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An Effective Approach to VMD at One Loop Order and the Departure from Ideal Mixing for Vector Mesons

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Abstract

We examine the mechanisms producing departures from ideal mixing for vector mesons within the context of the Hidden Local Symmetry (HLS) model. We show that kaon loop transitions between the ideal combinations of the ω and ϕ mesons necessitate a field transformation in order to get the mass eigenstates. It is shown that this transformation is close to a rotation for processes involving, like meson decays, on–shell ω and ϕ mesons. The HLS model predicts a momentum dependent, slowly varying mixing angle between the ideal states. We examine numerically the consequences of this for radiative and leptonic decays of light mesons. The mean $\omega - \phi$ mixing angle is found smaller than its ideal value; this is exhibited separately in radiative and in leptonic decays. Effects of nonet symmetry breaking in the vector sector are compared to those produced by the field rotation implied by the HLS model.

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I. INTRODUCTION

The Hidden Local Symmetry (HLS) Model in both its non–anomalous [2] and anomalous (FKTUY) sectors [3] is a powerful tool for analyzing experimental data, by providing a clear framework with the fewest possible number of free parameters. For instance, it allows a 3–parameter description of the $I = 1$ pion form factor; this gives a statistically optimal description in an energy interval running from threshold to the ϕ mass. This has been shown by Ref. [4] in analyzing the world data set for $e^+e^- \to \pi^+\pi^-$ annihilation collected in Ref. [5]. The exercise has been repeated as successfully with the data set recently collected by the CMD–2 Collaboration on the VEPP–2M collider at Novosibirsk [6].

However, in order to go beyond, while staying within the HLS–FKTUY framework, one needs to define a consistent scheme of symmetry breakings. Without SU(3) breaking, the HLS model cannot successfully describe the kaon form factors; without nonet symmetry breaking in the pseudoscalar (PS) sector, it cannot be used reliably to describe radiative decays of light mesons. The BKY mechanism [7,8] is a consistent way to introduce SU(3) breaking in both the vector (V) and PS sectors. It has been shown recently [9] that the BKY SU(3) breaking in the PS sector is in perfect agreement with all accessible predictions of Chiral Perturbation Theory [10–12] at first order in the breaking parameters. In order to reach this conclusion, the needed ingredients were only the BKY breaking in the PS sector (referred to hereafter as X_A breaking), the kinetic energy term of the non–anomalous HLS Lagrangian and the $P\gamma\gamma$ Lagrangian of Wess, Zumino and Witten [13]. Thus, this part is on secure grounds.

When dealing with PS mesons, the question of nonet symmetry breaking (NSB) cannot be avoided, as clear from Refs. [14,15] for instance. It was already introduced in the physics of single photon radiative decays (12 modes) by O'Donnell long ago [16], relying basically only on group theoretical considerations, but outside the context of effective Lagrangians. Thus, Ref. [16] already proved that NSB can be dealt with simply in both the V and PS sectors. However, the way to include both NSB kinds consistently within a Lagrangian (HLS for instance) is not necessarily a trivial matter.

Recently, we proposed [1] a model to describe, within the HLS framework, all radiative decays of light mesons ($VP\gamma$ or $P\gamma\gamma$ processes) and their leptonic decays ($\rho^0/\omega/\phi \to e^+e^-$ processes). Ref. [1] included SU(3) breaking in the V and PS sectors in the manner of BKY [7,8] and proposed a way to include simultaneously NSB in the PS sector. Slightly later, Ref. [9] showed that this mixed NSB and X_A breaking scheme can be derived rigorously by supplementing the non–anomalous HLS Lagrangian with a piece coming from the ChPT Lagrangian, $\mathcal{L}_1$ [10,11].

In this context, the BKY SU(3) breaking in the V sector (referred to as X_V) acts only in leptonic decays. As the data description was already quite satisfactory [1,9], a possible NSB in the vector sector was not examined; indeed, it could be guessed that this would be beyond the accuracy of the available data, at least if NSB in the V sector shows up only at the coupling constant level, as inferred in Ref. [16].

Except for SU(3) breaking effects, the model in Ref. [1] recovers the structure exhibited long ago by O'Donnell [16]. Fundamental parameters common to Refs. [1,16] are two mixing angles. δ_P has a clear origin [17] and is essentially generated by SU(3) breaking in the PS sector. Ref. [9] has shown that the traditional single PS mixing angle θ_P is algebraically related with the ChPT angle [10,11] by $\theta_8 \simeq \theta_P/2$, at least at leading order in breaking parameters.

Another fundamental ingredient of this model is an angle, δ_V, which describes departures of the ω/ϕ system from ideal mixing by bringing some strange quark content into the ω meson and, conversely, some non–strange component into the ϕ meson. This angle was already a basic ingredient in the axiomatic model of O'Donnell [16]. Its origin within effective Lagrangians has not yet been worked out throroughly, even if it is clearly connected with $\omega \leftrightarrow \phi$ transitions [8,18] inherent to VMD models like the HLS model [2].

Therefore, it looks relevant to study in some details how $\omega \leftrightarrow \phi$ transitions in the HLS model generate mixing of the ideal combinations into the physical ω and ϕ; whether this correspondence is stricly speaking a rotation, as commonly assumed [16,1], or a more complicated transformation is also an interesting issue. It is also interesting to explore the origin of such a transformation, and see whether this could be attributable to some sort of NSB in the V sector.

We address these topics using the method of effective Lagrangians, as higher orders in the HLS model would meet the problem of its renormalizability, which is not our concern. At each step, however, we have examined analyticity properties in connection with what can be inferred from the Analytic S–matrix Theory [19]. Finally, we shall comment on the way VMD has to be broken [1] in order to account effectively for the observed ratio of yields $[K^{*0} \rightarrow K^0\gamma]/[K^{*\pm} \rightarrow K^{\pm}\gamma] \simeq 2$ which is naturally possible within other contexts [16,20].

The outline of the paper is as follows. In Section II we briefly remind the reader of the BKY SU(3) breaking scheme we use. In Section III, we define the effective Lagrangian piece which accounts for loop corrections; this is done essentially by relying on the results provided by the Schwinger–Dyson equation in the ω/ϕ sector. In Section IV, we show that the diagonalization of the vector mass term at one–loop order is well approximated by a simple rotation, albeit momentum dependent, for on–shell ω and ϕ mesons; we also discuss somewhat the concept of mass–shell for vector mesons. In Section V, we remind how NSB in the PS sector is generated; we also discuss some variants of a possible NSB in the V sector which are numerically studied. In Section VI, we recall a Lagrangian for the anomalous sector, equivalent to FKTUY when no NSB in the V sector exists; however, it allows one to recover exactly the O'Donnell model in its full generality. Section VII recalls the breaking procedure which permits to include both K^* decay modes within the set of successfully fitted radiative modes; we argue that this might imply a renormalization of the vector fields presently missing in the BKY breaking scheme.

Section VIII is devoted to the numerical analysis of our model of radiative and leptonic decays under various conditions, noticeably with the momentum–dependent δ_V which can be inferred from the HLS model. Effects of vector meson self–energies on vector meson masses are briefly exemplified in Section IX. The fit results obtained are commented on in Section X, while Section XI is devoted to conclusions. Finally, analytic expressions for some loops, vector meson self–energies and transition amplitudes, coupling constants in the general case, are gathered in the Appendices in order to make easier reading of the main text.

II. SU(3) BREAKING OF THE NON–ANOMALOUS HLS LAGRANGIAN

Let us denote by P and V the bare pseudoscalar and vector field matrices under the assumption of nonet symmetry. With the convention we use in this paper, they can be found in Ref. [1] (see Eqs. (6) and (9) there; notice that our ideal ϕ field, denoted ϕ_I, is $-s\overline{s}$). Let us denote by A the electromagnetic field and let $Q = \mathrm{Diag}(2/3, -1/3, -1/3)$ be the quark charge matrix.

Let us also denote by $X_A = \mathrm{Diag}(1, 1, \ell_A)$ and $X_V = \mathrm{Diag}(1, 1, \sqrt{\ell_V})$ the SU(3) breaking matrices in, respectively, the PS and V sectors following from the BKY mechanism [7,8]. The unbroken limit is obtained by stating $\ell_A = \ell_V = 1$. Then, the SU(3) broken HLS Lagrangian can be written [7,8]

$$\mathcal{L} = \mathcal{L}_A + a\mathcal{L}_V \tag{1}$$

where a is the standard HLS parameter [2]. Following the breaking scheme proposed in Ref. [8] (see Section III.D therein), we have

$$\left\{ \begin{array}{lll} \mathcal{L}_A & = & \mathrm{Tr}[\partial P X_A \partial P X_A + 2ie(PQA - AQP) X_A \partial P X_A] \\ & & \\ \mathcal{L}_V & = & \mathrm{Tr}[f_\pi^2((gV - eQA)X_V)^2 + i(gV - eQA)X_V(\partial P P - P \partial P)X_V] \end{array} \right. \tag{2}$$

where f_π is the pion decay constant (92.42 MeV [21]), g is the universal coupling strength of vector mesons and e is the unit electric charge. The corresponding Lagrangian can be found expanded[1] in Ref. [8] as Eq. (A5).

The properties of the kaon form factor at $s = 0$ imply [7,8] that[2] $\ell_A \equiv [f_K/f_\pi]^2$. This is quite interesting as it implies that SU(3) breaking in the PS sector does not result in additional free parameters ($\ell_A = 1.5$); this has been checked with experimental data in Ref. [1]. Likewise, ℓ_V is connected with the ratio of the vector meson Higgs–Kibble (HK) masses [7,8] $m_\phi^2/m_\omega^2 = m_\phi^2/m_\rho^2 = \ell_V$; practically, this relation is, however, less interesting than the previous one since it implies that the Lagrangian (HK) masses can be extracted reliably from data.

In order to restore the kinetic energy term of the PS mesons to its canonical form, the X_A breaking results into a field renormalization [7,8]

$$P_{ren} = X_A^{1/2} P X_A^{1/2} \tag{3}$$

On the other hand, the X_V breaking [7,8] results in a mass breaking in the vector meson

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[1] In Ref. [8], the fields named ω and ϕ correspond to what we call here ω_I and $-\phi_I$.

[2] We might use indifferently some notations related to each other [1,9] $\ell_A = z = 1/Z$ for backward compatibility.

sector, but not in renormalization[3] of the vector fields. After this breaking, the mass term of neutral vector mesons becomes [8]

$$\mathcal{M} = \frac{af_\pi^2 g^2}{2}[\rho^2 + \omega_I^2 + \ell_V \phi_I^2] \tag{4}$$

We define for further use $m_\rho^2 = ag^2 f_\pi^2$. It should be clearly stated here once more that these masses might have little to do with the observed peak positions as reported in Ref. [21]. This is due to the difficulty of estimating unambiguously effects of the real part of vector meson self–energies. This clearly follows from the form factor studies in Refs. [22,23] and will also be illustrated below. On the other hand, the definition itself of vector meson masses is not unambiguous, as exemplified in Refs. [18,24–26]; this question is discussed in Sections III and IX.

III. AN ANALYTIC APPROACH TO VMD AT ONE–LOOP ORDER

As soon as one considers non–leading effects produced by the various SU(3) breaking procedures, consistency implies to account also for other non–leading effects which proceed from the HLS Lagrangian itself, irrespective of any breaking procedure. In this Section, we examine the contributions originating from loops and show how they can be incorporated *effectively* into the HLS Lagrangian. The purpose is to define a coherent framework, able to allow a phenomenological study of experimental data.

First of all, even if non–leading, loops produce observable effects in measurable (and measured) quantities. An illustrative example is the pion form factor; in this case, the shape observed is essentially an effect of dressing the ρ^0 meson propagator, noticeably an effect of the pion loop. As clear from Refs. [4,22], the detailed structure of the pion form factor is perfectly reproduced by the pion loop, and the $\pi\pi$ phase shift too. So, a coherent phenomenological handling of data implies to account properly for these leading loop effects. If pion form factor studies [4,22] give a clear hint that loop effects in vector meson self–energies play an essential role, loop corrections to the $\rho\pi\pi$ vertex do not show up. In the following such kind of one–loop effects are neglected.

In the present work, we focus on loops generated by the non–anomalous HLS Lagrangian. Thus, wo do not consider loops generated by the anomalous VVP Lagrangian (VP or $P\gamma$) or double loop effects produced for instance by the VPPP anomalous Lagrangian. They do not change qualitatively the picture; their possible effects in the realm of decays will be commented on at the relevant places.

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[3]The Yang–Mills kinetic term is added to the HLS Lagrangian but does not follow from its construction. On the other hand, we shall see in Section VII that data might give a hint in favor of a renormalization of the vector fields. Any renormalization of these would obviously break the relation between ℓ_V and the vector meson masses.

