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Dynamics of the Coherent State in Quantum Amplifier

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Abstract

Evolution of the density matrix in quantum amplifier is described for arbitrary initial conditions. Explicit form is given for the initial condition which is a superposition of coherent states.

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I. INTRODUCTION

Amplification by a linear phase-independent quantum amplifier can be modeled [1] as interaction of radiation with the two-level system being in an equilibrium with the thermal bath. Equation for the density matrix ρ describing variation of the density matrix in time is well known [1],

$$\dot{\rho} = -gN_+[aa^+\rho + \rho aa^+ - 2a^+\rho a] - gN_-[\rho a^+ a + a^+ a\rho - 2a\rho a^+]. \tag{1}$$

Here g is parameter of interaction and $N_\pm$a are population of upper/lower levels. The model of the amplifier implies fixed in time inverse population of two levels $N_+ > N_-$. The solution is usually characterized in terms of the lowest moments of the distribution or by the diagonal components of the density matrix. This is, for example, sufficient for study the signal to noise ratio. However, the explicit form of the density matrix may be required by some problems. In this note, the density matrix is derived for the arbitrary initial condition. The explicit form of the density matrix is found for the initial conditions given by a superposition of coherent states.

II. SOLUTION

Let us consider density matrix $\rho(t) = |n' > \rho(n', n, t) < n|$, $\rho(n', n, t) =< n'|\rho|n >$ in the n-photon basis $|n >$, $a|n >= |n > \sqrt{n}$, and define function F,

$$\rho(n', n, t) = \frac{F(N, m, t)}{\sqrt{n!\, n'!}}, \tag{2}$$

where $N = (n + n')/2$ and $m = (n - n')/2$. We consider components with $m >= 0$. Components $m < 0$ can be obtained from the components with $m > 0$ by complex conjugation, $F(N, -m, t) = F^*(N, m, t)$. Eq. (1) for the function $F(N, m, t)$ takes form

$$\dot{F}(N, m, t) = -[(N+1)F(N, m, t) - (N^2 - m^2)F(N-1, m, t)] - p[NF(N, m, t) - F(N+1, m, t)], \tag{3}$$

where dot means derivative over dimensionless time $\tau = 2N_+t$, and $p = N_-/N_+$. For an amplifier $p < 1$. All terms in Eq. (3) have the same dependence on m which is, therefore, a constant of motion. To solve Eq. (1) we use Mellin transform in z and Laplace transform in time

$$F(N,m,t) = \int_{-i\infty}^{i\infty} d\lambda \int_0^{\infty} dz z^{N-1} f_m(z,\lambda) e^{\lambda\tau}. \tag{4}$$

Functions $f_m(z,\lambda)$ are defined by the equation

$$z f_m''(z,\lambda) + [(1+p)z - 1] f_m'(z,\lambda) + [\frac{1-m^2}{z} - \lambda - 1 + pz] f_m(z,\lambda) = 0, \tag{5}$$

where prime means derivative over z. Eq. (5) has to be solved with the boundary conditions

$$z^{N-1} f_m(z,\lambda)|_0^\infty = 0, \quad z^N \frac{\partial f_m(z,\lambda)}{\partial z}|_0^\infty = 0. \tag{6}$$

The general solution of Eq. (4) is given in terms of the confluent hypergeometric functions:

$$f_m(z,\lambda) = z e^{-z} \{A_+(m,\lambda) z^m \Phi[m + \frac{\lambda+1-p}{1-p}, 2m+1, \zeta] + (m- > -m)\}, \tag{7}$$

where $\zeta = (1-p)z$.

The boundary condition at $z = 0$ is satisfied for positive n and n'. The boundary condition at infinity requires either $p > 0$ or $N + \lambda < \pm m$ what follows from asymptotic behavior at large z,

$$\Phi(\alpha,\gamma,z) - > z^{-\alpha} + e^z z^{\alpha-\gamma}. \tag{8}$$

Let us assume that $A_\pm(m,\lambda)$ have simple poles at λ given by $m + (\lambda+1-p)/(1-p) = -k$, where $k = 0, 1, 2...$ For integer $k >= 0$, both terms in Eq. (7) are proportional to Laguerre's polynomials $L_k^m(z)$,

$$L_k^m(z) = \frac{(k+2m)!}{k!(2m)!} \Phi[-k, 2m+1, z]. \tag{9}$$

With this assumption,

$$f_m(z,t) = \int d\lambda f_m(z,\lambda) e^{\lambda\tau} = \sum_k B(m,k) e^{-(k+m+1)(1-p)\tau} z^{m+1} e^{-z} L_k^{2m}(\zeta), \tag{10}$$

where $B(m,k)$ are constants proportional to the residues of the poles.

Let $f_0(z,m)$ be the Mellin transform of the initial condition $F(N,m,0)$,

$$f_0(z,m) = \int_{-i\infty}^{i\infty} \frac{dN}{2\pi i} z^{-N} F(N,m,0). \tag{11}$$

Eq. (10) at $t=0$ can be solved for $B(m,k)$ using orthogonality of the functions $\phi_k^m(z)$,

$$\phi_k^m(z) = \sqrt{\frac{k!}{(k+2m)!}} e^{-z/2} z^m L_k^{2m}(z), \tag{12}$$

$$\int_0^\infty dz \phi_k^m(z) \phi_{k'}^m(z) = \delta_{k,k'}. \tag{13}$$

Then,

$$B(m,k) = \frac{k!(1-p)^{2m+1}}{(2m+k)!} \int_0^\infty dz f_0(z,m) z^{m-1} L_k^{2m}(\zeta) e^{pz}. \tag{14}$$

