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Conformal expansions and renormalons*

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Abstract: The coefficients in perturbative expansions in gauge theories are factorially increasing, predominantly due to renormalons. This type of factorial increase is not expected in conformal theories. In QCD conformal relations between observables can be defined in the presence of a perturbative infrared fixed-point. Using the Banks-Zaks expansion we study the effect of the large-order behavior of the perturbative series on the conformal coefficients. We find that in general these coefficients become factorially increasing. However, when the factorial behavior genuinely originates in a renormalon integral, as implied by a postulated skeleton expansion, it does not affect the conformal coefficients. As a consequence, the conformal coefficients will indeed be free of renormalon divergence, in accordance with previous observations concerning the smallness of these coefficients for specific observables. We further show that the correspondence of the BLM method with the skeleton expansion implies a unique scale-setting procedure. The BLM coefficients can be interpreted as the conformal coefficients in the series relating the fixed-point value of the observable with that of the skeleton effective charge. Through the skeleton expansion the relevance of renormalon-free conformal coefficients extends to real-world QCD.

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1 Introduction

The large-order behavior of a perturbative expansion in gauge theories is inevitably dominated by the factorial growth of renormalon diagrams [1, 2, 3, 4]. In the case of quantum chromodynamics (QCD), the coefficients of perturbative expansions in the QCD coupling α_s can increase dramatically even at low orders. This fact, together with the apparent freedom in the choice of renormalization scheme and renormalization scales, limits the predictive power of perturbative calculations, even in applications involving large momentum transfer where α_s is effectively small.

A number of theoretical approaches have been developed to reorganize the perturbative expansions in an effort to improve the predictability of perturbative QCD. For example, optimized scale and scheme choices have been proposed, such as the method of effective charges [ECH] [5], the principle of minimal sensitivity [PMS] [6], and the Brodsky-Lepage-Mackenzie [BLM] scale-setting prescription [7] and its generalizations [8]–[20]. More recent development [4] include resummation of the formally divergent renormalon series and parameterization of related higher-twist power-suppressed contributions.

In general, a factorially divergent renormalon series arises when one integrates over the logarithmically running coupling $\alpha_s(k^2)$ in a loop diagram. Such contributions do not occur in conformally invariant theories, which have a constant coupling. Of course, in the physical theory, the QCD coupling does run. Nevertheless, relying on a postulated "dressed skeleton expansion", we shall show that a conformal series is directly relevant to physical QCD predictions.

In quantum electrodynamics the dressed skeleton expansion can replace the standard perturbative expansion. The skeleton diagrams are defined as those Feynman graphs where the three-point vertex and the lepton and photon propagators have no substructure [21]. Thanks to the QED Ward identity, the renormalization of the vertex cancels against the lepton self-energy, while the effect of dressing the photons in the skeleton diagrams by vacuum polarization insertions can be computed by integrating over the Gell-Mann Low effective charge $\bar{\alpha}(k^2)$. The perturbative coefficients defined from the skeleton graphs themselves are conformal – they correspond to the series in a theory with a zero β function. Therefore they are entirely free of running coupling effects such as renormalons. Each term in the dressed skeleton expansion resums renormalon diagrams to all orders in a renormalization scheme invariant way. The resummation ambiguity, which is associated with scales where the coupling becomes strong, can be resolved only at the non-perturbative level.

In QCD, a skeleton expansion can presumably be constructed based on several different dressed Green functions (see for example [22]). It is yet unclear[§], although much more interesting, whether there exists an Abelian-like skeleton expansion, with only one effective charge function. A diagrammatic construction of such a skeleton expansion using the "pinch technique" [23, 24, 25] has been established only through

[§]The basic difficulties, comparing with the Abelian case, are the presence of gluon self interaction diagrams and the essential difference between vacuum polarization insertion and charge renormalization.

two-loop order. The corresponding skeleton effective charge $\bar{\alpha}_s(k^2)$, which is defined from "vacuum-polarization-like" contributions, has been identified and shown to be gauge invariant. This technique may eventually provide a definition to all orders. In this paper, we shall simply postulate that an Abelian-like skeleton expansion can be defined at arbitrary order in QCD. As in QED, we can then identify running coupling effects to all orders in perturbation theory, and treat them separately from the conformal part of the perturbative expansion.

The conformal coefficients which appear in the assumed skeleton expansion are free of renormalons and are therefore expected to be better behaved. They also have a simple interpretation in the presence of a perturbative infrared fixed-point, as may occur in multi-flavor QCD: they are the coefficients in the series relating the fixed-point value of the observable under consideration with that of the skeleton effective charge. As a consequence, these coefficients can be obtained from the standard perturbative coefficients using the Banks-Zaks expansion [28, 29], where the fixed-point coupling is expanded in powers of β_0.

The conformal series can be seen as a template [9] for *physical* QCD predictions, where instead of the fixed coupling one has at each order a weighted average of the skeleton effective charge $\bar{\alpha}_s(k^2)$ with respect to an observable- (and order-) dependent momentum distribution function. The momentum integral corresponding to each skeleton term is renormalization-scheme invariant. It can be evaluated up to power-suppressed ambiguities, which originate in the infrared and are resolved by taking explicitly non-perturbative effects into account. Thus the skeleton expansion gives a natural framework in which renormalon resummation and the analysis of non-perturbative power corrections are performed together [26, 27].

As an alternative to evaluating the dressed skeleton integral, one can approximate it by the coupling at the BLM scale [7], in analogy to the mean-value theorem [11]. By going to higher orders in the perturbative expansion, this approximation can be systematically improved, although it is not yet clear how to deal with renormalon ambiguities and power-corrections in this approach. It is useful to form QCD predictions by relating the effective charges of one physical observable to another at their respective scales. These "commensurate scale relations" [12] can be obtained by algebraically eliminating the intermediate skeleton effective charge. The coefficients of the perturbative series for such commensurate scale relations are again conformal coefficients, as guaranteed by the transitivity property of the renormalization group. Thus we can once more use the conformal theory as a template for the perturbative expansion relating any two observables in QCD. The effect of the remaining non-conformal contributions, including the renormalon ambiguity, is shifted into the scales of the QCD coupling. In the case of the Crewther relation [30, 31, 18], which connects the effective charges of the e^+e^- annihilation cross section to the Bjorken and Gross-Llewellyn Smith sum rules for deep inelastic scattering, the conformal series is simply a geometric series. This example highlights the power of characterizing QCD perturbative expansions in terms of conformal coefficients.