A. Effects of Loops on Vector Meson Masses

Let us neglect all electromagnetic contributions from the HLS Lagrangian[4] and work at one–loop order; we also denote from now on ρ the ρ^0 field when there is no ambiguity. Amputated from its Lorentz tensor part, the dressed ρ propagator $D(s)$ is given by the Schwinger–Dyson equation [18]; we have at one loop (g^2) order

$$D^{-1}(s) = D_0^{-1}(s) - \Pi_{\rho\rho}(s) \tag{5}$$

where the inverse bare propagator is $D_0^{-1}(s) = s - m_\rho^2$ and $\Pi_{\rho\rho}(s)$ is the ρ meson self–energy which contains contributions from the pion and both kaon loops [18]. The ρ self–energy is given in Appendix D and can be explicitly constructed using formula from Appendix A. This can be *effectively* obtained from the HLS Lagrangian by adding a piece $\Pi_{\rho\rho}(s)\rho^2/2$ which turns out to modify the ρ mass term coefficient to $[m_\rho^2 + \Pi_{\rho\rho}(s)]/2$. This (effectively) modified Lagrangian $\mathcal{L}'(s)$ gives automatically the dressed ρ propagator requested by phenomenology and coincides with the solution to the Schwinger–Dyson equation. Below the two–pion threshold we have $\mathcal{L}'(s) = \mathcal{L}'^{\dagger}(s)$, for any real s. Above, the hermiticity condition is naturally extended to $\mathcal{L}'(s) = \mathcal{L}'^{\dagger}(s^*)$, where the symbol $*$ denotes complex conjugation; this property, known as hermitian analyticity [19,27], is indeed fulfilled by $\mathcal{L}'(s)$.

Therefore, using $\mathcal{L}'(s)$ turns out to include directly one–loop corrections into the HLS Lagrangian in a way consistent with the Schwinger–Dyson equation and in agreement with the analyticity assumption. The HLS Lagrangian can be *effectively* modified correspondingly in order to include also one–loop effects associated with charged ρ's and with all K^* fields. One should stress that these additional pieces *a priori* depend on the invariant mass squared s; they are given in Appendix D and can be constructed using the results in Appendices A and B.

As the considered fields are effective fields, this s dependence is not really an issue and should only reveal non–local effects due to the fact that vector fields are functions of more fundamental ones (quark and gluon degrees of freedom). Additionally, this s–dependence is a natural feature at hadron level; it tells that the mass of a resonance does not fulfill $s - m_V^2 = 0$ any longer, but $s - m_V^2 - \Pi_{VV}(s) = 0$. In the case of the ρ and K^* mesons, this proves that the physical mass is shifted into the complex s–plane and that the connection between the HK mass m_V^2 and the real part of the pole is somewhat lost (see Ref. [18] for the ρ meson).

B. The special case of the ω and ϕ mesons

In the ω_I/ϕ_I sector, the situation is slightly more complicated. In this case, the Schwinger–Dyson equation is the 2×2 matrix equation

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[4]Within the VMD framework, the photon has a special status. Indeed, the Lagrangian expressed in terms of physical fields has a quadratic term which contains mixed terms $\gamma\rho^0$, $\gamma\omega$, $\gamma\phi$. It is the essence of VMD to keep these transition terms. For a simultaneous handling of ρ^0 and photon within the HLS model, see Ref. [18].

$$D^{-1}(s) = \begin{pmatrix} s - m^2_{\omega_I} & 0 \\ & \\ 0 & s - m^2_{\phi_I} \end{pmatrix} - \begin{pmatrix} \Pi_{\omega_I\omega_I}(s) & \Pi_{\omega_I\phi_I}(s) \\ & \\ \Pi_{\phi_I\omega_I}(s) & \Pi_{\phi_I\phi_I}(s) \end{pmatrix} \tag{6}$$

which exhibits the dressing of the mass terms as for ρ and K^* mesons, but also a non–diagonal term. The pieces generating these loops at order g^2 are given by the following $V - PS$ interaction term from the $\mathcal{L}_V$ part of the HLS Lagrangian [8]

$$\begin{aligned} \mathcal{T}_1 = & -\frac{iag}{4} Z[\omega_I + \sqrt{2}\ell_V \phi_I] \left[K^+ \overset{\leftrightarrow}{\partial} K^- + K^0 \overset{\leftrightarrow}{\partial} \overline{K}^0 \right] \\ & -\frac{iag}{4} \rho [Z \left(K^+ \overset{\leftrightarrow}{\partial} K^- - K^0 \overset{\leftrightarrow}{\partial} \overline{K}^0 \right) + 2\pi^+ \overset{\leftrightarrow}{\partial} \pi^-] \end{aligned} \tag{7}$$

where we have renormalized the kaon fields according to Eq. (3) above ($Z = [f_\pi/f_K]^2$).

At first non–leading order, this term generates $K\overline{K}$ and $\pi^+\pi^-$ loops. Both $K\overline{K}$ loops generate the ω_I and ϕ_I self–energies $\Pi_{\omega_I\omega_I}(s)$ and $\Pi_{\phi_I\phi_I}(s)$. They also produce $\omega_I \leftrightarrow \phi_I$ transitions with a non–zero amplitude $\Pi_{\omega_I\phi_I}(s)$. The ρ self–energy $\Pi_{\rho\rho}(s)$ receives contributions from the pion loop and from both kaon loops, as stated above. However, as SU(2) symmetry is assumed, there is no $\rho \leftrightarrow \omega_I$ or $\rho \leftrightarrow \phi_I$ transition loops. Indeed, in these cases, charged and neutral kaon loops exactly compensate [18]. This is why we have decoupled the ρ^0 channel while writing Eq. (6).

In contrast with ρ and K^* fields which remain mass eigenstates with only one–loop modified masses, Eq. (6) shows that ω_I and ϕ_I are no longer mass eigenstates at one–loop order. The physical masses are (complex) solutions of $\mathrm{Det}[D^{-1}(s)] = 0$. This equation can also be written $[s - \lambda_\omega(s)][s - \lambda_\phi(s)] = 0$, in terms of the eigenvalues of

$$M^2 = \begin{pmatrix} \Pi_{\omega_I\omega_I}(s) + m^2_{\omega_I} & \Pi_{\omega_I\phi_I}(s) \\ & \\ \Pi_{\phi_I\omega_I}(s) & \Pi_{\phi_I\phi_I}(s) + m^2_{\phi_I} \end{pmatrix} \tag{8}$$

They depart from the HK masses ($\lambda_\omega = m^2_{\omega_I}$ and $\lambda_\phi = m^2_{\phi_I}$) because of loops and become running. The momentum–dependent analytic matrix[5]

$$G(\delta_V) = \begin{pmatrix} \cos\delta_V & \sin\delta_V \\ & \\ -\sin\delta_V & \cos\delta_V \end{pmatrix} \tag{9}$$

with

$$\tan 2\delta_V(s) = \frac{2\Pi_{\omega_I\phi_I}(s)}{m^2_\rho(1-\ell_V) + (\Pi_{\omega_I\omega_I}(s) - \Pi_{\phi_I\phi_I}(s))} \tag{10}$$

[5]The matrix $G(\delta_V)$ can be handled as if it were actually an orthogonal matrix, even for complex values of s or for real s above $4m^2_K$. Here and in the following, trigonometric functions should be understood as their underlying exponential expressions. Then, all usual trigonometric relations apply, even for complex arguments, reminding that $\cos iu = \cosh u$, $\sin iu = i \sinh u$, etc. . .

diagonalizes Eq. (6). The first term in the denominator is actually $m^2_{\omega_I} - m^2_{\phi_I}$, in terms of the HK masses coming out from the BKY broken HLS Lagrangian [8]. Indeed, we have

$$D_G^{-1} = [GDG^{-1}]^{-1} = sI - GM^2G^{-1} = sI - \begin{pmatrix} \lambda_\omega(s) & 0 \\ & \\ 0 & \lambda_\phi(s) \end{pmatrix} \tag{11}$$

(we have always $G^{-1}(\delta_V) = G(-\delta_V)$ as if G were always an actual orthogonal matrix), and the eigenvalues

$$\begin{aligned} \lambda_\omega(s) &= [m^2_{\omega_I} + \Pi_{\omega_I\omega_I}(s)]\cos^2\delta_V + [m^2_{\phi_I} + \Pi_{\phi_I\phi_I}(s)]\sin^2\delta_V + \Pi_{\omega_I\phi_I}(s)\sin 2\delta_V \\ \lambda_\phi(s) &= [m^2_{\phi_I} + \Pi_{\phi_I\phi_I}(s)]\cos^2\delta_V + [m^2_{\omega_I} + \Pi_{\omega_I\omega_I}(s)]\sin^2\delta_V - \Pi_{\omega_I\phi_I}(s)\sin 2\delta_V \end{aligned} \tag{12}$$

are the running squared masses associated with the physical eigenstates of the HLS Lagrangian at one–loop order. In scattering processes the corresponding dressed propagators are $D_\omega^{-1}(s) = s - \lambda_\omega(s)$ and $D_\phi^{-1}(s) = s - \lambda_\phi(s)$.

The "trigonometric" functions entering Eqs. (12), can be expressed easily in terms of the right–hand side of Eq. (10) using

$$\cos 2\delta_V(s) = \frac{1}{(1+\tan^2 2\delta_V(s))^{1/2}} \quad , \ \sin 2\delta_V(s) = \frac{\tan 2\delta_V(s)}{(1+\tan^2 2\delta_V(s))^{1/2}} \tag{13}$$

A detailed study of the singularities of the eigenvalues and eigenvectors as real–analytic functions is beyond the scope of this paper. Let us only mention that renormalization conditions (see below) on self–energies might have to compensate the simple zeros of $1 + \tan^2 2\delta_V(s)$ located in the physical sheet of the scattering amplitudes. On the other hand, $\mathrm{Det}[D^{-1}(s)]$ is tightly connected with the D function of the former N/D formalism [19,27,28] and has certainly all desired analyticity properties.

The *physical* ω mass is solution of $s - \lambda_\omega(s) = 0$, while the *physical* ϕ mass is separately solution of $s - \lambda_\phi(s) = 0$. Physical ω and ϕ fields correspond to these mass eigenstates; they are obviously associated with the eigenvectors of the matrix M^2 of Eq. (8). Each physical field can be identified by inspecting the behavior of the eigensolutions at $\delta_V \to 0$. This identification is meaningful as the effective value of δ_V is small, a few degrees only [1].

C. Two Regimes For The $\omega - \phi$ Mixing

It is easy to check that all features described in the two previous Subsections can be obtained by adding the following *effective* piece to the HLS Lagrangian

$$\begin{aligned} \mathcal{L}_{\rm loops}(s) = \ & \frac{1}{2}[\Pi_{\rho\rho}(s)\rho^2 + \Pi_{\omega_I\omega_I}(s)\ \omega_I^2 + \Pi_{\phi_I\phi_I}(s)\ \phi_I^2 + 2\Pi_{\omega_I\phi_I}(s)\ \omega_I\phi_I] \\ & + [\Pi_{\rho\rho}(s)\rho^+\rho^- + \Pi_{K^*K^*}(s)(K^{*+}K^{*-} + K^{*0}\overline{K}^{*0})] \end{aligned} \tag{14}$$

All needed loops and self–energies are gathered in Appendices A–D; couplings constants can be identified by examining the Lagrangian in Eq. (A5) of Ref. [8] and are not listed here. They should be modified according to field renormalization procedures. Therefore, the use of $\mathcal{L}_{\rm loops}(s)$ turns out to account directly for one–loop effects into the propagators, in a way consistent with the Schwinger–Dyson equation. This Lagrangian piece fulfills the hermitian analyticity condition [19] $\mathcal{L}_{\rm loops}(s) = \mathcal{L}^{\dagger}_{\rm loops}(s^*)$.

In the approximation where electromagnetic contributions and anomalous Lagrangian terms [3] are neglected, all loops involved in the ω/ϕ sector are basically the $K\overline{K}$ loop only. Let us denote by $\Pi(s)$ the $K\overline{K}$ loop amputated from the coupling constants to vector mesons and state $\Pi_{VV'} = g_{VK\overline{K}} g_{V'K\overline{K}} \Pi(s)$ for any pair of vector mesons (V, V') coupling to kaon pairs.

There are clearly two regimes for the $\omega - \phi$ mixing, depending on the value of s involved. As clear from Appendix A, $\Pi(s)$ is real for real s such that $s \leq 4m_K^2$ and acquires an imaginary part when $s \geq 4m_K^2$. The $\omega_I - \phi_I$ transitions affect the dressed propagators already at one–loop order, as illustrated above. However, the same transition loops affect also ω_I and ϕ_I as external legs. The diagonalization of the full mass term

$$\mathcal{M}(s) = \frac{a f_\pi^2 g^2}{2} [\omega_I^2 + \ell_V \phi_I^2] + \frac{1}{2} [\Pi_{\omega_I\omega_I}(s)\ \omega_I^2 + \Pi_{\phi_I\phi_I}(s)\ \phi_I^2 + 2\Pi_{\omega_I\phi_I}(s)\ \omega_I\phi_I] \tag{15}$$

of the Lagrangian $\mathcal{L}_{\rm loop} + \mathcal{L}_{HLS}$, defines the effective physical fields and also diagonalizes the Schwinger–Dyson equation. The physical ω and ϕ fields have running squared masses $\lambda_\omega(s)$ and $\lambda_\phi(s)$ respectively.

For real $s \leq 4m_K^2$, $G(\delta_V)$ is indeed a (mass–dependent) rotation while above it is not. However, the transformation G is still valid and fulfills $G^{-1}(\delta_V) = G(-\delta_V)$. $\delta_V(s)$ can always be split up into its real and imaginary parts $\delta_V(s) = \alpha_V(s) + i\beta_V(s)$ which are no longer (separately) analytic functions; $\alpha_V(s)$ and $\beta_V(s)$ can be derived from Eq. (10).

