigma/2} I_{l_\Sigma}(2p\sqrt{\lambda\lambda'}). \tag{48}$$

Remaining integrals can be calculated using series for the Bessel function

$$Tr[\hat{\rho}(t)] = \sum_{l_\Sigma=0}^{\infty} \sum_{n=0}^{\infty} \frac{(n+l_\Sigma)!}{n! l_\Sigma!} \frac{p^{2n}(2p)^{l_\Sigma}}{(1+p)^{2(n+l_\Sigma+1)}}. \tag{49}$$

The sum $\sum_{l=0}^{\infty} (n+l)! x^l/(n! l!) = (1-x)^{-n-1}$ and the sum over n is geometric series.

This gives, finally, the desirable result $Tr[\hat{\rho}(t)] = 1$.

B Density matrix of radiation

The radiation density matrix at the exit of the undulator can be obtained averaging Eq. (37) over the state of the bunch,

$$\hat{\rho}_{rad}(t) = \rho_{rad}(l'_\Sigma, l_\Sigma) |l'_\Sigma >< l_\Sigma|, \tag{50}$$

where

$$\rho_{rad}(l'_\Sigma, l_\Sigma) = \frac{1}{\sqrt{l_\Sigma! l'_\Sigma!}} \int \frac{d\psi d\psi'}{(2\pi)^2} e^{-i(l'_\Sigma\psi' - l_\Sigma\psi)} e^{i\omega t(l_\Sigma - l'_\Sigma)} \int d\lambda d\lambda' e^{-\lambda-\lambda'} \hat{O}_{\lambda\kappa} \hat{O}_{\lambda'\kappa'} \sum_q F_{loc}(q,q). \tag{51}$$

The last factor, $\sum_q F_{loc}(q,q)$ is still given by Eq. (45). Eq. (46) and the following discussion allow us to simplify $\rho_{rad}(l'_\Sigma, l_\Sigma)$ and, after integration over ψ, ψ', it becomes the diagonal matrix

$$\rho_{rad}(l'_\Sigma, l_\Sigma) = \delta_{l_\Sigma, l'_\Sigma} \frac{1}{l_\Sigma!} \int d\lambda d\lambda' e^{-(1+p)(\lambda+\lambda')} (4\lambda\lambda')^{l_\Sigma/2} I_{l_\Sigma}(2p\sqrt{\lambda\lambda'}). \tag{52}$$

Integration here is the same as in the previous section giving

$$\rho_{rad}(l'_\Sigma, l_\Sigma) = \delta_{l_\Sigma, l'_\Sigma} \frac{(2p)^{l_\Sigma}}{(1+2p)^{l_\Sigma+1}}. \tag{53}$$

The average number of radiated photons $< a^+a >= Tr[\hat{\rho}_{rad} a^+ a] = \sum_{l_\Sigma} l_\Sigma \rho_{rad}(l_\Sigma, l_\Sigma)$ is $< a^+a >= 2p = N_B(gt)^2$. Hence, parameter of expansion $(gt)^2$ in Eq. (44) is the number of photons radiated in the undulator per particle. More exactly, the number of photons is $a^+a\Omega$, where Ω is the phase volume of the mode $\Omega = Vd^3k/(2\pi)^3$. In the lab frame, $d^3k_L = \pi k_L^3 (d\omega_L/\omega_L) d\alpha^2$, where $d\omega_L/\omega_L = 1/(2N_u)$ is the relative frequency spread, and $\pi\alpha^2$ is solid angle of the mode, $\alpha = \sqrt{\lambda_L/L_u} = (1/\gamma_0)\sqrt{(1+K_0^2)/(2N_u)}$ [7]. In the moving frame where $k_L = 2\gamma k$, $k = \gamma k_u$, and relativistic factor $\gamma = \gamma_0/\sqrt{1+K_0^2}$, $d^3k = \pi k^3/[N_u^2(1+K_0^2)]$. This gives number of photons in the mode: $k\ a^+a\Omega = (gt)^2\Omega = (\pi/4)(\frac{e^2}{hc})K_0^2/(1+K_0^2)$.

In the same way, $< (a^+a)^2 >= \sum_{l_\Sigma} l_\Sigma^2 \rho_{rad}(l_\Sigma, l_\Sigma)$ or $< (a^+a)^2 >= x(d/dx)x(d/dx)(x^l/(1+x)^{l+1})|_{x=2p} = 2(2p)^2 + 2p$. Hence, the rms spread of number of photons increases with the average number of photons. Eq. (53) reproduces the thermal statistics of radiation [6].