Eqs. (10) and (14) define

$$f_m(z,t) = \int_0^\infty dz' K_m(z,z',\tau) f_0(z',m), \tag{15}$$

where the Green's function

$$K_m(z,z',\tau) = \sum_k \frac{k!(1-p)}{(k+2m)!} (\zeta')^{m-1} \zeta^{m+1} e^{-z+pz'} e^{-(1-p)(m+k+1)\tau} L_k^{2m}(\zeta) L_k^{2m}(\zeta'). \tag{16}$$

Summation here can be carried out using [2]

$$\sum_n \frac{n!}{(n+m)!} z^n L_n^m(x) L_n^m(y) = \frac{(xyz)^{-m/2}}{1-z} e^{-z(x+y)/(1-z)} I_m(\frac{2\sqrt{xyz}}{1-z}), \tag{17}$$

where I_m is modified Bessel function of the second kind, and $|z| < 1$. With notation $\xi = e^{(1-p)\tau}$, we get

$$K_m(z,z',\tau) = (1-p)\frac{z}{z'}\frac{\xi}{1-\xi} e^{-z\frac{1-\xi p}{1-\xi} - z'\frac{\xi-p)}{1-\xi}} I_{2m}[\frac{2(1-p)}{1-\xi}\sqrt{zz'\xi}]. \tag{18}$$

Note, $K_m(z,z',t) -> \delta(z-z')$ in the limit $t->0$, $\xi->1$.

$F(N,m,t)$ can be written as

$$F(N,m,t) = \int_0^\infty dz' G_m(N,z',\tau) f_0(z',m), \tag{19}$$

where the kernel $G(N,z',t) = \int_0^\infty K_m(z,z',t) z^{N-1} dz$ is given in terms of the Laguerre's polynomials:

$$G_m(N,z',\tau) = (N-m)!(\frac{1-p}{z'})(\frac{\xi}{1-\xi})(\frac{1-\xi}{1-\xi p})^{N+1} b^m L_{N-m}^{2m}(-b) e^{b-z'(\xi-p)/(1-\xi)}, \tag{20}$$

where

$$b = \frac{(1-p)^2 \xi z'}{(1-\xi)(1-\xi p)}, \quad \xi = e^{-(1-p)\tau}. \tag{21}$$

Eqs. (19),(20), and (11) describe dynamics for arbitrary initial conditions.

III. EVOLUTION OF A COHERENT STATE

For the sake of simplicity, let us consider the full inverse population, $p = 0$. Eq. (20) is simplified in this case:

$$G_m(N, z', \tau) = (N-m)!\frac{\xi}{z'}(1-\xi)^N b^m L_{N-m}^{2m}(-b), \tag{22}$$

where

$$b = \frac{\xi z'}{(1-\xi)}, \quad \xi = e^{-2N+gt}. \tag{23}$$

Consider initial coherent state

$$\rho(n', n, 0) = \frac{\alpha^{n'}(\alpha^*)^n}{\sqrt{n!n'!}} e^{-|\alpha|^2}, \quad F(N, m, 0) = (\frac{\alpha^*}{\alpha})^m |\alpha^*\alpha|^N e^{-|\alpha|^2}. \tag{24}$$

Eq. (11) gives

$$f_0(z, m) = (\frac{\alpha^*}{\alpha})^m |\alpha|^2 e^{-|\alpha|^2}\delta(z - |\alpha|^2). \tag{25}$$

Then, Eqs. (19), (22) define $\rho(n', n, t) = F(N, m, t)/\sqrt{n!n'!}$,

$$F(N, m, t) = (N-m)!(\frac{\alpha^*}{\alpha})^m |\alpha|^{2m} e^{-|\alpha|^2}\xi^{m+1}(1-\xi)^{N-m} L_{N-m}^{2m}(-\frac{\xi}{1-\xi}|\alpha|^2). \tag{26}$$

Let us, for example, find the lowest moment of the distribution. The amplitude

$$< a(t) >= \sum_{n,n'} \rho(n', n, t) < n|a|n' >= \sum_n \sqrt{n}\rho(n, n-1, t), \tag{27}$$

or

$$< a(t) >= \sum_n \frac{F(n-1/2, -1/2, t)}{(n-1)!}. \tag{28}$$

Initial amplitude $< a(0) >= \alpha$. Using

$$\sum_n z^n L_n^m(x) = (1-z)^{-m-1} e^{\frac{xz}{1-z}}, \tag{29}$$

it is easy to find

$$< a(t) >= \frac{\alpha}{\sqrt{\xi}}. \tag{30}$$

Hence, $G_A = 1/\xi$ is power gain (amplification factor) of the amplifier. In the same way it is easy to get higher order moments. In particular, $< a(t)^2 >= \alpha^2 G$, the well known result which can be obtained directly from Eq. (1).

Similarly, somewhat more general initial state

$$F(N,m,0) = (\frac{\alpha_1^*}{\alpha_2})^m (\alpha_1^* \alpha_2)^N \tag{31}$$

can be considered. Then, for $Re[\alpha_1^* \alpha_2] > 0$, we get in terms of power gain:

$$F(N,m,t) = (N-m)!(\frac{\alpha_1^*}{\alpha_2})^m (\alpha_1^* \alpha_2)^m (\frac{1}{G_A})^{m+1} (1-\frac{1}{G_A})^{N-m} L_{N-m}^{2m}(-\frac{\alpha_1^* \alpha_2}{G_A - 1}). \tag{32}$$

In the limit $t->0$, $L_{N-m}^{2m}(x)->(-x)^{N-m}/(N-m)!$, and the initial condition Eq. (31) is satisfied.

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REFERENCES

[1] M.O. Scully and M. S. Zubairy, Quantum Optics, Cambridge University Press, 1997

[2] I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Series, and Products, Academic Press, 1980, p. 1038