The mass matrix M^2 is rendered diagonal by physical fields which are combinations of the ideal states. It is easy to check that these effective physical fields are

$$\begin{pmatrix} \omega \\ \\ \phi \end{pmatrix} = \begin{pmatrix} \cos\delta_V & \sin\delta_V \\ \\ -\sin\delta_V & \cos\delta_V \end{pmatrix} \begin{pmatrix} \omega^I \\ \\ \phi^I \end{pmatrix} \tag{16}$$

With this definition of physical fields, the mass term Eq. (15) of the modified HLS Lagrangian becomes canonical[6]

$$\mathcal{M}(s) = \frac{1}{2} [\lambda_\omega(s) \omega^2 + \lambda_\phi(s) \phi^2] \tag{17}$$

and fulfills $\mathcal{M}(s) = \mathcal{M}^{\dagger}(s^*)$. It is worth noting that, in the effective approach we follow, the physical ω and ϕ fields behave like real–analytic functions of s and are associated with running masses which are also real–analytic functions of s.

Eq. (16) is valid for any value of s. For complex s or for real s above $4m_K^2$, this can be written more traditionally

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[6] The s dependence of the fields is understood. One could write this expression in a more symmetric way, taking into account that $\omega(s) \equiv \omega^{\dagger}(s^*)$ and $\phi(s) \equiv \phi^{\dagger}(s^*)$.

$$\begin{pmatrix} \omega \\ \\ \phi \end{pmatrix} = \begin{pmatrix} \cosh\beta_V & i\sinh\beta_V \\ \\ -i\sinh\beta_V & \cosh\beta_V \end{pmatrix} \begin{pmatrix} \cos\alpha_V & \sin\alpha_V \\ \\ -\sin\alpha_V & \cos\alpha_V \end{pmatrix} \begin{pmatrix} \omega^I \\ \\ \phi^I \end{pmatrix} \tag{18}$$

in terms of the real and imaginary parts of δ_V. For real s below $4m_K^2$, β_V vanishes and $G(\delta_V)$ is a rotation of angle as given by Eq. (10). This can also be written

$$\tan 2\delta_V(s) = \frac{2\sqrt{2}aZ^2\ell_V\Pi(s)}{8f_\pi^2(1-\ell_V)+aZ^2(1-2\ell_V^2)\Pi(s)} \tag{19}$$

in terms of the basic HLS parameters from the broken Lagrangian. $\Pi(s)$ is the generic kaon loop already defined and depends on a subtraction polynomial $P(s)$ (see Appendices A and D) discussed in the following Subsection. Departures of the $\omega-\phi$ mixing from a pure rotation are exhibited in Eq. (18) and will be discussed below.

D. Renormalization Conditions on Loops

The self–energies we have defined are each given by a dispersion relation on the imaginary part of a loop function; they are analytic functions of s defined on a 2–sheeted Riemann surface with a second order branch point at threshold. This implies some analytic properties for propagators [19,27] which are not examined here; let us only mention that physical meson masses correspond to propagator poles in the unphysical sheet of a 2–sheeted Riemann surface onto which amplitudes are defined[7].

Integrability conditions on the dispersion integral imply that the dispersion relation should be at least twice subtracted, which gives rise to a subtraction polynomial $P(s)$ of (at least) first degree with (at least) two unknown coefficients which have to be fixed by stating some renormalization conditions: $P(s)=d_0+d_1s$. The minimum number of subtractions is mandatory in order for the integral to make sense; however, the *actual* number of subtractions can be larger and depends on the assumed behavior at infinity of the function considered. If one performs k subtractions, giving rise to a polynomial of degree $k-1$, it is useful (but not mandatory) to require $\lim_{s\to 0}[\Pi(s)-P(s)] \simeq \mathcal{O}(s^k)$.

It is quite usual for all meson self–energies to fix the constant terms d_0 of the polynomials $P(s)$ to zero, in order to ensure that the photon remains massless [22]. This condition is more stringent than needed, as masslessness of the photon implies only that some combination of the d_0's is zero[8]. However, assuming that the d_0 parameter associated with each possible

[7]Roughly speaking, poles associated with a resonance come by pairs; specializing to 2–body decay channels as in the HLS model, the pole effectively associated with the resonance is located at some s_R in the unphysical sheet close to the physical region, while there exists a so–called shadow pole [19] located at s_R^* in the same Riemann sheet but generally far from the physical region. This is not necessarily true if resonance poles are located close to some threshold.

[8]Each relevant loop gives rise to a polynomial $P(s)$ and all such polynomials are different from each other.

loop is zero implies that each loop fulfills $\lim_{s\to 0}\Pi_{VV'}=0$. This condition has the virtue to ensure that each (running) vector meson mass coincides with its HK mass value at the chiral point. This is a strong condition which can be assumed, at least provisionally, because of its aesthetic character.

On the other hand, the renormalization condition $d_0=0$ is also appropriate for the transition loop $\Pi_{\omega_I\phi_I}(s)$. Indeed, the condition $\Pi_{\omega_I\phi_I}(s=0)=0$ allows to derive the amplitudes [1,9] for $P\gamma\gamma$ processes from the FKTUY Lagrangian [3] PVV by making $\delta_V(s=0)=0$. Stated otherwise, internal ω or ϕ lines computed at $s=0$ coincides with ω_I or ϕ_I respectively. This was explicitly noted in Ref. [1] (see last paragraph of Section V here). In the mass region of vector resonances, the condition $\delta_V=0$ is no longer fulfilled for obvious reasons (see Eq. (19)).

Therefore, the s–dependence of δ_V is a fundamental property as it allows to make consistent a zero value for mixing angle at $s=0$ with non–zero values in the mass region of vector resonances.

Concerning the ω/ϕ mixing which is presently our main concern, it remains to fix the rest of the polynomial $P(s)$. As there is no clear and unambiguous statement about the remaining coefficients, it follows that the phase shift $\delta_V(s)$ is somewhat free, or that its value can be used as additional renormalization condition. As explicit in Eq. (18), we have assumed $\Pi_{VV'}=g_{VK\overline{K}}g_{V'K\overline{K}}\Pi(s)$; this is actually a strong assumption which serves only to lessen the number of free parameters (or of needed renormalization conditions) in physical problems, as the subtraction polynomials (not the rest!) might be not related[9] by simple rescaling by the appropriate product of coupling constants. This point is commented on slightly more in Appendix D.

IV. VECTOR MESON MASS–SHELL AND ON–SHELL $\omega-\phi$ MIXING

Particle mass–shell is a well defined concept for objects like PS mesons. For others, like vector mesons, this concept is somewhat more embarrassing. Indeed, one can choose to define it as zero of the real part of the inverse propagator of this particle; this is recommended by the Particle Data Group [21] but not free from ambiguities, depending on coupled channels with thresholds above the vector particle mass. The HK mass of a vector meson, even if the most relevant from the point of view of effective Lagrangians, is somewhat indirect as it does not correspond directly to a measured quantity. It will be seen below that the HK masses of ω and ϕ mesons are both below the two–kaon thresholds.

From the rigorous point of view of S–matrix Theory [19], the relevant mass–shell concept is rather the pole location of the propagator. Focusing on the ω/ϕ sector, we have seen that the mass squared involves solutions to an eigenvalue problem and should satisfy $s-\lambda_\omega(s)=0$ and $s-\lambda_\phi(s)=0$. So the pole is certainly located in the complex plane. The issue is basically the same for the ρ meson [18] or the K^*'s, and is quite general [26].

There is no reported [21] information on the pole position for the ω and ϕ mesons. However, using the reported masses (M_V) and widths (Γ_V) [21], one can state $s_V=M_V^2-$

[9] If indeed related, the proof is anyway not completely straightforward.

$iM_V\Gamma_V$ with some unknown precision. We thus can guess that the poles for the ω and ϕ mesons are very close to the physical region and, additionally, that the real part of the ϕ pole is very close to the 2–kaon thresholds.

Neglecting its imaginary part compared to its real part, the ω on–shell pole is certainly much below the 2–kaon thresholds. Therefore, it is appropriate to state $\delta_\omega(s_\omega) \simeq \delta_\omega(M_\omega^2)$, and get a real phase.

In the same approximation, the ϕ mass is however, slightly above the 2–kaon thresholds and then in a region where $\delta_\phi(s_\phi) \simeq \delta_\phi(M_\phi^2)$ carries a tiny imaginary part. It is quite instructive to inspect the loop function $\Pi(s)$ as given in Eq. (A8) from Appendix A (with $s_0 = 4m_K^2$) and look at its behavior in the vicinity of the observed ϕ mass. The logarithm and the imaginary part are of order $\simeq 0.05$ GeV^2 compared to the polynomial part ($\simeq 0.3$ GeV^2). Therefore, it is *a priori* justified to neglect the imaginary part of the loop function while staying at the ϕ mass–shell; somewhat farther beyond this point, this statement would have surely to be revisited.

Therefore, even if somewhat accidental[10], the matrix $G(\delta_V)$ which gives the ϕ and ω eigenstates is indeed close to a pure rotation matrix when using the pole definition for the mass–shell. Anticipating our results, we have checked that the subtraction polynomial which is fitted in radiative and leptonic decays does not change this picture. If one rather defines the mass–shell through the HK masses, $G(\delta_V)$ is a pure rotation matrix.

Additionally, we have also neglected in the present approach other e^2 contributions like the loops $\pi^0\gamma$, $\eta\gamma$, $\eta'\gamma$ which follow from the FKTUY Lagrangian, as given in Ref. [1]. These contribute to provide small imaginary parts (of order e^2) to the self–energies we consider for $s \geq m_{\pi^0}^2$, $s \geq m_\eta^2$, $s \geq m_{\eta'}^2$. Most VP loops neglected only contribute to self-energies strictly speaking and we shall argue below why their influence is small. Inspecting the anomalous VVP Lagrangian (see Eqs. (A8) or (A14) in Ref. [1]), the neglected contributions to $\Pi_{\omega_I\phi_I}(s)$ are K^*K loops which have HK threshold masses above the ϕ meson; thus, they would contribute as real quantities in our mass region of interest. We shall argue below that such effects are practically harmless, essentially because the mixing function is really slowly varying in the mass region involved in meson decays.

Some of these neglected loops can be computed in closed form (see Appendix C). We have finally neglected two–loop effects produced from the FKTUY Lagrangian couplings and double loop effects produced by the $VPPP$ FKTUY Lagrangian [3]. Thus, concerning $\Pi_{\omega_I\phi_I}(s)$, the most important neglected effects are either two–loops or of order e^2.

So, whatever the mass–shell definition, when working with on–shell ω and ϕ mesons, the transformation from ideal to physical fields is very close to a rotation and, then, making this approximation is justified. However, as a consequence of VMD at one–loop, $\delta_V(s)$ is s–dependent. This is practically equivalent to having *two* different mixing angles $\delta_\omega = \delta_V(m_\omega^2)$, $\delta_\phi = \delta_V(m_\phi^2)$ functionally related. The question to which extent these two angles differ is addressed below; however, the statistical quality of the fits in Ref. [1] with only one such mixing angle allows to infer that they are probably close together. On a related topic, let us recall that a momentum–dependent mixing angle for pseudoscalar mesons has been recently considered [29].

[10]Small widths and/or small imaginary part for the kaon loop are clearly accidental features.

For practical purposes in fit procedures, we have preferred computing the phase $\delta_V(s)$ as the HK masses of the ω and ϕ mesons[11]. This sets the ω mass close to its PDG value and the ϕ mass slightly below the 2–kaon thresholds. As these HK masses are real, the matrix G is indeed orthogonal.

When dealing with scattering processes, *i.e.* over a broad range for s extending possibly far above $s = 4m_K^2$, the running of the mixing angle $\delta_V(s)$ is a feature which should play some role.

V. NONET SYMMETRY BREAKING (NSB) IN THE HLS MODEL

In Ref. [9], it was shown that the HLS Lagrangian can undergo NSB in the PS sector by adding to Eq.(1) the following term

$$\mathcal{L}' = \frac{1}{2}[\mu^2\eta_0^2 + \lambda\partial_\mu\eta_0\partial^\mu\eta_0] \tag{20}$$

where η_0 denote the singlet PS field. Such contributions can be inferred from Chiral Perturbation Theory [10,11] (ChPT).

At the level of radiative decays, the additional kinetic energy term implies to modify the renormalization condition Eq. (3) to [9]

$$P = X_A^{-1/2}(P_8^{ren} + xP_1^{ren})X_A^{-1/2} \tag{21}$$

using obvious notations, and by defining the PS NSB parameter $x = 1/\sqrt{1+\lambda}$. Then exact nonet symmetry is defined by $x = 1$. In order to perform both SU(3) breaking and NSB in the PS sector, we refer here to the change of fields in Eq. (21), which diagonalizes the PS kinetic energy in $\mathcal{L} + \mathcal{L}'$ at first order. It has been shown that this fits [9] perfectly all related information from ChPT [10–12] concerning the η/η' sector at first order in the breaking parameters. One can then consider quite reliable the full PS breaking scheme represented by Eq. (21).