The main purpose of this work is to study the consequences of the assumed Abelian-like skeleton expansion. We therefore start in section 2 by recalling the concept of the skeleton expansion in the Abelian case [21] and stating the main assumptions concerning the non-Abelian case. We continue, in section 3, by reviewing the standard BLM scale-setting procedure and recalling the ambiguity of the procedure beyond the next-to-leading order. We then show how this ambiguity is resolved upon assuming a skeleton expansion, provided we work in the appropriate renormalization scheme, the "skeleton scheme", and require a one-to-one correspondence between the terms in the BLM series and the dressed skeletons. We then concentrate (section 4) on the coefficients which remain after performing BLM scale-setting. We derive a relation between these BLM coefficients and the conformal coefficients defined in the infrared limit in the conformal window, where a non-trivial perturbative fixed-point exists [32]–[36].

Having made the connection with the conformal coefficients, we recall in section 5 the standard way to calculate such coefficients, namely the Banks-Zaks expansion. We also present there an alternative derivation which makes use of the explicit log-structure in the perturbative series. In sections 6 and 7 we investigate whether conformal coefficients are affected by the factorial increase of the perturbative coefficients. We know that renormalons arise due to the running coupling, and thus conformal expansions should be free of renormalons. On the other hand, conformal coefficients correspond to specific combinations of the perturbative coefficients, and thus it is non-trivial how the former can be free of renormalons when the latter are dominated by them. In section 6 we study simple examples of a single Borel pole or Borel cut which serve as models for the large-order behavior characterizing renormalons. We find that, in general, in these examples conformal coefficients do become factorially increasing. In section 7 we show that assuming a skeleton expansion conformal coefficients are, almost by definition, renormalon-free. We then construct a more specific example to contrast with section 6, where the coefficients are generated by a renormalon integral. We show how the renormalons are cancelled in the corresponding conformal relation.

In section 8 we make the connection between our general arguments and previous observations concerning the smallness of conformal and Banks-Zaks coefficients. In section 9 we look at the effective charge approach from the point of view of the skeleton expansion and present a simple relation between the two at the next-to-next-to-leading order level (β_2). We also calculate there the $\beta_0 = 0$ limit of the skeleton coupling β function coefficient $\bar{\beta}_2$. The conclusions are given in section 10.

2 Renormalons and the skeleton expansion

Consider a Euclidean QED observable $a_R(Q^2)$, which depends on a single external space-like momentum Q^2 and is normalized as an effective charge. The perturbative expansion in a generic renormalization scheme is then given by,

$$a_R(Q^2) = a(\mu^2) + r_1 a(\mu^2)^2 + r_2 a(\mu^2)^3 + \cdots, \tag{1}$$

where $a = \alpha/\pi$ and μ is the renormalization scale.

The perturbative series can be reorganized and written in the form of a skeleton expansion

$$a_R(Q^2) = R_0(Q^2) + s_1 R_1(Q^2) + s_2 R_2(Q^2) + \cdots, \tag{2}$$

where the first term, R_0, corresponds to a single dressed photon: it is the infinite set of "renormalon diagrams" obtained by all possible vacuum polarization insertions into a single photon line. The second term, $s_1 R_1$, corresponds to a double dressed-photon exchange and so on. In QED, vacuum polarization insertions amount to charge renormalization. Thus R_0 can be written as

$$R_0(Q^2) \equiv \int_0^\infty \bar{a}(k^2)\, \phi_0\left(k^2/Q^2\right) \frac{dk^2}{k^2} \tag{3}$$

where k^2 is the virtuality of the exchanged photon, $\bar{a}(k^2)$ is the Gell-Mann Low effective charge representing the full propagator, and ϕ_0 is the (observable dependent) Feynman integrand for a single photon exchange diagram, which is interpreted as the photon momentum distribution function [14]. Similarly, R_1 is given by

$$R_1(Q^2) \equiv \int_0^\infty \bar{a}(k_1^2)\, \bar{a}(k_2^2)\, \phi_1\left(k_1^2/Q^2, k_2^2/Q^2\right) \frac{dk_1^2}{k_1^2} \frac{dk_2^2}{k_2^2} \tag{4}$$

and so on.

For convenience the normalization of ϕ_i in $R_i(Q^2)$ has been set to 1 such that the $R_i(Q^2)$ in (2) have an expansion $R_i(Q^2) = \bar{a}(Q^2)^{i+1} + \cdots$. For example, the normalization of $\phi_0(k^2/Q^2)$ in R_0 is

$$\int_0^\infty \phi_0\left(k^2/Q^2\right) \frac{dk^2}{k^2} = 1. \tag{5}$$

In QED fermion loops appear either dressing the exchanged photons or in light-by-light type diagrams, where they are attached to four or more photons (an even number). Barring the latter, the dependence on the number of massless fermion flavors N_f is fully contained in the Gell-Mann Low effective charge. It follows that the skeleton coefficients s_i as well as the momentum distribution functions ϕ_i are entirely free of N_f dependence. Light-by-light type diagrams have to be treated separately, as the starting point of new skeleton structures.