C Density matrix of the bunch

The density matrix of the bunch $\hat{\rho}_{bunch}(t)$ can be obtained from Eq. (37) as $\sum_{l_\Sigma=0}^{\infty} \rho(q', q, l_\Sigma, l_\Sigma)$.

$$\hat{\rho}_{bunch}(t) = \rho_\perp(\vec{q}'_\perp, \vec{q}_\perp)\rho_B(q', q,)|q' >< q|, \tag{54}$$

where

$$\rho_B(q', q) = \sum_{l_\Sigma} \frac{1}{l_\Sigma!} \int \frac{d\psi d\psi'}{(2\pi)^2} e^{-il_\Sigma(\psi'-\psi)} \int d\lambda d\lambda' e^{-\lambda-\lambda'} \hat{O}_{\lambda\kappa} \hat{O}_{\lambda'\kappa'} F_{loc}(q', q). \tag{55}$$

Here $F_{loc}(q', q) = \Pi_{i=1}^{N_B} f_i(q'_i, q_i)$, and f_i are defined by Eq. (39).

For illustration, let us consider only diagonal terms, $\rho_B(q, q)$. In this case, $f_i(q, q)$ and F_{loc} are different from those in Eq. (43) and Eq. (45) only by additional factor $\rho_l^0 = \frac{h\sqrt{2\pi}}{L\Delta_B} e^{-\frac{1}{2\Delta_B^2}(q_i + 2hkl)^2}$.

Neglecting small terms $o(gt)^4$, we get similar to Eq. (44)

$$f_i(q'_i, q_i) = \frac{h\sqrt{2\pi}}{L\Delta_B}\{[1 - g^2t^2(\lambda\kappa + \lambda'\kappa')]e^{-\frac{1}{2\Delta_B^2}q_i^2} + \tag{56}$$

$$+g^2t^2[\lambda\lambda' e^{i(\psi'-\psi)}e^{-\frac{1}{2\Delta_B^2}(q+2hk)^2}+\kappa\kappa' e^{-i(\psi'-\psi)}e^{\frac{1}{2\Delta_B^2}(q-2hk)^2}]. \tag{57}$$

With the same accuracy,

$$F_{loc}(q,q)=[\Pi_i\rho_0(q_i)]e^{-2p(\lambda\kappa+\lambda'\kappa')}e^{2p[\lambda\lambda' e^{i(\psi'-\psi)}\sigma_-+\kappa\kappa' e^{-i(\psi'-\psi)}\sigma_+]}, \tag{58}$$

where $\rho_0(q)=\frac{h\sqrt{2\pi}}{L\Delta_B}e^{-\frac{1}{2\Delta_B^2}q^2}$, and

$$\sigma_\pm=\frac{1}{N_B}\sum_{i=1}^{N_B}e^{-\frac{1}{2\Delta_B^2}(\pm 4hkq_i+(2hk)^2)}. \tag{59}$$

Using Eq. (46)and Eq. (28), we get

$$\rho_B(q,q)=[\Pi_i\rho_0(q_i)]\sum_{l_\Sigma}\frac{1}{l_\Sigma!}\int d\lambda d\lambda' e^{-(1+p)(\lambda+\lambda')}(4\lambda\lambda')^{l_\Sigma/2}I_{l_\Sigma}(2ps\sqrt{\lambda\lambda'})[\frac{\sigma_-}{\sigma_+}]^{l_\Sigma/2}, \tag{60}$$

where $s=\sqrt{\sigma_-\sigma_+}$. Integrals can be calculated as in Eq. (48), what gives finally

$$\rho_B(q,q)=[\Pi_i\rho_0(q_i)]\frac{1}{1+2p(1-\sigma_-)+p^2[1-\sigma_-\sigma_+]}. \tag{61}$$

Approximately,

$$\sigma_\pm=1-\frac{2(hk)^2}{\Delta_B^2}+(2\frac{(hk)^2}{N_B\Delta_B^4}\sum q_i^2\mp\frac{2hk}{N_B\Delta_B^2}\sum_i q_i. \tag{62}$$

Expanding $\rho_B(q,q)$ in series over hk,

$$\rho_B(q,q)=[\Pi_i\rho_0(q_i)]\{1-\frac{(2hk)^2}{\Delta_B^2}p(1+p)-\frac{4hkp}{N_B\Delta_B^2}\sum_i q_i+\frac{(2hk)^2p(1+p)}{N_B\Delta_B^4}\sum_i q_i^2+o(\frac{(2hkp)^2}{N_B^2\Delta_B^4}(\sum_i q_i)^2)\}, \tag{63}$$

it is easy to check that the norm is preserved $\sum_q\rho_B(q,q)=1$ with the accuracy of terms $o(N_b(gt)^4)$ we neglected. The average $<q>=-4hkp/N_B$, and $<q^2>=\Delta_B^2$. This corresonds to energy loss $<q>=-2hk(gt)^2$. The rms energy spread does not changed.

Hence, modulation in the bunch phase space by radiation is due to non-diagonal terms we neglected so far. They will be considered in the next paper.

V. CONCLUSION

The main results by Becker-McIver for weak laser-beam interaction are reproduced including statistics of the undulator radiation. Explicit form of the density matrix is given. The density matrices of the radiation and of the bunch are obtained as projections of the density matrix of the system. Although fluctuations of number of radiated photons are strong, they do not increase the rms energy spread of the bunch. It is found that small non-diagonal terms of the density matrix have to be taken into account to describe micro-correlations induced in a bunch due to radiation.

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