In the vector sector, the way to introduce NSB is unclear. It is even not clear whether it can be done in full accordance with the conceptual framework of the HLS model. One could imagine that an appropriate breaking term of the HLS Lagrangian would be an additional singlet mass term

$$\mathcal{L}'' = \mu_0^2(\sqrt{2}\omega_I - \phi_I)^2 \tag{22}$$

[11] Actually, the situation for the ϕ meson is even more uncertain than sketched above; within the HLS model, and pushing aside electromagnetic interactions, $K\overline{K}$ scattering in ($I = 0$, $l = 1$)–wave is a 2–channel problem with thresholds at $K^0\overline{K}^0$ and K^+K^-; the domain of definition is a 4–sheeted Riemann surface. Even if of electromagnetic origin, the distance of the thresholds ($\simeq 8$ MeV) is not negligible compared to the distance of the PDG mass for the ϕ to the $K^0\overline{K}^0$ threshold ($\simeq 24$ MeV). Therefore, branch points and poles are gathered in a tiny neighborhood; this makes the local topology of the 4–sheeted Riemann surface influencing and, at the level of a few MeV's, the physical ϕ pole position cannot be guessed reliably from its reported mass [21].

and/or a change of field of the form

$$V = V_8^{ren} + yV_1^{ren} \tag{23}$$

which corresponds to having g as coupling constant for the vector octet matrix and yg for the singlet one. Departures from $y = 1$ would then flag NSB in the vector sector.

However, if one focuses on deriving $P\gamma\gamma$ amplitudes from the VVP Lagrangian, as done in Ref. [1], the difficulty is that this derivation is impossible – or at least not obvious – except if one assumes that μ_0^2 and y are running and that $\mu_0^2(s = 0) = 0$ and $y(s = 0) = 1$. Stated otherwise: the above mentioned derivation is (trivially) possible only if vector NSB vanishes at the chiral point. It should be noted that both X_A and X_V breakings on the one hand [7,8] and the PS NSB breaking [1] mentioned above on the other hand, do not require such a condition within the HLS framework. This has been checked explicitly in [1]. However, the connection between $PV\gamma$ and PVV amplitudes, named Extended VMD in [3], is an extension of the usual VMD assumption which might have to be reconsidered, if motivated; it is worth mentioning that Ref. [3] does not recommend the possible further extension of the VMD assumption to box anomalies (γPPP).

Considering the HLS model as an effective model, one can, nevertheless, investigate the question of NSB in the V sector to see whether some support can be found in experimental data.

A term like Eq. (22) results essentially in an additional contribution to the phase shift δ_V and is hard to disentangle from the loop effects which naturally follow from the HLS Lagrangian at first non–leading order. It could only break the relation between the phase shift δ_V and the loop functions, allowing thus for more freedom in fits.

A breaking term like Eq. (23) can be generated (in a presently unknown way) by breaking the vector Yang–Mills term which is beyond the scope of the HLS model[12]. What is important to note is that, whatever this unknown procedure, it should summarize into a relation like Eq. (23) at leading order in breaking parameters.

We limit ourselves to examining the couple of vector NSB mechanisms given by Eqs. (22) and (23).

VI. RADIATIVE DECAYS OF LIGHT MESONS

One can construct axiomatically the decay amplitudes for the processes $PV\gamma$ assuming $SU(3) \times U(1)$ group structure for PS and V mesons. It has been done in the classic paper of O'Donnell [16] (see also the Appendix in [30] where some misprints have been corrected). It can be checked easily that the coupling constants $g_{VP\gamma}$ of [16] can be derived from the $VP\gamma$ term of the following anomalous Lagrangian

$$\mathcal{L}_{WZW} = K\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left[\partial_\mu(eQA_\nu + gV_\nu)\partial_\rho(eQA_\sigma + gV_\sigma)P\right] \tag{24}$$

[12]We mean that some kind of relation should then exist for y, in correspondence with the one relating the PS NSB parameter x and the (basic) kinetic energy breaking parameter λ recalled above.

using Eqs. (21) and (23) above (with $\ell_A = 1$ in order to limit oneself to NSB only). The coefficient $K = -3/(4\pi^2 f_\pi)$ is a normalization fixed by requiring the coupling constant for $\pi^0\gamma\gamma$ to be the usual one. Eq. (24) gives the connection between coupling constants for $P\gamma\gamma$, $PV\gamma$ and PVV processes as expected from the Extended VMD assumption of Ref. [3].

The notations in the present work (or in Ref. [1]) and in Ref. [16] are connected by $g_{V_8P_8\gamma} = G = -3eg/(8\pi^2 f_\pi)$, $g_{V_8P_1\gamma} = xG$ and $g_{V_1P_8\gamma} = yG$. Therefore, Eq. (23) holds whatever is the precise underlying relation between y and more basic (and unknown) vector NSB parameters.

This is quite an interesting pattern. Indeed, if vector NSB is only hidden inside δ_V, it seems beyond any unambiguous phenomenological evidence. However, if it affects *also* separately the vector coupling constant by a factor y, then departures from $y = 1$ can be explored.

Moreover, applying Eq. (24) by assuming additionally $\ell_A \neq 1$ is legitimate. Indeed, as recalled above, Ref. [9] has shown that the X_A breaking scheme [7,8] is in accord with all accessible predictions of ChPT [10–12].

The coupling constants for radiative decays have been computed starting from Eq. (24) assuming NSB and SU(3) breaking in both sectors. Additionally, vector meson decays to lepton pairs have been computed assuming $y \neq 1$; in this case, corrections are quite negligible but might be considered in order to be complete. All formulae of relevance are given in Appendix E. In the limit $y = 1$, they coincide with the corresponding expressions derived in Ref. [1] starting from the FKTUY Lagrangian. In the general case where y could differ from unity, Eq. (24) can be used as as alternative way to express the Extended VMD assumption.

We also work in the framework of the so–called K^* model commented on below. It introduces an additional breaking procedure [1] in order to account for the observed K^* radiative decay rates and a dimensionless breaking parameter ℓ_T fit as [1] $\ell_T = 1.19 \pm 0.06$.

VII. COMMENTS ABOUT THE K^* MODEL

Referring to [1] for details, it has been shown that the above described breaking mechanisms (V and PS NSB's, $X_{A,V}$ breakings) altogether account for all leptonic and radiative decays, except for $K^{*\pm} \to K^\pm\gamma$. More precisely, no way has been found to allow for the observed ratio of yields $[K^{*0} \to K^0\gamma]/[K^{*\pm} \to K^\pm\gamma] \simeq 2$; it cannot be else than $\simeq 4$ within the approach of O'Donnell [16] or starting from Eq. (24) supplemented with the breaking procedures already described. Quite interestingly, the non–relativistic quark model (NRQM) [16] allows for more freedom, depending on the ratio of the quark magnetic moments $r = \mu_s/\mu_d$

$$\frac{G_{K^{*0}K^0\gamma}}{G_{K^{*\pm}K^\pm\gamma}} = -\frac{1+r}{2-r} \tag{25}$$

More recently, motivated by the surprisingly large success of NRQM, G. Morpurgo has shown that the NRQM predictions K^* radiative decay coupling constants [20] are valid in low energy QCD, provided one assumes that gluonic contributions are negligible in this energy range. This property plays also some role in leptonic decays [32]. Therefore, it is of concern to see how a relation as appropriate as Eq. (25) could be derived within the VMD framework.

In order to account for the observed K^* relative rate, Ref. [1] tried first introducing the breaking procedure (named there X_W breaking) proposed by Bramon, Grau and Pancheri [31]. This turned out to introduce another breaking matrix $X_W = \text{Diag}(1,1,1+c_W)$ in the FKTUY Lagrangian [3]; this can be symbolically written $\text{Tr}[X_W VVP]$. However, the degree of algebraic correlations introduced by SU(3) among all single photon radiative decay modes is such that all fits return $X_W = 1$ and do not allow for any improvement concerning the observed ratio $[K^{*0} \to K^0\gamma]/[K^{*\pm} \to K^\pm\gamma] \simeq 2$.

In order to change this picture, Ref. [1] proposed to perform the replacement $V \Rightarrow X_T V X_T$ in addition to the BGP breaking; X_T is assumed to have the classical SU(3) breaking structure $X_T = \text{Diag}(1,1,1+c_T)$. In this case, one succeeds in fitting both observed K^* radiative decay modes. Additionally, it was phenomenologically observed [1] that $X_W X_T^4 = 1$ is well fulfilled by the full set of radiative decays; this makes all coupling constants independent of the $X_W - X_T$ breaking, except for the K^* modes which become

$$\left\{\begin{array}{lll} G_{K^{*0}K^0\gamma} = & - & G' \dfrac{\sqrt{\ell_T}}{3}(1+\dfrac{1}{\ell_T}) \\ G_{K^{*\pm}K^\pm\gamma} = & & G' \dfrac{\sqrt{\ell_T}}{3}(2-\dfrac{1}{\ell_T}) \end{array}\right. \tag{26}$$

where $G' = -3eg/(8\pi^2 f_K)$ and $\ell_T = (1+c_T)^2$. The ratio which can be derived from Eqs. (26) is in obvious correspondence with Eq. (25) and allows to identify $r \leftrightarrow 1/\ell_T$. To our knowledge, the mechanism proposed in Ref. [1], and just sketched, is the single one able to reproduce the NRQM–Morpurgo relation for the K^* decay rates in a VMD framework.

The change $V \Rightarrow X_T V X_T$ is in clear correspondence with the change of (PS) fields imposed by the BKY X_A breaking mechanism (see Eq. (3) above). It resembles what could be a vector field renormalization, presently lacking within the BKY breaking framework [7,8]. If this were so, the K^* sector is the single one where this effect can be unambiguously visible. Whether it is possible to derive it rigorously within the HLS framework deserves some effort which should be supported by new data confirming the reported level for the $K^{*\pm}$ radiative decay rate.

It is worth mentioning the following remarks

- The phenomenological observation [1] $X_W X_T^4 = 1$ can be interpreted if Eq. (24) could be rewritten when the $X_W - X_T$ breaking mechanism is at work

$$\mathcal{L}'_{WZW} = K\epsilon^{\mu\nu\rho\sigma}\text{Tr}\left\{X_W\,[X_T\partial_\mu(eQA_\nu+gV_\nu)X_T]\,[X_T\partial_\rho(eQA_\sigma+gV_\sigma)X_T]\;P\right\} \tag{27}$$

  In order that the A^2 term still gives the 2–photon decay amplitudes predicted by the original WZW Lagrangian [13], $X_W X_T^4 = 1$ becomes indeed a necessary condition.

- If the replacement $V \Rightarrow X_T V X_T$ is indeed the renormalization condition for the vector fields, then ℓ_V in the standard formulae (given in Appendix E or in Ref. [1]) actually hides as many powers of ℓ_T as the number of ϕ_I fields the term involves. For leptonic decays ℓ_V should then be understood as $\ell_V\ell_T$. If the above replacement were theoretically motivated, it would indicate that the ϕ HK mass squared is not [7,8] $m_\rho^2\ell_V$ but rather $m_\rho^2\ell_V\ell_T^2$; this enforces our standpoint that the relevance of any *a priori* value for the true ℓ_V is pending and might have to wait for a final answer concerning vector field renormalization in the HLS–BKY framework.

- If one relies on the derivation by Morpurgo of Eq. (25), gluonic contributions should be suppressed. Therefore, as Eq. (26) is experimentally favored [1], one may infer that the K^* sector of radiative decays disfavors a significant interplay of vector gluonic singlet component in the mass region of vector resonances.

VIII. NUMERICAL ANALYSIS

In studies published elsewhere [9], it was shown that the PS mixing angle θ_P and the PS NSB parameter x were fulfilling

$$\tan\theta_P = \sqrt{2}\frac{1-z}{2+z}x \tag{28}$$

exceptionally well ($z = [f_K/f_\pi]^2$). This relation is a consequence of the small value of the decay constant F_η^0 recently defined in ChPT [10,11]. Quite interestingly, it projects onto the PS mixing angle θ_P all departures from nonet symmetry. From the point of view of numerical analysis, this also allows to reduce the number of free parameters by one unit without any change in the fit quality. In all fits referred to below, this condition has been either relaxed or requested. In all cases where fits returned a good probability this additional requirement has been found to leave the χ^2 unchanged; setting this condition mechanically improves the probability, as it turns out to increasing the number of degrees of freedom for exactly the same χ^2 value. The set of data submitted to fits are the 14 radiative decays[13] $VP\gamma$ and $P\gamma\gamma$ and the 3 leptonic decays $\rho/\omega/\phi \to e^+e^-$ taken from the Review of Particle Properties [21]; this represents 17 data points, *i.e.* the largest sample ever submitted successfully to a fit. All formulae used for fits can be found in Appendix E. We do not reproduce the reconstructed branching fractions as they are indistinguishable from the final results in Ref. [1] or from those in Table 2 of Ref. [9].

In order to explore the question of vector meson mixing, we have examined several analysis strategies.

1/ Approximating the right-hand–sides of Eq. (10) by a constant. This is nothing but the approach developed in Ref. [1] with one constant mixing angle δ_V. In this case we have either left y free or fixed it to 1.

- Setting $y = 1$ (no vector NSB), we get $\chi^2/dof = 9.14/11$ (61% probability) as in Ref. [9] after stating Eq. (28). From a statistical point of view, this result can be considered as optimum. The number of fit parameters is 6, out of which 4 are especially devoted to the 14 radiative decays (δ_V, δ_P, the universal vector coupling constant g and the parameter ℓ_T, specific to the K^* sector) and 2 (a, ℓ_V) concern solely the leptonic sector.

- Leaving y and δ_V free, we get $\chi^2/dof = 8.82/10$ (55 % probability); the χ^2 is practically unchanged but the probability is slightly degraded by having one more free parameter

[13] This counting leaves aside the process $\pi^0 \to \gamma\gamma$ which would only fix the value of the decay constant f_π already taken from [21] as for f_K.