The skeleton expansion (2) is a renormalization group invariant expansion: each term is by itself scheme invariant. This is in contrast with the standard scale and scheme dependent perturbative expansion (1). The renormalons in (1) can be obtained upon expansion of the dressed skeleton terms in (2) in some scheme. Let us consider first the leading skeleton (3) and examine, for simplicity, its expansion in $\bar{a}(Q^2)$. We assume that $\bar{a}(k^2)$ obeys the renormalization group equation,

$$\frac{d\bar{a}(k^2)}{d\ln k^2} = -\left(\beta_0 \bar{a}(k^2)^2 + \beta_1 \bar{a}(k^2)^3 + \bar{\beta}_2 \bar{a}(k^2)^4 + \cdots\right) \equiv \bar{\beta}(\bar{a}) \tag{6}$$

where β_0 is negative in QED and positive in QCD. Then $\bar{a}(k^2)$ can be expanded as

$$\bar{a}(k^2) = \bar{a}(Q^2) + \beta_0 t \bar{a}(Q^2)^2 + \left(\beta_1 t + \beta_0^2 t^2\right) \bar{a}(Q^2)^3 + \left(\bar{\beta}_2 t + \frac{5}{2}\beta_1 \beta_0 t^2 + \beta_0^3 t^3\right) \bar{a}(Q^2)^4 + \cdots \tag{7}$$

where $t \equiv -\ln(k^2/Q^2)$. Inserting this in eq. (3) under the integration sign we obtain

$$\begin{aligned} R_0(Q^2) = \bar{a}(Q^2) \quad &+ \quad r_1^{(1)}\beta_0 \bar{a}(Q^2)^2 + \left(r_2^{(2)}\beta_0^2 + r_1^{(1)}\beta_1\right)\bar{a}(Q^2)^3 \\ &+ \quad \left(r_3^{(3)}\beta_0^3 + \frac{5}{2}r_2^{(2)}\beta_1\beta_0 + r_1^{(1)}\bar{\beta}_2\right)\bar{a}(Q^2)^4 + \cdots \end{aligned} \tag{8}$$

where

$$r_i^{(i)} \equiv \int_0^\infty \left[-\ln\left(k^2/Q^2\right)\right]^i \phi_0(k^2/Q^2)\,\frac{dk^2}{k^2}. \tag{9}$$

We note that in the large N_f (large β_0) limit*, the perturbative coefficients $r_i = r_i^{(i)}$ and thus†

$$a_R(Q^2)\Big|_{\text{large }\beta_0} = \bar{a}(Q^2)\left[\sum_{i=0}^{\infty} r_i^{(i)}\left(\beta_0\bar{a}(Q^2)\right)^i + \mathcal{O}\left(1/\beta_0\right)\right]. \tag{10}$$

At large orders $i \gg 1$, both small and large momentum regions become dominant in (9), giving rise to the characteristic renormalon factorial divergence ($r_i^{(i)} \sim i!$). As mentioned above, this is believed to be the dominant source of divergence of the perturbative expansion (1). On the other hand, in the skeleton expansion (2) the renormalons are by definition resummed and so the remaining coefficients s_i should be free of this divergence. These coefficients are expected to increase much slower leading to a better behaved expansion.

As mentioned in the introduction, the generalization of the Abelian skeleton expansion to QCD is not straightforward. Diagrammatically, the skeleton expansion in QCD has a simple realization only in the large N_f limit where gluon self-interaction contributions are negligible so that the theory resembles QED‡. In the framework of renormalon calculus, one returns from the large N_f limit to real world QCD by replacing N_f with the linear combination of N_f and N_c which appears in the leading coefficient [38] of the β function,

$$\beta_0 = \frac{1}{4}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right). \tag{11}$$

This replacement, usually called "naive non-Abelianization" [39, 14, 15, 16], amounts to taking into account a gauge invariant set of diagrams which is responsible for the one-loop running of the coupling constant. To go beyond the "naive non-Abelianization" level and construct an Abelian-like skeleton expansion in QCD, one needs a method to

*In QCD, Abelian correspondence in the large N_f limit requires that the coefficient $\bar{\beta}_i$ of the skeleton coupling β function (6) would not contain N_f^{i+1}. It has to be a polynomial of order N_f^i in N_f. This would guarantee that in the large N_f limit $\bar{\beta}(\bar{a})$ is just the one-loop β function. Note that while some schemes (e.g. $\overline{\text{MS}}$ and static potential effective charge) have this property, generic effective charges (defined through observable quantities) do not. This property of the skeleton scheme is used making the identification of $r_i^{(i)}$ in (8) as the large N_f coefficients.

†We comment that the sub-leading terms in $1/N_f$ in (8) of the form $\beta_1\beta_0^{i-2}$ were computed to all-orders in [15]. However, other terms which involve higher order coefficients of the β function contribute at the same level in $1/N_f$.

‡This can also be understood from the $N_c \to 0$ limit discussed in ref. [37].

identify skeleton structures and to isolate vacuum-polarization-like insertions which are responsible for the running of the coupling at any order. The pinch technique [23, 24, 25] may provide a systematic way to make this identification. The resulting set of skeleton structures would surely be larger than in the Abelian theory. It may include, for example, fermion loops attached to an odd number of gluons, which vanish in the Abelian limit. Like Abelian light-by-light type diagrams, these structures should be treated separately. As opposed to the Abelian theory, where light-by-light type diagrams are distinguished by their characteristic dependence on the charges, in the non-Abelian case these structures may not be separable based only on their group structure. We shall assume that there is a unique way to identify skeleton structures in QCD and a gauge invariant way to "dress" them corresponding to the skeleton effective charge. Then, upon excluding specific classes of diagrams, e.g. of the type described above, we expect the form of eq. (2) with N_f independent s_i and ϕ_i to be relevant to QCD.

We stress that the coupling constant $\bar{a}(k^2)$ in (3) is understood to be a *specific* effective charge, in analogy to the Gell-Mann Low effective charge in QED. This "skeleton effective charge" $\bar{a}(k^2)$ should be defined diagrammatically order by order in perturbation theory. In the framework of the pinch technique, $\bar{a}(k^2)$ has been identified at the one-loop level[§], e.g. it is related to the $\overline{\mathrm{MS}}$ coupling by

$$\bar{a}(k^2) = a_{\overline{\mathrm{MS}}}(\mu^2) + \left[-\beta_0\left(\log\frac{k^2}{\mu^2} - \frac{5}{3}\right) + \frac{N_c}{3}\right] a_{\overline{\mathrm{MS}}}(\mu^2)^2 + \cdots \tag{12}$$

Recently, there have been encouraging developments [25] in the application of the pinch technique beyond one-loop. This would hopefully lead to a systematic identification of the "skeleton effective charge" at higher orders, namely the determination of higher order coefficients ($\bar{\beta}_i$ for $i \geq 2$) of the β function $\bar{\beta}(\bar{a}) = d\bar{a}/d\ln k^2$. This β function should coincide with the Gell-Mann Low function upon taking the Abelian limit $C_A = 0$ (see ref. [37]).