($i.e.$ one less degree of freedom). In this case we get $y = 1.012 \pm 0.022$, a quite insignificant departure from no NSB in the V sector; the vector mixing angle $\theta_V = 31.57^\circ \pm 0.62^\circ$ is still found below ideal mixing, that is $\delta_V \simeq -3.70^\circ \pm 0.62^\circ$, quite significantly negative (about a 6 σ effect). This result is in agreement with the phase of Dillon and Morpurgo [32] which relies on only leptonic decays of vector mesons. Removing the leptonic decay modes from the fit leaves the χ^2 unchanged ($\chi^2/dof = 8.52/10$, 38% probability) and provides the same values for δ_V (in magnitude and sign) and for y.

- Finally, we have forced $\delta_V = 0$ and left y free, in order to see whether explicit vector NSB could alone do the work generally attributed [1,32] to angular departures from ideal mixing. In this case departure from nonet symmetry is fit as large in the V sector ($y = 0.893 \pm 0.005$) as in the PS sector ($x = 0.901 \pm 0.017$); however, the fit quality is unacceptably degraded ($\chi^2/dof = 41.81/11$, 2. 10^{-5} probability).

  Comments : Concerning the vector mixing angle, several conclusions follow from these fits. First, whatever its precise origin, the effects of angular departures from ideal mixing are fundamental; indeed, whatever the way used in order to circumvent it, the gain which can be attributed to it is of the order of 30 units in the χ^2, a quite significant effect for a single parameter. Moreover, our various fits show that radiative decays and leptonic decays carry *separately* the same information about the magnitude and sign of δ_V. The same sign information has been reached by the quite independent approach of Ref. [32] using only leptonic decays. We come back on this point in Section X.

  Explicit departures from nonet symmetry for vector mesons (y) are statistically insignificant (about 0.5 σ) and can clearly be ignored. Stated otherwise, if there is nonet symmetry breaking in the vector sector, it cannot be the manifest U(3) breaking à la O'Donnell [16], as data sharply favor $g_{V_8 P_8 \gamma} = g_{V_1 P_8 \gamma}$. Quite interestingly, Refs. [20,32] reach a parent conclusion in a framework quite different from ours; they express it by stating that gluonic annihilations should have negligible contributions in light meson decays.

2/ We consider the vector mixing angle δ_V as the s–dependent function given in Eq. (10). In this case, through the kaon loop, it depends on a well defined function and an arbitrary polynomial (see Eq. (D3)). The behavior of $P(s)$ is minimally $d_0 + d_1 s$; however, several attempts led us to go one unit beyond minimality in subtracting the dispersion relation and then choose $P(s) = d_0 + d_1 s + d_2 s^2$; the final loop function used is given in Eq. (D3). In Section IV, we have shown that the appropriate renormalization condition for the constant term here was $d_0 = 0$. On the other hand, analysis of fit results has shown that the renormalization condition $d_1 = 0$ is numerically appropriate, despite that d_1 and d_2 happens to be highly correlated (99%).

Therefore, the function we use for $\delta_V(s)$ depends on one single free parameter (d_2) and on a non–trivial well defined logarithm function. This single parameter practically generates 2 mixing angles $\delta_V(m_\omega^2)$ and $\delta_V(m_\phi^2)$ functionally related with each other. In part **1** just above, the corresponding free parameter was a single mixing angle, so the situation is somewhat different. Interestingly, this functional dependence correlates the HLS parameter a and the BKY breaking parameter ℓ_V to the sector of radiative decays (see Eq. (19)), which is obviously not the case when having a single constant vector mixing angle (see **1/** above). As we have seen that explicit NSB in the V sector does not provide any improvement, we have set $y = 1$ in all fits referred to hereafter.

- We have first performed fits with the full expression in Eq. (19) computed at the HK masses for ω_I and ϕ_I as they occur in Eq. (4), using at each minimization step the current values for a, g and ℓ_V. The fit quality reached is $\chi^2/dof = 7.89/11$ (72% probability). Compared with the (first mentioned) reference fit in (**1/**), the gain in probability is not statistically significant. However, as it is the same data set and the same number of free parameters, this could point towards some evidence that departure from ideal mixing is indeed observed mass–dependent. Only much improved data could allow to go farther.

  The fit parameter values are $a = 2.44 \pm 0.04$, $G = 0.703 \pm 0.002$ GeV^{-1} (the relation between G and g is given in Eq. (E1)), the pseudoscalar mixing angle is $\theta_P = -10.30^\circ \pm 0.20^\circ$ as always above. The main purpose of Ref. [9] was to show that this fits perfectly with all ChPT predictions, and the usual (ChPT) mixing angle θ_8 fulfills $\theta_8 \simeq 2\theta_P$.

  The breaking parameter is $\ell_V = 1.42 \pm 0.03$ and we get $d_2 = (0.147 \pm 0.008)\ 10^{-2}$ GeV^{-2}. All other values are nearly identical to the corresponding ones in the fits with one vector mixing angle [1]. The NSB x parameter value corresponding to the PS mixing angle is fit to $x = 0.900 \pm 0.017$.

  Therefore the varying vector mixing angle is consistent with the full data set; its values are tightly connected with the very small (but significantly non–zero) value for d_2. Using the parameters above and their errors Eq. (19) allows to compute $\delta_V(m_\omega^2) = -2.22^\circ \pm 0.21^\circ$ and $\delta_V(m_\phi^2) = -3.44^\circ \pm 0.30^\circ$. The errors here are of course not independent, they nevertheless allow to understand why a single mixing angle works so well.

- As we know that anomalous ω_I couplings are important and different of those for ϕ_I, the connection done in Eq. (19) between $\Pi_{\omega_I\phi_I}(s)$ and $(\Pi_{\omega_I\omega_I}(s) - \Pi_{\phi_I\phi_I}(s))$ might be broken in real life. On the other hand one can expect that $m_{\phi_I}^2 - m_{\omega_I}^2 >> |\Pi_{\omega_I\omega_I}(s) - \Pi_{\phi_I\phi_I}(s)|$ in the region of meson resonances. Therefore we have redone the fits by removing the loop contribution in the denominator of the expression for $\delta_V(s)$ in Eqs. (10) or (19).

  The fit obtained is also quite good ($\chi^2/dof = 7.94/11$, 72% probability) and the numerical results do not appreciably differ from those already mentioned. Therefore the mixing angle is not too much affected by uncertainties on ω_I and ϕ_I self–energies related with the neglected loops. Additionally, we do not detect any need to decouple the mixing angle from the $\Pi_{\omega_I\phi_I}(s)$ loop expression, as it could be if there were a significant vector NSB contribution to the mixing angle function.

IX. LOOP EFFECTS IN MASS VALUES

It is not the purpose of the present paper to perform a detailed study of the contribution of self–energies to observed values of vector meson masses. However, we can limit ourselves to mentioning the effects of kaon loops, as this trivially follows from the above fits.

The effective masses can be approximated by [18]

$$m^2_{Veff.} = m^2_V + \Pi_{VV}(m^2_V) \tag{29}$$

It has been checked numerically that rotations have a negligible effect here and have not been included.

Using the fit parameter values it is easy to get the following numbers (units are MeV)

$$\left\{ \begin{array}{ll} m_\omega^{HK} = 814.6 \pm 6.6 & m_\omega^{eff.} = 817.6 \pm 6.6 \\ & \\ m_\phi^{HK} = 969.8 \pm 14.1 & m_\phi^{eff.} = 986.1 \pm 14.2 \end{array} \right. \tag{30}$$

Then the effect of kaon loops is to shift moderately the ω mass upwards (by 3 MeV), while the shift is important for the ϕ (about 16 MeV, 4 times its width!). In order to be really conclusive the other (anomalous) couplings should be taken into account, but we see already that effects of real part of self–energies are qualitatively sufficient to push the effective mass of the ϕ far from its HK value and closer to the observed peak value. This illustrates our statement that observed masses might be quite different from the masses in the Lagrangian (we also refer to [22,23] and to [33]).

X. DEPARTURES FROM IDEAL MIXING IN ω AND ϕ DECAYS

The origin of departures of the ω/ϕ system from ideal mixing has been investigated in this paper by analyzing several mechanisms separately and together. The benchmark is the set of all radiative decays of light mesons (14 processes $VP\gamma$ and $P\gamma\gamma$) and leptonic decays of vector mesons (3 modes).

The central part of the various models is the BKY SU(3) breaking scheme. Its reliability is obviously a crucial condition.

The breaking scheme [8,1] in the PS sector seems reliable as the connection between the HLS model broken in this way and expectations from ChPT [10–12] is well reproduced [9]. On the other hand, the way nonet symmetry is broken in the V sector is in accordance with basics of group theory as illustrated by the correspondence between the model we propose and the standard formulation of O'Donnell [16].

The BKY SU(3) breaking [7,8] in the vector sector is harder to evaluate directly; it results essentially in shifting apart the ω_I and the ϕ_I HK masses and in a slight modification of the leptonic decay rates. However, the leptonic widths of the vector mesons depend on this breaking parameter (named ℓ_V above) and also on the HLS parameter a (see Eqs. (E6)); if this breaking procedure were not appropriate, one may guess that fits would return a value for a inconsistent with its value fit in other independent data sets.

However, the present fit with a varying vector mixing angle gives $a = 2.44 \pm 0.04$ (close to the result with a fixed angle: $a = 2.50 \pm 0.03$ [1]), in surprisingly good agreement with the value coming out from fit to the $e^+e^- \to \pi^+\pi^-$ world data [5] [4] ($a = 2.37 \pm 0.02$), or to the most recent (and independent) data set [6] ($a = 2.38 \pm 0.02$). One should rather remark that a varying mixing angle (with Z at its physical value 2/3) makes leptonic and radiative decays providing a value for a quite consistent with pion form factor studies.

In the pion form factor, the prominent feature accounted for by the HLS a parameter is the strength of a direct coupling $\gamma\pi^+\pi^-$ relative to the ρ^0 contribution; its effect extends from threshold to about 1 GeV. The pion form factor is free of any influence of SU(3) breaking (noticeably X_V) and then the fit value for a is free of any correlation with ℓ_V. Therefore, there is also no manifest reason to suspect that the BKY SU(3) breaking in the vector sector could be questionable.

In order to be more exhaustive, we have tried including nonet symmetry breaking in the vector sector in two different ways. Finally, we have considered the challenging effect produced by kaon loops in generating $\omega_I \leftrightarrow \phi_I$ transitions which forces to rotate the ideal fields in order to diagonalize the vector mass term.

Having performed a crossed study of all possible effects together or separately, we reached the following conclusions:

- Whatever its origin, an angle δ_V exhausts (by far) the best fit quality, and this quality is statistically optimum. An *explicit* vector NSB (y) is unable to produce a comparable effect.

- A mass–dependent phase shift $\delta_V(s)$ behaves as well, and even somewhat better, without introducing more freedom in the fits. This could be considered as a slight evidence in favor of an observed mass–dependence of the mixing angle, in functional accordance with loop expressions in the HLS model.

- Leaving free the manifest NSB parameter y cannot mimic the effect of δ_V (constant or not). Moreover the value for y returned by all fits is consistent with no vector NSB.

- Whatever the context, δ_V is negative ($\delta_V \simeq -3^\circ$), confirming within VMD the analysis of Dillon and Morpurgo [32], who share certainly the same conventions as ours. When there are effectively two such angles, both are found negative and close together. For completeness, this sign for δ_V is tightly connected with our definition of $\phi_I = -s\overline{s}$ and with the signs in the matrix $G(\delta_V)$. This corresponds to an ω/ϕ mixing angle slightly smaller than its ideal value ($\theta_V \simeq 32^\circ$). The fit probabilities are always of the order 60% to more than 70%.

- We have carefully tried to find secondary acceptable minimum χ^2 solutions. The aim was to look for a solution with somewhat different parameter values and noticeably a positive value for δ_V (and, thus, a vector mixing angle greater than $\simeq 35^\circ$). We never reached a χ^2 better than about 40 units when forcing δ_V to stay positive or zero. This means a fit probability of the order 10^{-5}.

So, the conclusion about the mixing angle coming out from fits to radiative and leptonic decays is stable under a large variety of conditions. It is obtained within a highly constrained scheme with very few parameters and quite good probabilities (above the 60% level). The data used come from different kind of experiments and can be widely considered statistically independent of each other.

Actually, it is quite trivial to prove, from within the non–anomalous HLS Lagrangian alone, that an average δ_V is surely negative while relying on only leptonic decays. Our definition $\phi = -s\bar{s}$ being implicit, it is trivial to show that the $V-\gamma$ coupling constants (see Eqs. (E6)) of the BKY broken HLS Lagrangian fulfill[14]

$$f_{\omega\gamma}\cos\delta_V - f_{\phi\gamma}\sin\delta_V = \frac{f_{\rho\gamma}}{3} \quad , \; f_{\omega\gamma}\sin\delta_V + f_{\phi\gamma}\cos\delta_V = \frac{f_{\rho\gamma}}{3}\sqrt{2}\ell_V \tag{31}$$

up, possibly, to higher order terms in vector NSB (see Eqs. (E7)) and without any influence of radiative decay models. The first relation can be considered as an equation for δ_V in terms of leptonic decay data [21] (units are GeV^2): $f_{\rho\gamma} = 0.119 \pm 0.003$, $f_{\omega\gamma} = (3.586 \pm 0.060)\ 10^{-2}$ and $f_{\phi\gamma} = (7.933 \pm 0.114)\ 10^{-2}$. It is trivial to solve it and get $\delta_V = -2.79^\circ \pm 0.84^\circ$.

Therefore, our results merely illustrate that the sign information for δ_V, hidden in radiative decays is in perfect agreement with the sign which can be obviously exhibited in leptonic decays. So, in the HLS approach, the algebraic value for δ_V follows from the radiative *and* from the leptonic sectors *separately*.

However, there is also a result by Achasov, Kozhevnikov and Shestakov [34] which predicted a correlation between the signs of $R = [f_{\phi\gamma}G_{\phi\rho\pi}]/[f_{\omega\gamma}G_{\omega\rho\pi}]$ and the interference pattern in the neighborhood of the ϕ meson in the annihilation processes $e^+e^- \to \pi^+\pi^-\pi^0$. From the results reported in [35], the conclusion was that R should be negative and this should imply a positive value for $\delta_V(s)$. Recent analyses of the $e^+e^- \to \pi^+\pi^-\pi^0$ cross section by Achasov *et al.* [36] seem to confirm their conclusion. The origin of this disagreement has not been explored and could point toward an interesting puzzle.

Indeed, taking into account the stability of our fit results inside the HLS framework and the cross–check represented by the independent analysis of Dillon and Morpurgo [32], we consider our small negative value for δ_V unavoidable when using (even separately) radiative and leptonic decays within the HLS framework. A varying mixing angle leading to the same result, this checks that the neglected loop contributions are indeed numerically small. Their influence could however be accounted for practically by the numerical values for d_1 and d_2.

XI. CONCLUSION

We have studied in detail the origin of departures of the ω/ϕ system from ideal mixing in the non–anomalous HLS Model. Whatever its origin, the mechanism at work is clearly the existence of $\omega_I \leftrightarrow \phi_I$ transitions. The simplest origin of these transitions are the kaon loops to which ω/ϕ couple. Because they involve on–shell particles, in resonance decays the ω/ϕ mixing occurs essentially through a field rotation, as traditionally assumed. Departures from rotation could be presently observable in scattering processes involving far off mass–shell ω and ϕ mesons as internal lines.

In order to make this usable for phenomenological purposes, we have defined an effective Lagrangian piece containing all loops of the non–anomalous HLS Lagrangian only. Summing up the standard broken Lagrangian with this effective piece confirms that $\omega_I \leftrightarrow \phi_I$ transitions are inherent to the HLS model, broken or not, in accordance with the Schwinger–Dyson equation. The transformation from ideal to physical fields has been studied.

[14] These relations correct each for a misprint in unnumbered relations in Ref. [1] nearby Eqs. (22) and (23) (which are instead both correct).

A possibly new result is that this mixing angle appears as a well-defined momentum–dependent function $\delta_V(s)$, depending also on a subtraction polynomial $P(s)$. Its constant coefficient (d_0) should be zero in order to recover the $P\gamma\gamma$ decay amplitudes from the FKTUY Lagrangian. One could imagine that it could be fixed from ChPT information concerning the kaon form factors; here we have fitted it.

We have left aside the consequences for scattering amplitudes and focused on the radiative and leptonic decays of light mesons. We have thus shown that the model proposed in Ref. [1] was perfectly consistent in either of its two possible variants: constant or invariant–mass dependent δ_V. Data give some slight evidence in favor of this dependence, however a constant value for δ_V is, presently, a good approximation. Our result compares well with the independent analysis by Dillon and Morpurgo using a quite different framework. The value for a is found in good agreement with independent 0pion form factors studies; the agreement is somewhat better with varying mixing angle than with a fixed one.

ω/ϕ mixing is generated by loop effects, without any help of symmetry breaking. Following some trend, we have nevertheless examined whether a successful description of radiative and leptonic decays could be reached (or improved) by adding nonet symmetry breaking (NSB) in the vector sector of the HLS Lagrangian. Instead of the remarkable effect of NSB in the PS sector, we have found no indication, statistically significant, that vector NSB could help in a better understanding of the data *as a whole* (we mean the 17 decay modes considered altogether, as it should). If it exists, vector NSB could however be hidden inside δ_V and thus hard to disentangle from genuine loop effects; these are, however, widely sufficient in order to understand qualitatively and quantitatively all the data we have examined at their present level of accuracy.

From a specific point of view, all the variants explored (vector NSB, fixed or varying δ_V) converge towards a ω/ϕ mixing angle slightly below its ideal value. Stated otherwise, a constant δ_V is at about $-3°$; a varying one is equivalent to two such angles (one at the ω mass, one at the ϕ mass) but both are negative, close together and also $\simeq -3°$.

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APPENDIX A: THE $K\overline{K}$ OR $\pi\pi$ LOOP EXPRESSION

The loop expressions for a vector particle decaying into two pseudoscalar mesons of equal masses can be computed by means of dispersion relations [22] or by using Pauli–Villars regulators [23]. We derive here this expression without performing any explicit integration, by relying on properties of analytic functions.

Let us denote by V a particular vector meson and by P and $\overline{P}$ the pseudoscalars of the pair to which they couple; the common mass to the pseudoscalars is denoted m_P. Let us also denote $\Pi(s)$ the $P\overline{P}$ loop function.

From general principles $\Pi(s)$ is a real analytic function of s (*i.e.* fulfilling $\Pi(s) = \Pi^*(s^*)$), real below the threshold located at $s_0 = 4m_P^2$. Its imaginary part above s_0 can be computed using Cutkotsky rules or by means of the partial width $V \to P\overline{P}$ ($\text{Im}\Pi(s) = -\sqrt{s}\Gamma(s)$). We thus have

$$\text{Im}\Pi(s) = -\frac{g_{VP\overline{P}}^2}{48\pi}\frac{(s-s_0)^{3/2}}{s^{1/2}} \tag{A1}$$

The analytic function $\Pi(s)$ fulfills (at least) a twice subtracted dispersion relation, as clear from power counting in the expression for $\text{Im}\Pi(s)$ above. This equation can be written

$$\Pi(s) = P(s) + \frac{s^2}{\pi}\int_{s_0}^{\infty} \frac{\text{Im}\Pi(z)}{z^2(z-s+i\epsilon)}dz \tag{A2}$$

exhibiting that the single cut on the physical sheet lies along the physical region $s \geq s_0$. $P(s)$ denotes a polynomial of (at least) first degree. The coefficients in $P(s)$ should be real and fixed by means of (external) renormalization conditions, as in Chiral Perturbation Theory (ChPT). As noted above, the minimal degree of $P(s)$ is 1; however, the actual degree of this polynomial (and, hence, the actual number of subtractions to the dispersion relation above) depends on the assumed behavior of $\Pi(s)$ at infinity[15]. The coefficients of this (arbitrary) polynomial need then to be fixed using renormalization conditions such as the values of $\Pi(s)$ and/or its derivatives at some point. This can be chosen as the point $s = 0$, if one likes to connect with ChPT; other ways are possible which will not be examined here (see Refs. [22,23]).

On the other hand, the theory of analytic function teaches that, if we can find *one* analytic function $\Pi(s)$ such that Eq. (A2) holds, then this solution is unique up to a polynomial (or an entire function) real for real s. That is, two arbitrary analytic solutions to Eq. (A2) differ only by a polynomial with real coefficients. In this section, and in the following ones dealing with loop computations, we specialize to a minimally subtracted dispersion relation, and then $P(s)$ is of first degree.

Now, let us define the function $K(s)$ by

$$\Pi(s) = \frac{g_{VP\overline{P}}^2}{48\pi}\frac{s-s_0}{s}K(s) + P(s) \tag{A3}$$

where $P(s)$ is the polynomial already defined; $K(s)$ is real for real s below threshold and we have

$$\begin{cases} \text{Im}K(s) = -s^{1/2}(s-s_0)^{1/2} \quad , \quad (s \geq s_0) \\ K(s) = c_0 + c_1 s + c_2 s^2 + \dfrac{s^3}{\pi}\displaystyle\int_{s_0}^{\infty} \frac{\text{Im}K(z)}{z^3(z-s+i\epsilon)}dz \end{cases} \tag{A4}$$

The dispersion relation for $K(s)$ should be subtracted three time in order to remove the constant term at $s = 0$ which would produce a simple pole for $\Pi(s)$ as clear from the

[15]Depending on this, $P(s)$ could also be some entire function of s, real for real s. All the general properties listed here follow from the standard analytic S–matrix theory [19]; they apply obviously to all amplitudes constructed from any acceptable Lagrangian.

first Eq. (A3). This is the minimum number of subtractions consistent with integrability conditions.

One can construct easily such a function, denoted $\varphi(s)$, real in the interval $0 \leq s \leq s_0$. It is

$$\varphi(s) = -\frac{i}{\pi} s^{1/2}(s_0 - s)^{1/2} \ln \frac{(s_0 - s)^{1/2} + i s^{1/2}}{(s_0 - s)^{1/2} - i s^{1/2}} \tag{A5}$$

It can be rewritten :

$$\varphi(s) = \frac{2}{\pi} s^{1/2}(s_0 - s)^{1/2} \arctan \sqrt{\frac{s}{(s_0 - s)}} \quad , \; 0 \leq s \leq s_0 \tag{A6}$$

Eq. (A5) can easily be continued above s_0 (by winding clockwise around this point by an angle of π radians) and below $s_c = 0$, the crossed threshold (by winding counter–clockwise by an angle of π radians). This gives the function $K(s)$ on the whole real axis. The constants c_i in rels. (A4) are fixed by requiring that

$$\lim_{s \to 0} K(s) = 0 \quad , \quad \lim_{s \to 0} \frac{d}{ds} K(s) = 0 \quad \text{and} \quad \lim_{s \to 0} \frac{d^2}{ds^2} K(s) = 0. \tag{A7}$$

Then, assuming the minimal number of subtractions, the general solution for $\Pi(s)$ is, for real s

$$\left\{ \begin{array}{lll} & & \Pi(s) = d_0 + d_1 s + Q(s) \\ & & Q(s) = \dfrac{g^2_{VP\overline{P}}}{24\pi^2} \left[G(s) + s_0 - \dfrac{4}{3} s \right] \\ s \leq 0 & : & G(s) = -\dfrac{1}{2} \dfrac{(s_0 - s)^{3/2}}{(-s)^{1/2}} \ln \dfrac{(s_0 - s)^{1/2} - (-s)^{1/2}}{(s_0 - s)^{1/2} + (-s)^{1/2}} \\ 0 \leq s \leq s_0 & : & G(s) = -\dfrac{(s_0 - s)^{3/2}}{s^{1/2}} \arctan \sqrt{\dfrac{s}{(s_0 - s)}} \\ s \geq s_0 & : & G(s) = -\dfrac{1}{2} \dfrac{(s - s_0)^{3/2}}{s^{1/2}} \left[\ln \dfrac{s^{1/2} - (s - s_0)^{1/2}}{s^{1/2} + (s - s_0)^{1/2}} \right] \\ & & -\dfrac{i\pi}{2} \dfrac{(s - s_0)^{3/2}}{s^{1/2}} \end{array} \right. \tag{A8}$$

The behavior of $\Pi(s)$ near $s = 0$ is simply $d_0 + d_1 s + \mathcal{O}(s^2)$, where $Q(s)$ behaves like $\mathcal{O}(s^2)$ near the origin. This result coincides with the one of Ref. [22]. By performing more subtractions, one could also choose to fix externally the s^2 behavior of the loop near the origin, etc ...

The results here apply directly to loops like $\pi\pi$ or $K\overline{K}$ and to the ρ, ω and ϕ self–energies with an appropriate choice of the specific $VP\overline{P}$ coupling constant.

APPENDIX B: THE $K\pi$ LOOP EXPRESSION

It is not of common custom to give the loop expression for vector particles coupling to a pair of unequal mass PS mesons (see however [37]). Its derivation is tightly connected with the previous case. We limit ourselves to the minimally subtracted case, as before; it is trivial (and tedious) to go beyond.