Being scheme invariant and free of renormalon divergence, the skeleton expansion (2) seems much favorable over the standard perturbative QCD expansion (1). This advantage may become crucial in certain applications, e.g. for the extraction of α_s from event shape variables [27]. However, in the absence of a concrete all-order diagrammatic definition for the skeleton expansion in QCD, the use of it directly as a calculational tool is limited to the leading skeleton term. On the other hand, the BLM scale-setting procedure, which is well defined up to arbitrary large order, can be considered as a manifestation of the skeleton expansion. As we shall see, it is possible in this framework to study sub-leading terms, which carry the correct normalization of sub-leading terms in the skeleton expansion, provided the skeleton scheme is used. Currently, since the skeleton effective charge has not been identified, the choice of scheme in the BLM procedure remains an additional essential ingredient.

[§]This means that the corresponding QCD scale $\bar{\Lambda}$ is identified.

3 BLM scale-setting

The BLM approach [7] is motivated by the skeleton expansion. The basic idea is that the dressed skeleton integral (3) can be well approximated by $R_0 \simeq \bar{a}(\mu^2) + \cdots$ provided that the renormalization scale μ is properly chosen. Indeed, by the mean value theorem [11], there exists a scale k_0 such that

$$R_0(Q^2) = \int_0^\infty \bar{a}(k^2)\,\phi_0\left(k^2/Q^2\right)\frac{dk^2}{k^2} = \bar{a}(k_0^2)\int_0^\infty \phi_0(k^2/Q^2)\frac{dk^2}{k^2} = \bar{a}(k_0^2) \tag{13}$$

where the last step follows from the assumed normalization for ϕ_i (5).

A first approximation to k_0 is given by the *average virtuality of the exchanged gluon*,

$$k_{0,0}^2 = Q^2 \exp\left(\int_0^\infty \ln\frac{k^2}{Q^2}\phi_0(k^2/Q^2)\frac{dk^2}{k^2} \Big/ \int_0^\infty \phi_0(k^2/Q^2)\frac{dk^2}{k^2}\right) = Q^2\exp\left(-r_1^{(1)}\right) \tag{14}$$

where $r_1^{(1)}$ is the next-to-leading coefficient of a_R in the large β_0 limit (9). The scale (14) is called the "leading order BLM scale". It can be determined directly from the N_f dependent part of the next-to-leading coefficient (r_1) in the perturbative series of the observable in terms of $\bar{a}(Q^2)$,

$$a_R(Q^2) = \bar{a}(Q^2) + r_1\bar{a}(Q^2)^2 + r_2\bar{a}(Q^2)^3 + \cdots. \tag{15}$$

Thanks to the linear N_f dependence of r_1, it can be uniquely decomposed into a term linear in β_0, which is related to the leading skeleton, and a free term

$$r_1 = r_1^{(0)} + r_1^{(1)}\beta_0, \tag{16}$$

where both $r_1^{(1)}$ and $r_1^{(0)}$ are N_f independent. After BLM scale-setting, with $k_{0,0}^2$ given by (14), one has

$$a_R(Q^2) = \bar{a}(k_{0,0}^2) + r_1^{(0)}\bar{a}(k_{0,0}^2)^2 + \cdots. \tag{17}$$

Thus, technically, the BLM scale-setting procedure amounts, at leading order, to eliminating the β_0 dependent part from the next-to-leading order coefficient. Note that although the leading order BLM scale $k_{0,0}$ of (14) has a *precise* meaning as the average gluon virtuality it is just the lowest order approximation to k_0 of eq. (13). In other words, aiming at the evaluation of the leading skeleton term (3) it is just the first step. Based on higher orders in the perturbative expansion this approximation can be systematically improved (see eq. (33) below).

3.1 Multi-scale BLM and skeleton expansion correspondence

A BLM series [12] can be written up to arbitrary high order

$$a_R(Q^2) = a(k_0^2) + c_1 a(k_1^2)^2 + c_2 a(k_2^2)^3 + c_3 a(k_3^2)^4 + \cdots \tag{18}$$

where k_i^2 are, in general, different scales proportional to the external scale Q^2 (as in (14)) and c_i are N_f independent coefficients. The intuition behind this generalization is that each skeleton term in (2) is approximated by a corresponding term in the multi-scale BLM series: each skeleton term may have different characteristic momenta. This one-to-one correspondence with the skeleton expansion requires that the coupling a will be the skeleton effective charge $a = \bar{a}$ such that

$$R_i(Q^2) = \bar{a}(k_i^2)^{i+1}. \tag{19}$$

In this case the coefficients of sub-leading terms in (18) should coincide with the coefficients of the sub-leading skeleton terms, namely $c_i = s_i$.

More generally, a BLM series can be written in an arbitrary scheme: then the coupling a in (18) can be either defined in a standard scheme like $\overline{\mathrm{MS}}$ or, as suggested in [12], be another measurable effective charge. In the latter case, (18) can be used to compare experimental data of two observables directly and thus test perturbative QCD without any intermediate renormalization scheme.

Let us recall how the BLM scale-setting procedure is performed beyond the next-to-leading order [12, 8], yielding an expansion of the form (18). Suppose that the perturbative expansion of $a_R(Q^2)$ in terms of $a(Q^2)$ is given by¶

$$a_R(Q^2) = a(Q^2) + r_1 a(Q^2)^2 + r_2 a(Q^2)^3 + r_3 a(Q^2)^4 + \cdots \tag{20}$$

Based on the fact that r_i are polynomials of order i in N_f and that β_0 and β_1 are linear in N_f, we can write r_1 as in (16) and

$$r_2 = r_2^{(0)} + r_2^{(1)} r_1^{(0)} \beta_0 + r_2^{(2)} \beta_0^2 + r_1^{(1)} \beta_1 \tag{21}$$

where $r_i^{(j)}$ are N_f independent. The reason for the β_1 dependent term in (21) shall become clear below. Expanding $a(k_i^2)$ in terms of $a(Q^2)$ similarly to eq. (7), the next-to-next-to-leading order BLM series (18) can be written as

$$a_R(Q^2) = a(Q^2) + (c_1 + t_0\beta_0)\, a(Q^2)^2 + \left(c_2 + 2t_1 c_1 \beta_0 + t_0\beta_1 + t_0^2\beta_0^2\right) a(Q^2)^3. \tag{22}$$