In order to fix notations, we identify this case with $K^* \to K\pi$. The imaginary part of the loop is

$$\mathrm{Im}\Pi(s) = -\frac{g^2_{K^*K\pi}}{24\pi}\frac{(s-s_0-s_c)}{\sqrt{s}}p \tag{B1}$$

where we have defined $s_0 = (m_K + m_\pi)^2$ and $s_c = (m_K - m_\pi)^2$, the direct and crossed thresholds. p is the cms momentum of the decay products $p = 1/2\sqrt{(s-s_0)(s-s_c)/s}$. Let us also define, as previously, the function $K(s)$ and the subtraction polynomial $P(s)$ by

$$\Pi(s) = \frac{g^2_{K^*K\pi}}{48\pi}\frac{s-(s_0+s_c)}{s}K(s) + P(s) \tag{B2}$$

where the function $K(s)$ obeys the three time subtracted dispersion of Eq. (A4) with

$$\mathrm{Im}K(s) = -(s-s_0)^{1/2}(s-s_c)^{1/2} \quad , \; s \geq s_0 \tag{B3}$$

The subtraction constants c_i are chosen such as $K(s) \simeq \mathcal{O}(s^3)$ when $s \to 0$. The function $K(s)$ can be easily constructed as before between s_0 and s_c, and continued below s_c and above s_0. Let us now define

$$\left\{\begin{array}{l} A = s_0 + s_c - \dfrac{1}{2\sqrt{s_0 s_c}}\left[(s_0+s_c)^2 + 2s_0 s_c\right]\ln\dfrac{m_\pi}{m_K} \\[2ex] B = -\left[\dfrac{(s_0+s_c)^2 + 4s_0 s_c}{4s_0 s_c}\right] - \dfrac{1}{8}\dfrac{s_0+s_c}{(s_0 s_c)^{3/2}}\left[(s_0-s_c)^2 - 2s_0 s_c\right]\ln\dfrac{m_\pi}{m_K} \end{array}\right. \tag{B4}$$

Then, for real s, the final solution for $\Pi(s)$ is given by :

$$
\begin{cases}
 & \Pi(s) = d_0 + d_1 s + Q(s) \\
 & Q(s) = \dfrac{g^2_{K^* K\pi}}{48\pi^2}\left[G(s) + A + Bs\right] \\
s \leq s_c \quad : & G(s) = (s-(s_0+s_c))\dfrac{(s_0-s)^{1/2}(s_c-s)^{1/2}}{s}\ln\left[\dfrac{(s_0-s)^{1/2}-(s_c-s)^{1/2}}{(s_0-s)^{1/2}+(s_c-s)^{1/2}}\right] \\
 & +\dfrac{\sqrt{(s_0 s_c)}(s_0+s_c)}{s}\ln\dfrac{m_\pi}{m_K} \\
s_c \leq s \leq s_0 \quad : & G(s) = 2(s-(s_0+s_c))\dfrac{(s_0-s)^{1/2}(s-s_c)^{1/2}}{s}\arctan\left[\dfrac{s-s_c}{s_0-s}\right]^{1/2} \\
 & +\dfrac{\sqrt{(s_0 s_c)}(s_0+s_c)}{s}\ln\dfrac{m_\pi}{m_K} \\
s \geq s_0 \quad : & G(s) = -(s-(s_0+s_c))\dfrac{(s-s_0)^{1/2}(s-s_c)^{1/2}}{s}\ln\left[\dfrac{(s-s_c)^{1/2}-(s-s_0)^{1/2}}{(s-s_c)^{1/2}+(s-s_0)^{1/2}}\right] \\
 & +\dfrac{\sqrt{(s_0 s_c)}(s_0+s_c)}{s}\ln\dfrac{m_\pi}{m_K} \\
 & -i\pi\dfrac{(s-s_0)^{1/2}(s-s_c)^{1/2}}{s}(s-(s_0+s_c))
\end{cases}
\tag{B5}
$$

where the function $Q(s)$ behaves like s^2 near the chiral point. This result is a non trivial extension of the previous case. Going to other PS meson pairs is easily performed by changing m_π and m_K by resp. the lightest and heaviest meson mass.

APPENDIX C: SOME NEGLECTED LOOPS

It could be useful for future developments including anomalous contributions, or for other purposes, to have at disposal VP and γP loop expressions which can play some (presently minor) role in estimating self–energies. Double loop expressions PPP cannot be computed in closed form; they might also contribute little to meson self–energies [33].

1. VP Loops

Their main effect, at the present level of accuracy of the data, could be the contribution for $\rho\pi$ to ω_I self–energy. It is presently overwhelmed by the HK mass values. Other possible contributions to self–energies or transition amplitudes would involve an intermediate η or η' mesons, which push the thresholds quite high compared to the ϕ mass. The masses for vector mesons here should be the HK masses. We skip detailed proofs, as they follow closely the lines in the two Sections above.

Let us fix notations by focusing on $\rho \to \omega\pi$. Stating $s_c = (m_\omega - m_\pi)^2$ and $s_0 = (m_\omega + m_\pi)^2$. We have

$$\mathrm{Im}\Pi(s) = -\frac{g^2_{\omega\rho\pi}}{96\pi}\frac{(s-s_0)^{3/2}(s-s_c)^{3/2}}{s} \tag{C1}$$

and define the function $K(s)$ by

$$\Pi(s) = \frac{g^2_{\omega\rho\pi}}{96\pi}\frac{(s-s_0)(s-s_c)}{s}K(s) + P(s) \tag{C2}$$

where $P(s)$ is a polynomial of (at least) degree 1 if $\Pi(s)$ obeys (at least) a twice subtracted dispersion relation. This function $K(s)$ obeys the dispersion relation in Eq. (A4) above with

$$\mathrm{Im}K(s) = -(s-s_0)^{1/2}(s-s_c)^{1/2} \tag{C3}$$

The procedure above applies and, setting

$$\begin{cases} A = s_0 s_c \left[-1 + \dfrac{3}{2}\dfrac{s_0+s_c}{\sqrt{s_0 s_c}}\ln\dfrac{m_\pi}{m_\omega}\right] \\ B = \dfrac{5}{4}(s_0+s_c) - \dfrac{3}{8}\dfrac{(s_0+s_c)^2+4s_0s_c}{\sqrt{s_0s_c}}\ln\dfrac{m_\pi}{m_\omega} \end{cases} \tag{C4}$$

the final solution for $\Pi(s)$ on the real axis, with the minimum number of subtractions is

$$\begin{cases} & \Pi(s) = d_0 + d_1 s + Q(s) \\ & Q(s) = \dfrac{g^2_{\omega\rho\pi}}{96\pi^2}\left[G(s) + A + Bs\right] \\ s \le s_c \quad : & G(s) = \dfrac{(s_0-s)^{3/2}(s_c-s)^{3/2}}{s}\ln\left[\dfrac{(s_0-s)^{1/2}-(s_c-s)^{1/2}}{(s_0-s)^{1/2}+(s_c-s)^{1/2}}\right] \\ & -\dfrac{(s_0s_c)^{3/2}}{s}\ln\dfrac{m_\pi}{m_\omega} \\ s_c \le s \le s_0 \quad : & G(s) = -2\dfrac{(s_0-s)^{3/2}(s-s_c)^{3/2}}{s}\arctan\left[\dfrac{s-s_c}{s_0-s}\right]^{1/2} \\ & -\dfrac{(s_0s_c)^{3/2}}{s}\ln\dfrac{m_\pi}{m_\omega} \\ s \ge s_0 \quad : & G(s) = -\dfrac{(s-s_0)^{3/2}(s-s_c)^{3/2}}{s}\ln\left[\dfrac{(s-s_c)^{1/2}-(s-s_0)^{1/2}}{(s-s_c)^{1/2}+(s-s_0)^{1/2}}\right] \\ & -\dfrac{(s_0s_c)^{3/2}}{s}\ln\dfrac{m_\pi}{m_\omega} \\ & -i\pi\dfrac{(s-s_0)^{3/2}(s-s_c)^{3/2}}{s} \end{cases} \tag{C5}$$

2. $P\gamma$ Loops

This contribution could be, for some applications, less academic than the previous one. It is additionally a singular limit of the VP case ($s_c \to s_0$). Let us specialize to $\omega \to \pi\gamma$. The crossed and direct threshold now coincide at $s_0 = m_\pi^2$. We have

$$\mathrm{Im}\Pi(s) = -\frac{g^2_{\omega\pi\gamma}}{96\pi}\frac{(s-s_0)^3}{s} \tag{C6}$$

which follow continuously from the case just above. We still define the function $K(s)$ by

$$\Pi(s) = \frac{g^2_{\omega\pi\gamma}}{96\pi}\frac{(s-s_0)^3}{s}K(s) + P(s) \tag{C7}$$

which fulfills the dispersion relation Eq. (A4) with

$$\mathrm{Im}K(s) = -1 \quad , \ s \geq s_0 \tag{C8}$$

A specific solution to this equation below threshold is the function

$$\phi(s) = \frac{1}{\pi}\ln\frac{(s_0 - s)}{s_0} \tag{C9}$$

real for $s \leq s_0$ and which is analytically continued above s_0 to

$$\phi(s) = \frac{1}{\pi}\ln\frac{(s - s_0)}{s_0} - i \tag{C10}$$

Then the solution for $\Pi(s)$ is

$$\left\{\begin{array}{ll} & \Pi(s) = d_0 + d_1 s + Q(s) \\ & Q(s) = \dfrac{g^2_{\omega\pi\gamma}}{96\pi^2}[G(s) - s_0^2 + \dfrac{5}{2}s_0 s] \\ s \leq s_0 & : \ G(s) = -\dfrac{(s_0 - s)^3}{s}\ln\dfrac{(s_0 - s)}{s_0} \\ s \geq s_0 & : \ G(s) = \dfrac{(s - s_0)^3}{s}\ln\dfrac{(s - s_0)}{s_0} - i\pi\dfrac{(s - s_0)^3}{s} \end{array}\right. \tag{C11}$$

One should note the disappearance of the square root branch point compared to the VP case. Therefore, the collapse of s_0 and s_c pulls the logarithmic branch point, originally far inside the unphysical sheet, to the threshold.

APPENDIX D: SELF–ENERGIES AND TRANSITION AMPLITUDES

For the purpose of the present work, we consider all loops computed in the first two Appendices above amputated from their couplings constants (stated otherwise, all expression there are considered with unit coupling constants). Now let us denote $\Pi(s)$ the kaon loop and $\Pi'(s)$ the pion loop. Let us also denote $\Pi''(s, \pi/\eta/\eta')$ the loops functions for the three PS meson pairs $K - (\pi/\eta/\eta')$.

In exact SU(2) limit, the self–energies for charged and neutral ρ mesons are the same

$$\Pi_{\rho\rho}(s) = 2g^2_{\rho K\overline{K}}\Pi(s) + g^2_{\rho\pi\pi}\Pi'(s) \tag{D1}$$

Likewise, in the same exact SU(2) limit, for all K^* mesons the self–energies are

$$\Pi_{K^*K^*}(s) = (1+\sqrt{2})g^2_{K^*K\pi}\Pi''(s,\pi) + g^2_{K^*K\eta}\Pi''(s,\eta) + g^2_{K^*K\eta'}\Pi''(s,\eta') \tag{D2}$$

The coupling constants involved in Eqs. (D1) and (D2) can be read off the Lagrangian given in Ref. [8] and are not explicited here. There is no more contributions involving ρ or K^* mesons, as soon as one neglects terms of order e^2 and all anomalous terms. There is also no transition from ρ (or K^*) meson to any other meson in the same approximation.

Let us redefine the generic kaon loop $\Pi(s)$ as in Eq. (A8) by going one unit beyond the minimal subtraction scheme for reasons explained in the body of the paper. We have

$$\left\{\begin{array}{rcl} \Pi(s) &=& P(s) + \dfrac{1}{24\pi^2}\left[G(s) + s_0 - \dfrac{4}{3}s + \dfrac{1}{5}\dfrac{s^2}{s_0}\right] \\ \\ P(s) &=& d_0 + d_1 s + d_2 s^2 \end{array}\right. \tag{D3}$$

where $s_0 = 4m_K^2$ and $G(s)$ is given by Eq (A8). Correspondingly to introducing d_2s^2 we have subtracted the s^2 behavior in $G(s)$.

We give now the functions $\Pi_{\omega_I\omega_I}(s)$, $\Pi_{\phi_I\phi_I}(s)$ and $\Pi_{\omega_I\phi_I}(s)$ which come at next–to–leading order in influencing the non–anomalous HLS Lagrangian mass term. The bare Lagrangian piece in Eq. (7) implies that the kaon loop in each self–energy comes in both their charged and neutral modes; they both coincide with $\Pi(s)$ defined above, because of exact SU(2) symmetry. Then, we have

$$\left\{\begin{array}{rcl} \Pi_{\omega_I\omega_I}(s) &=& 2g^2_{\omega_I K\overline{K}}\Pi(s) \\ \Pi_{\omega_I\omega_I}(s) &=& 2g^2_{\phi_I K\overline{K}}\Pi(s) \\ \Pi_{\omega_I\phi_I}(s) &=& 2g_{\phi_I K\overline{K}}g_{\omega_I K\overline{K}}\Pi(s) \end{array}\right. \tag{D4}$$

where the factor of 2 accounts for the two kaon loops (charged or neutral modes). All dependences in the coupling constants $VK\overline{K}$ are explicit. The coupling constants in Eq. (D4) can be read off Eq. (7). One should finally note that we assume the same subtraction polynomial $P(s) = d_0+d_1s+d_2s^2$ in all three functions. This is highly constraining and might be somewhat relaxed. Indeed, one could consider separately dispersion relations for the functions $\Pi_{\omega_I\omega_I}(s)$, $\Pi_{\phi_I\phi_I}(s)$ and $\Pi_{\omega_I\phi_I}(s)$ which could then undergo different renormalization conditions. This turns out to remark that a condition like $\mathrm{Im}\Pi_{\omega_I\omega_I}(s) = \lambda \mathrm{Im}\Pi_{\phi_I\phi_I}(s)$ (λ being a real constant) for $s \geq s_0$ does imply $\Pi_{\omega_I\omega_I}(s) = \lambda \Pi_{\phi_I\phi_I}(s)$, *up to a polynomial with real coefficients*. We have nevertheless preferred considering as basic the dispersion relation for the elementary $K\overline{K}$ loop. Moreover, contributions of order e^2 and other anomalous contributions are neglected.

APPENDIX E: COUPLING CONSTANTS FOR RADIATIVE DECAYS

Starting from the Lagrangian in Eq. (24), and using the breaking procedure defined by Eqs.(21) and (23), one can compute the coupling constants for all radiative and leptonic decays of relevance. Let us define

$$G = -\frac{3eg}{8\pi^2 f_\pi} \quad , \quad G' = -\frac{3eg}{8\pi^2 f_K} \quad , \quad Z = f_\pi/f_K \tag{E1}$$

We shall use in all formulae a single vector mixing angle δ_V. In the framework where it is approximated by a constant phase (like in Ref. [1]), it is the common mixing angle which expresses departures from ideal mixing. In the case when one considers a mass dependent mixing, one has to make the replacement $\delta_V \Rightarrow \delta_V(s = m_\omega^2)$ in all expressions for $g_{P\omega\gamma}$, while the replacement is $\delta_V \Rightarrow \delta_V(s = m_\phi^2)$ in all expressions for $g_{P\phi\gamma}$. In this case, we practically have *two* different vector mixing angles δ_V^ω and δ_V^ϕ functionally related.