Writing the scale-shifts $t_i \equiv \ln(Q^2/k_i^2)$ as a power series in the coupling

$$t_i \;\equiv\; t_{i,0} \;+\; t_{i,1}\, a(Q^2) +\; t_{i,2}\, a(Q^2)^2 + \cdots \tag{23}$$

where $t_{i,0}$ are assumed to be N_f independent, we get

$$\begin{aligned} a_R(Q^2) \;&=\; a(Q^2) + (c_1 + t_{0,0}\beta_0)\, a(Q^2)^2 \\ &+\; \left(c_2 + (2t_{1,0}c_1 + t_{0,1})\beta_0 + t_{0,0}\beta_1 + t_{0,0}^2\beta_0^2\right) a(Q^2)^3. \end{aligned} \tag{24}$$

¶We work now in a generic scheme but at a difference with (1) we start here with the renormalization scale $\mu = Q$ thereby simplifying the formulas that follow. Since the scale is tuned in the BLM procedure, this initial choice is of little significance. The only place where the arbitrary renormalization scale is left at the end is in the power series for the scales-shifts, eq. (23) below.

An order by order comparison of (24) and (20) yields the scale shifts $t_0 = \ln(Q^2/k_0^2)$ and $t_1 = \ln(Q^2/k_1^2)$ and the coefficients c_1 and c_2 in terms of r_1 and r_2 and the coefficients of the β function of $a(Q^2)$. The comparison at the next-to-leading order gives

$$c_1 = r_1^{(0)} \tag{25}$$

and

$$t_{0,0} = r_1^{(1)}. \tag{26}$$

The comparison at the next-to-next-to-leading order for the β_i independent piece gives

$$c_2 = r_2^{(0)} \tag{27}$$

while for the β_0 dependent piece it gives

$$t_{0,1} + 2t_{1,0}r_1^{(0)} + \beta_0 \left(r_1^{(1)}\right)^2 = r_2^{(1)}r_1^{(0)} + \beta_0 r_2^{(2)}. \tag{28}$$

Thanks to the explicit β_1 dependent term introduced in (21), the equality of the corresponding piece there to that in (24) is satisfied based on the next-to-leading order result (26). To proceed we need to specify $t_{0,1}$ and $t_{1,0}$ such that eq. (28) is satisfied. Having two free parameters with just one constraint there is apparently no unique solution. Two natural possibilities are the so called multi-scale BLM prescription [12],

$$\begin{aligned} t_{0,1} &= \beta_0 \left[r_2^{(2)} - \left(r_1^{(1)}\right)^2\right] \\ t_{1,0} &= \frac{1}{2} r_2^{(1)} \end{aligned} \tag{29}$$

and the single-scale BLM prescription [8] where $t_{1,0} \equiv t_{0,0}$ and

$$t_{0,1} = \beta_0 \left[r_2^{(2)} - \left(r_1^{(1)}\right)^2\right] - 2r_1^{(1)}r_1^{(0)} + r_2^{(1)}r_1^{(0)}. \tag{30}$$

Having in mind the original motivation for BLM, it is interesting to examine the case where the scheme of a coincides with the skeleton effective charge $\bar{a}$. Then we would like to have a one-to-one correspondence (19) between the terms in the BLM series (18) and those of the skeleton expansion (2). The multi-scale procedure is consistent with this requirement: the leading term $\bar{a}(k_0^2)$ in the BLM series (18) represents only the leading skeleton term R_0 in (2), since the scale-shift

$$t_0 = r_1^{(1)} + \left[r_2^{(2)} - \left(r_1^{(1)}\right)^2\right] \beta_0 \bar{a}(Q^2) \tag{31}$$

involves only coefficients which are leading in the large β_0 limit and originate in ϕ_0 (cf. eq. (9)). On the other hand the single-scale procedure violates this requirement, since there t_0 involves (30) terms which are sub-leading in β_0 and *do not belong* to the leading skeleton term R_0. In fact, in order to guarantee that the scale-shift t_0

would represent just the leading skeleton R_0 we are bound to choose $t_{0,1}$ proportional to β_0 and thus the solution (29) is uniquely determined.

We see that a unique scale-setting procedure at the next-to-next-to-leading order (r_2) is implied by the requirement that the scale-shift t_0 should represent the leading skeleton R_0. In order to continue and apply BLM at the next order (r_3) we have to impose further constraints based on the structure of both R_0 and R_1.

3.2 BLM scale-setting for the leading skeleton

Let us first examine the structure of the scale-shift t_0 by applying BLM to a hypothetical observable that contains only an R_0 term of the form (3). Expanding the coupling $\bar{a}(k^2)$ under the integration sign in terms of $a(Q^2)$ we obtain (8). We would like to apply BLM to the latter series obtaining simply $\bar{a}(k_0^2)$, with $t_0 \equiv \ln(Q^2/k_0^2) = t_{0,0} + t_{0,1}\bar{a}(Q^2) + \cdots$. Expanding $\bar{a}(k_0^2)$ we obtain from (7),

$$\begin{aligned} \bar{a}(k_0^2) &= \bar{a}(Q^2) + \beta_0 t_{0,0}\bar{a}(Q^2)^2 + \left(\beta_0 t_{0,1} + \beta_1 t_{0,0} + \beta_0^2 t_{0,0}^2\right)\bar{a}(Q^2)^3 \\ &+ \left(\beta_0 t_{0,2} + \beta_1 t_{0,1} + 2\beta_0^2 t_{0,0}t_{0,1} + \bar{\beta}_2 t_{0,0} + \beta_0^3 t_{0,0}^3 + \frac{5}{2}\beta_0\beta_1 t_{0,0}^2\right)\bar{a}(Q^2)^4 + \cdots \end{aligned} \tag{32}$$

Comparing (8) with (32) we get

$$\begin{aligned} t_0 &= r_1^{(1)} + \left[r_2^{(2)} - \left(r_1^{(1)}\right)^2\right]\beta_0\bar{a}(Q^2) \\ &+ \left\{\left[r_3^{(3)} - 2r_1^{(1)}r_2^{(2)} + \left(r_1^{(1)}\right)^3\right]\beta_0^2 + \frac{3}{2}\left[r_2^{(2)} - \left(r_1^{(1)}\right)^2\right]\beta_1\right\}\bar{a}(Q^2)^2 + \cdots \end{aligned} \tag{33}$$

Here we recovered the two leading orders in t_0 of eq. (31). At order $\bar{a}(Q^2)^2$ we obtained an explicit dependence on both β_0 and β_1. The combination $r_2^{(2)} - \left(r_1^{(1)}\right)^2$ appearing at the next-to-leading order in t_0 has an interpretation as the width of the distribution ϕ_0, assuming the latter is positive definite (see [14, 17]). In general, eq. (33) can be written in terms of central moments of the distribution ϕ_0, defined by

$$M_n = \left\langle\left(\ln\frac{Q^2}{k^2} - \left\langle\ln\frac{Q^2}{k^2}\right\rangle_{\phi_0}\right)^n\right\rangle_{\phi_0} = \left\langle\left(\ln\frac{k_{0,0}^2}{k^2}\right)^n\right\rangle_{\phi_0} \tag{34}$$

for $n \geq 2$, where $M_1 = \left\langle\ln\frac{Q^2}{k^2}\right\rangle_{\phi_0} = \ln\frac{Q^2}{k_{0,0}^2}$ corresponds to $r_1^{(1)}$ in eq. (9). In terms of the central moments we have

$$\begin{aligned} t_0 &= M_1 + M_2\beta_0\,\bar{a}(Q^2) + \left\{\left[M_3 + M_1M_2\right]\beta_0^2 + \frac{3}{2}M_1\beta_1\right\}\bar{a}(Q^2)^2 + \cdots \\ &= M_1 + M_2\beta_0\,\bar{a}(k_{0,0}^2) + \left\{M_3\beta_0^2 + \frac{3}{2}M_1\beta_1\right\}\bar{a}(k_{0,0}^2)^2 + \cdots \end{aligned} \tag{35}$$

At large orders n the moments M_n become sensitive to extremely large and small momenta and thus develop renormalon factorial divergence, similarly to the standard perturbative coefficients in eq. (9). We thus see that in the BLM approach, the scale-shift itself is an asymptotic expansion, affected by renormalons.

3.3 BLM scale-setting for sub-leading skeletons

Next, let us consider an R_1 term, given by (4). Expanding the couplings $\bar{a}(k_1^2)$ and $\bar{a}(k_2^2)$ under the integral in terms of $\bar{a}(Q^2)$ using (7), we get (cf. the expansion of R_0 in eq. (8))

$$R_1(Q^2) = \bar{a}(Q^2)^2 + \beta_0 r_2^{(1)} \bar{a}(Q^2)^3 + \left(r_3^{(2)}\beta_0^2 + r_2^{(1)}\beta_1\right)\bar{a}(Q^2)^4 + \cdots \tag{36}$$

where

$$\begin{aligned} r_2^{(1)} &\equiv \phi_1^{(1,0)} + \phi_1^{(0,1)} \\ r_3^{(2)} &\equiv \phi_1^{(2,0)} + \phi_1^{(1,1)} + \phi_1^{(0,2)} \end{aligned} \tag{37}$$

with

$$\phi_1^{(j,k)} \equiv \int_0^\infty \left[-\ln(k_1^2/Q^2)\right]^j \left[-\ln(k_2^2/Q^2)\right]^k \phi_1\left(k_1^2/Q^2, k_2^2/Q^2\right)\frac{dk_1^2}{k_1^2}\frac{dk_2^2}{k_2^2}. \tag{38}$$

The BLM scale-setting procedure can now be applied according to (19): $R_1(Q^2)$ given in eq. (36) should be written as $\bar{a}(k_1^2)^2$. Expanding $\bar{a}(k_1^2)^2$ in terms of $\bar{a}(Q^2)$ using (7) and $t_1 = t_{1,0} + t_{1,1}\bar{a}(Q^2) + \cdots$ we have

$$\bar{a}(k_1^2)^2 = \bar{a}(Q^2)^2 + 2t_{1,0}\beta_0\bar{a}(Q^2)^3 + \left(2t_{1,1}\beta_0 + 3t_{1,0}^2\beta_0^2 + 2t_{1,0}\beta_1\right)\bar{a}(Q^2)^4 + \cdots. \tag{39}$$

The comparison with (36) at the next-to-leading order implies

$$t_{1,0} = \frac{1}{2} r_2^{(1)}. \tag{40}$$

The comparison at the next-to-next-to-leading order then yields

$$2t_{1,1}\beta_0 + \frac{3}{4}\left(r_2^{(1)}\right)^2\beta_0^2 + r_2^{(1)}\beta_1 = r_3^{(2)}\beta_0^2 + r_2^{(1)}\beta_1 \tag{41}$$

which implies that $t_{1,1}$, just as $t_{0,1}$, is bound to be proportional to β_0. Finally we obtain the scale-shift for R_1,

$$t_1 = \frac{1}{2} r_2^{(1)} + \frac{1}{2}\left[r_3^{(2)} - \frac{3}{4}\left(r_2^{(1)}\right)^2\right]\beta_0\bar{a}(Q^2). \tag{42}$$

Similarly, applying BLM to R_2,

$$R_2 = \bar{a}(Q^2)^3 + r_3^{(1)}\beta_0\bar{a}(Q^2)^4 + \cdots, \tag{43}$$

we get

$$t_2 = \frac{1}{3} r_3^{(1)}. \tag{44}$$

3.4 Skeleton decomposition and its limitations

Let us now return to the case of a generic observable (20) and see that with these skeleton-expansion-correspondence constraints there is a unique BLM scale-setting procedure. The basic idea is that, given the existence of a skeleton expansion, it is possible to separate the entire series into terms which originate in specific skeleton terms. This corresponds to a specific decomposition of each perturbative coefficient r_i similarly to (16) and (21). Then the application of BLM to the separate skeleton terms, namely representing R_i by $\bar{a}(k_i^2)^{i+1}$, immediately implies a specific BLM scale-setting procedure for the observable. For example, when this procedure is applied up to order $\bar{a}(Q^2)^4$, the scale-shifts t_i for $i = 0, 1, 2$ are given by (33), (42) and (44), respectively.