Let us define the parameters h_i ($i = 1, \cdots 4$) which contain all information about breaking nonet symmetry in the PS and V sectors, while breaking SU(3) itself only in the PS sector. Indeed, as the BKY breaking mechanism does not result in a redefinition of the vector fields, the radiative decays are not sensitive to SU(3) breaking in the vector sector; additionally, the condition $X_W X_T^4$ removes all dependences on ℓ_T for all couplings constants except for K^*. Thus, we have

$$\left\{ \begin{array}{lcl} h_1 & = & \dfrac{(1+2y)(1-x)+2(y-1)(2+x)Z}{3} \\[2ex] h_2 & = & \dfrac{(1+2y)(1+2x)+4(y-1)(1-x)Z}{3} \\[2ex] h_3 & = & \dfrac{(y-1)(1-x)+(2+y)(2+x)Z}{3} \\[2ex] h_4 & = & \dfrac{(y-1)(1+2x)+2(2+y)(1-x)Z}{3} \end{array} \right. \tag{E2}$$

The full U(3) symmetry limit is $x = y = Z = 1$. The $VP\gamma$ coupling constants are

$$
\begin{cases}
G_{\rho^0\pi^0\gamma} = & \frac{1}{3}G \\
G_{\rho^\pm\pi^\pm\gamma} = & \frac{1}{3}G \\
G_{K^{*0}K^0\gamma} = & -\frac{G'}{3}\sqrt{\ell_T}(1+\frac{1}{\ell_T}) \\
G_{K^{*\pm}K^\pm\gamma} = & \frac{G'}{3}\sqrt{\ell_T}(2-\frac{1}{\ell_T}) \\
G_{\rho^0\eta\gamma} = & \frac{1}{3}G\left[\sqrt{2}(1-x)\cos\delta_P-(2x+1)\sin\delta_P\right] \\
G_{\rho^0\eta'\gamma} = & \frac{1}{3}G\left[\sqrt{2}(1-x)\sin\delta_P+(2x+1)\cos\delta_P\right] \\
G_{\omega\pi^0\gamma} = & \frac{1}{3}G\left[(1+2y)\cos\delta_V-\sqrt{2}(y-1)\sin\delta_V\right] \\
G_{\phi\pi^0\gamma} = & -\frac{1}{3}G\left[(1+2y)\sin\delta_V+\sqrt{2}(y-1)\cos\delta_V\right]
\end{cases}
\tag{E3}
$$

The other single photon radiative modes provide

$$
\begin{cases}
G_{\omega\eta\gamma} = \frac{1}{9}G\left[\sqrt{2}h_1\cos\delta_V\cos\delta_P-h_2\cos\delta_V\sin\delta_P-2h_3\sin\delta_V\cos\delta_P+\sqrt{2}h_4\sin\delta_V\sin\delta_P\right] \\
G_{\omega\eta'\gamma} = \frac{1}{9}G\left[h_2\cos\delta_V\cos\delta_P+\sqrt{2}h_1\cos\delta_V\sin\delta_P-\sqrt{2}h_4\sin\delta_V\cos\delta_P-2h_3\sin\delta_V\sin\delta_P\right] \\
G_{\phi\eta\gamma} = \frac{1}{9}G\left[-2h_3\cos\delta_V\cos\delta_P+\sqrt{2}h_4\cos\delta_V\sin\delta_P-\sqrt{2}h_1\sin\delta_V\cos\delta_P+h_2\sin\delta_V\sin\delta_P\right] \\
G_{\phi\eta'\gamma} = \frac{1}{9}G\left[-\sqrt{2}h_4\cos\delta_V\cos\delta_P-2h_3\cos\delta_V\sin\delta_P-h_2\sin\delta_V\cos\delta_P-\sqrt{2}h_1\sin\delta_V\sin\delta_P\right]
\end{cases}
\tag{E4}
$$

The 2–photon decay modes keep exactly their form as in Ref. [1]

$$
\begin{cases}
G_{\eta\gamma\gamma} = & -\frac{\alpha_{em}}{\pi\sqrt{3}f_\pi}\left[\frac{5-2Z}{3}\cos\theta_P-\sqrt{2}\frac{5+Z}{3}x\sin\theta_P\right] \\
G_{\eta'\gamma\gamma} = & -\frac{\alpha_{em}}{\pi\sqrt{3}f_\pi}\left[\frac{5-2Z}{3}\sin\theta_P+\sqrt{2}\frac{5+Z}{3}x\cos\theta_P\right] \\
G_{\pi^0\gamma\gamma} = & -\frac{\alpha_{em}}{\pi f_\pi}
\end{cases}
\tag{E5}
$$

Finally, the $V-\gamma$ couplings become

$$
\begin{cases}
f_{\rho\gamma} = & af_\pi^2 g \\
f_{\omega\gamma} = & \frac{f_{\rho\gamma}}{3}\left[h_5\cos\delta_V+h_6\sin\delta_V\right] \\
f_{\phi\gamma} = & \frac{f_{\rho\gamma}}{3}\left[-h_5\sin\delta_V+h_6\cos\delta_V\right]
\end{cases}
\tag{E6}
$$

where we have defined

$$\begin{cases} h_5 = \ 1 + \dfrac{2}{3}(y-1)(\ell_V - 1) \\ h_6 = \ \sqrt{2}\ell_V + \dfrac{2}{3}(y-1)(\ell_V - 1) \end{cases} \tag{E7}$$

The X_V breaking parameter ℓ_V has been defined in the main text. All relations between the coupling constants here and decay rates are exactly as defined in Ref. [1]. One can check that in the limit $y \to 1$, the present results coincide with those given in this reference. Finally, keeping only U(3) breaking to SU(3)×U(1) in both V and PS sectors, one can also check that the $VP\gamma$ coupling constants here coincide with the axiomatic results of O'Donnell [16].

It should be noted, from Eqs. (E7), that NSB in the vector sector for leptonic decays undergo a further suppression by SU(3) breaking. Finally, as noted in the main text, if the X_T breaking has to be understood as the relevant vector field renormalization, ℓ_V should be understood as $\ell_V \ell_T$ without further changes in the other relations above.

REFERENCES

[1] M. Benayoun, L. DelBuono, S. Eidelman, V. N. Ivanchenko and H. B. O'Connell, "Radiative decays, nonet symmetry and SU(3) breaking," Phys. Rev. **D59** (1999) 114027 [hep-ph/9902326].

[2] M. Bando, T. Kugo and K. Yamawaki, "Nonlinear Realization And Hidden Local Symmetries," Phys. Rept. **164** (1988) 217.
M. Bando, T. Kugo, S. Uehara, K. Yamawaki and T. Yanagida, "Is Rho Meson A Dynamical Gauge Boson Of Hidden Local Symmetry?," Phys. Rev. Lett. **54** (1985) 1215.

[3] T. Fujiwara, T. Kugo, H. Terao, S. Uehara and K. Yamawaki, "Nonabelian Anomaly And Vector Mesons As Dynamical Gauge Bosons Of Hidden Local Symmetries," Prog. Theor. Phys. **73** (1985) 926.

[4] M. Benayoun, S. I. Eidelman, K. R. Maltman, H. B. O'Connell, B. Shwartz and A. G. Williams, "New results in rho0 meson physics," Eur. Phys. J. **C2** (1998) 269 [hep-ph/9707509].

[5] L. M. Barkov *et al.*, "Electromagnetic Pion Form-Factor In The Timelike Region," Nucl. Phys. **B256** (1985) 365.

[6] R. R. Akhmetshin *et al.*, "Measurement of $e^+e^- \to \pi^+\pi^-$ cross-section with CMD-2 around ρ meson," hep-ex/9904027.

[7] M. Bando, T. Kugo and K. Yamawaki, "On The Vector Mesons As Dynamical Gauge Bosons Of Hidden Local Symmetries," Nucl. Phys. **B259** (1985) 493.

[8] M. Benayoun and H. B. O'Connell, "SU(3) breaking and hidden local symmetry," Phys. Rev. **D58** (1998) 074006 [hep-ph/9804391].

[9] M. Benayoun, L. DelBuono and H. B. O'Connell, "VMD, the WZW Lagrangian and ChPT: The third mixing angle," hep-ph/9905350.

[10] H. Leutwyler, "On the 1/N-expansion in chiral perturbation theory," Nucl. Phys. Proc. Suppl. **64** (1998) 223 [hep-ph/9709408].

[11] R. Kaiser and H. Leutwyler, "Pseudoscalar decay constants at large N(c)," hep-ph/9806336.

[12] T. Feldmann, "Quark structure of pseudoscalar mesons," Int. J. Mod. Phys. **A15** (2000) 159 [hep-ph/9907491].

[13] E. Witten, "Global Aspects Of Current Algebra," Nucl. Phys. **B223** (1983) 422.
J. Wess and B. Zumino, "Consequences Of Anomalous Ward Identities," Phys. Lett. **B37** (1971) 95.

[14] G. M. Shore and G. Veneziano, "Renormalization group aspects of eta-prime $\to$ gamma gamma," Nucl. Phys. **B381** (1992) 3.

[15] G. M. Shore, "Radiative eta' decays, the topological susceptibility and the Witten-Veneziano mass formula," hep-ph/9908217.

[16] P. J. O'Donnell, "Radiative Decays Of Mesons," Rev. Mod. Phys. **53** (1981) 673.

[17] J.F. Donoghue, E. Golowich, B.R. Holstein, "Dynamics of the Standard Model", Cambridge University Press, Cambridge 1996.

[18] M. Benayoun, H. B. O'Connell and A. G. Williams, "Vector meson dominance and the rho meson," Phys. Rev. **D59** (1999) 074020 [hep-ph/9807537].

[19] R.J. Eden, P.V. Landshoff, D.I. Olive, J.C. Polkinghorne, "The Analytic S–Matrix", Cambridge University Press, Cambridge UK (1966).

[20] G. Morpurgo, "General Parametrization Of The V $\to$ P Gamma Meson Decays," Phys. Rev. **D42** (1990) 1497.
[21] C. Caso *et al.*, "Review of particle physics," Eur. Phys. J. **C3** (1998) 1.
[22] F. Klingl, N. Kaiser and W. Weise, "Effective Lagrangian approach to vector mesons, their structure & decays," Z. Phys. **A356** (1996) 193 [hep-ph/9607431].
[23] Y. C. Shin, B. S. Han, M. K. Cheoun, K. Kim and I. Cheon, "Pion and kaon electromagnetic form factors in a SU(3)LxSU(3)R effective Lagrangian," nucl-th/9912015.
[24] S. Gardner and H. B. O'Connell, "Rho omega mixing and the pion form factor in the time-like region," Phys. Rev. **D57**, 2716 (1998) [hep-ph/9707385].
[25] S. Eidelman, "The rho (770): in Review of Particle Physics (RPP 1998)," Eur. Phys. J. **C3**, 364 (1998).
[26] M. R. Pennington, "Riddle of the scalars: Where is the sigma?" hep-ph/9905241.
[27] M. Benayoun, Ph. Leruste, J. Lancien, "Relativistic Extension of the Le Couteur–Newton formalism I", Il Nuov. Cim. **63A** (1969) 969.
[28] M. Jacob and G.F. Chew, "Strong Interaction Physics", W.A. Benjamin Inc, New York 1964.
[29] R. Escribano and J. M. Frere, "Phenomenological evidence for the energy dependence of the eta eta' mixing angle," Phys. Lett. **B459** (1999) 288 [hep-ph/9901405].
[30] M. Benayoun, P. Leruste, L. Montanet and J. L. Narjoux, "Meson radiative decays and anomaly physics: A Test of QCD," Z. Phys. **C65** (1995) 399.
[31] A. Bramon, A. Grau and G. Pancheri, "Radiative vector meson decays in SU(3) broken effective chiral Lagrangians," Phys. Lett. **B344** (1995) 240.
[32] G. Dillon and G. Morpurgo, "QCD parametrization of the rho gamma, omega gamma and phi gamma couplings: Why f (rho gamma): f (omega gamma) = 3:1 in spite of flavor breaking," Z. Phys. **C64** (1994) 467.
[33] M. A. Pichowsky, S. Walawalkar and S. Capstick, Phys. Rev. **D60** (1999) 054030 [nucl-th/9904079].
[34] N. N. Achasov, A. A. Kozhevnikov and G. N. Shestakov, "On The Possibility Of The Test Of The SU(3) Symmetry Breaking Models In The Reactions E+ E- $\to$ 3 Pi And E+ E- $\to$ Pi0 Gamma," JETP Lett. **21** (1975) 229.
[35] S. I. Dolinsky *et al.*, "Summary of experiments with the neutral detector at the e+ e- storage ring VEPP-2M," Phys. Rept. **202** (1991) 99.
[36] N. N. Achasov and A. A. Kozhevnikov, "Measuring the finite width and unitarity corrections to the Phi omega mixing amplitude," Phys. Rev. **D61** (2000) 054005 [hep-ph/9906520].
[37] I.T. Todorov, "Analytic Properties of Feynmann Diagrams in Quantum Field Theory", Pergamon Press, Oxford UK, 1971.