To demonstrate this argument let us simply add up the expanded form of the skeleton terms up to order $\bar{a}(Q^2)^4$ with R_0 given by (8), R_1 by (36) and R_2 by (43). For R_3 we simply have at this order $R_3 = \bar{a}(Q^2)^4$. Altogether we obtain,

$$\begin{aligned} a_R &= \bar{a} + \left[s_1 + r_1^{(1)}\beta_0\right]\bar{a}^2 \\ &+ \left[s_2 + s_1 r_2^{(1)}\beta_0 + r_2^{(2)}\beta_0^2 + r_1^{(1)}\beta_1\right]\bar{a}^3 \\ &+ \left[s_3 + s_2 r_3^{(1)}\beta_0 + s_1 r_3^{(2)}\beta_0^2 + r_3^{(3)}\beta_0^3 + r_1^{(1)}\bar{\beta}_2 + \frac{5}{2} r_2^{(2)}\beta_1\beta_0 + s_1 r_2^{(1)}\beta_1\right]\bar{a}^4 \end{aligned} \tag{45}$$

Here we identify the notation s_i which is the coefficient in front of the skeleton term R_i with $r_i^{(0)}$. We recognize the form of r_1 and r_2 as the decompositions introduced before in eq. (16) and (21) in order to facilitate the application of BLM. We see that the skeleton expansion structure implies a specific decomposition. Suppose for example we know r_1 through r_3 in the skeleton scheme. Eq. (45) then defines a unique way to decompose them so that each term corresponds specifically to a given term in the skeleton expansion. The decomposition of r_i includes a polynomial in β_0 up to order β_0^i,

$$s_i + \sum_{k=1}^{i} s_{i-k} r_i^{(k)} \beta_0^k \tag{46}$$

where $s_0 = 1$ by the assumed normalization. The other terms in r_i in (45) depend explicitly on higher coefficients of the β function $\bar{\beta}_j$ with $1 \le j \le i-1$. Up to order $\bar{a}(Q^2)^4$ these terms depend exclusively[‖] on coefficients $r_j^{(k)}$ which appeared at previous orders in the β_0 polynomials (46). Finally, we need to verify that a decomposition of the form (45) is indeed possible. For a generic observable a_R, the coefficient r_i is a polynomial of order i in N_f. Since also the β function coefficients $\bar{\beta}_i$ are polynomials of maximal order i, the decomposition of r_i according to (45) amounts to solving $i+1$ equations with $i+1$ unknowns: $r_i^{(k)}$ with $0 \le k < i$. Thus in general there is a unique solution.

[‖]As we shall see below, this is no longer true beyond this order, where the coefficients depend on moments which appeared at previous orders, but cannot be expressed in terms of the lower order coefficients themselves.

We see that based on the assumed skeleton structure, one can uniquely perform a "skeleton decomposition" and thus also BLM scale-setting which satisfies a one-to-one correspondence of the form (19) with the skeleton terms. By construction in this procedure the scale t_0 is determined exclusively by the large β_0 terms $r_i^{(i)}$ which belong to R_0 (see (33)), t_1 is determined by $r_i^{(i-1)}$ terms which belong to R_1 (see (42)), t_2 is determined by $r_i^{(i-2)}$ terms which belong to R_2, etc.

It should be stressed that *formally* the decomposition (45), and thus also BLM scale-setting, can be performed in any scheme: given the coefficients r_i up to order n, all the coefficients s_i and $r_i^{(j)}$ for $i \leq n$ and $j \leq i$ are uniquely determined. No special properties of the "skeleton effective charge" were necessary to show that the decomposition is possible. Even the assumption that for this effective charge the β function coefficients $\bar{\beta}_i$ are polynomials of order i can be relaxed. For example, the decomposition (45) can be formally performed in physical schemes where $\bar{\beta}_i$ are polynomials of order $i+1$. In this case, however, the interpretation of $r_i^{(j)}$ in terms of the log-moments of distribution functions is not straightforward. It is also clear that a one-to-one correspondence between BLM and the skeleton expansion (19) exists only if the coupling a is chosen as the skeleton effective charge $\bar{a}$.

Let us now address several complications that limit the applicability of the discussion above. First, we recall the assumption we made that the entire dependence of the perturbative coefficients on N_f is related to the running coupling. This means that any explicit N_f dependence which is part of the skeleton structure is excluded from (45). In reality there may be skeletons with fermion loops as part of the structure, which would have to be identified and treated separately.

Having excluded such N_f dependence, we have seen that up to order $\bar{a}(Q^2)^4$ a formal "skeleton decomposition" (45) of the perturbative coefficients can be performed algebraically without further diagrammatic identification of the skeleton structure. This is no longer true at order $\bar{a}(Q^2)^5$, where the "skeleton decomposition" requires the moments of the momentum distribution functions to be identified separately. Such an identification depends on a diagrammatic understanding of the skeleton structure. Looking at R_1, the coefficient of $\bar{a}(Q^2)^5$ in eq. (36) is

$$\begin{aligned} &\beta_0^3 \left[\phi_1^{(3,0)} + \phi_1^{(0,3)} + \phi_1^{(1,2)} + \phi_1^{(2,1)}\right] \\ &+ \quad \beta_1\beta_0 \left[2\phi_1^{(1,1)} + \frac{5}{2}\left(\phi_1^{(2,0)} + \phi_1^{(0,2)}\right)\right] + \beta_2\left[\phi_1^{(1,0)} + \phi_1^{(0,1)}\right]. \end{aligned} \tag{47}$$

Writing the $\bar{a}(Q^2)^5$ term in (45), one will find as before, that the terms which depend explicitly on higher coefficients of the β function $\bar{\beta}_l$ with $1 \leq l \leq 3$, contain only moments of the skeleton momentum distribution functions $\phi_i^{(j,k)}$ which appeared in the decomposition (45) in the coefficients of $\beta_0^{j+k}\,\bar{a}^{1+i+j+k}$ at the previous orders. However, the coefficient of $\beta_1\beta_0$ will depend on a new linear combination of moments, different from the one identified at order $\bar{a}(Q^2)^4$ (compare the coefficient of $\beta_1\beta_0$ in (47) with $r_3^{(2)}$ in eq. (37)). Thus, strictly based on the algebraic decomposition of the coefficients at previous orders there is no way to determine the coefficient of $\beta_1\beta_0$ at order $\bar{a}(Q^2)^5$. Additional information, namely the values of $\phi_1^{(1,1)}$, $\phi_1^{(2,0)}$ and $\phi_1^{(0,2)}$ is required. In the Abelian case, where the diagrammatic identification of the skeleton structure is transparent, it should be straightforward to calculate these moments separately. In the non-Abelian theory this not yet achievable.

The need to identify the skeleton structure, as a preliminary stage to writing the decomposition of the coefficients (and thus also to BLM scale-setting) may actually arise at lower orders *if several skeletons appear at the same order*. The simplest example in the Abelian theory is e^-e^- scattering with both t channel and u channel exchange. Several skeletons at the same order also occur in single-scale observables considered here. In this sense the assumed form of the skeleton expansion (2) is oversimplified and should be generalized to include several different $s_i R_i(Q^2)$ terms at any order i. For example, in the non-Abelian theory it is natural to expect that different group structures characterizing different vertices will be associated with different skeletons. Assume, for instance, that instead of a single sub-leading skeleton term $s_1 R_1(Q^2)$, we have a sum of two terms: $s_1^A R_1^A(Q^2)$ and $s_1^F R_1^F(Q^2)$ where s_1^A is proportional to C_A and s_1^F is proportional to C_F. Now both $R_1^A(Q^2)$ and $R_1^F(Q^2)$ have the structure of (4) where the corresponding momentum distribution functions ϕ_1^A and ϕ_1^F carry no group structure. In the decomposition of the coefficients (45) one then has,

$$\begin{aligned} a_R &= \bar{a} + \left[s_1 + r_1^{(1)}\beta_0\right]\bar{a}^2 \\ &+ \left[s_2 + \left(s_1^A r_{2,A}^{(1)} + s_1^F r_{2,F}^{(1)}\right)\beta_0 + r_2^{(2)}\beta_0^2 + r_1^{(1)}\beta_1\right]\bar{a}^3 + \cdots \end{aligned} \tag{48}$$

where $s_1 = s_1^A + s_1^F$ (cf. eq. (114)) and the pure numbers $r_{2,A}^{(1)}$ and $r_{2,F}^{(1)}$ are uniquely determined. Applying BLM scale-setting in this case, one should treat separately the two $\mathcal{O}(\bar{a}^2)$ skeletons, leading to a BLM series of the form

$$a_R(Q^2) = \bar{a}(k_0^2) + s_1^A \bar{a}(k_{1,A}^2)^2 + s_1^F \bar{a}(k_{1,F}^2)^2 + \cdots \tag{49}$$

Thus, provided the contribution of the different skeletons can be identified, the scales $k_{1,A}$ and $k_{1,F}$ can be determined.

In general, the color group structure of the coefficients is not sufficient to distinguish between the contributions of different skeletons. In the Abelian theory, at the $s_1 R_1$ level there are two skeletons: the planar two-photon exchange diagram ($s_1^p \equiv \sigma_1^p C_F$) and the non-planar diagram ($s_1^{np} \equiv \sigma_1^{np} C_F$) with crossed photons. In the Abelian case both skeletons are proportional to C_F, but in general they have different momentum distribution functions which correspond to the different momentum flow in the skeleton diagram. It is reasonable to expect that in the non-Abelian case there will be just one more skeleton term at this order, the one based on a three-gluon vertex, proportional to C_A. Given that the full β_0 independent next-to-leading order coefficient in the skeleton scheme is $s_1 = \sigma_A C_A + \sigma_F C_F$, where the Abelian piece is decomposed according to $\sigma_F = \sigma_1^p + \sigma_1^{np}$, the non-Abelian decomposition should be

$$s_1 = \left(\sigma_A + \frac{1}{2}\sigma_1^{np}\right) C_A + \sigma_1^{np}\left(C_F - \frac{1}{2}C_A\right) + \sigma_1^p C_F \tag{50}$$

where the combination $\left(C_F - \frac{1}{2}C_A\right)$ corresponding to the non-planar skeleton is determined by the large N_c limit**.

To summarize, we have seen that by tracing the flavor dependence of the perturbative coefficients in the skeleton scheme, one can identify the contribution of the different skeleton terms. This procedure allows one to "reconstruct" the skeleton expansion algebraically from the calculated coefficients as summarized by eq. (45). This decomposition implies a unique BLM scale-setting which has a one-to-one correspondence with the skeleton expansion. We also learned that there are several limitations to the algebraic procedure which can probably be resolved only by explicit diagrammatic identification of the skeleton structures and the skeleton effective charge. These limitations include the need to

a) treat separately contributions from skeleton structures which involve fermion loops (in the Abelian case these are just the light-by-light type diagrams)

b) identify separately the different moments $\phi_i^{(j,k)}$ of a given momentum distribution function which appear as a sum with any j and k such that $j+k=n$ in the perturbative coefficients of $\beta_0^n\,\bar{a}^{1+i+n}$

c) identify separately the contributions of different skeleton terms which happen to appear at the same order in $\bar{a}$.

4 BLM and conformal relations

Let us now consider the general BLM scale-setting method, where the scheme is not necessarily the one of the skeleton effective charge, and no correspondence with the skeleton expansion is sought for. Then any scale-setting procedure which yields an expansion of the form (18) with N_f independent c_i coefficients and scale-shifts that are power series in the coupling (23) is legitimate. We saw that under these requirements there is no unique procedure for setting the BLM scale beyond the leading order ($k_{0,0}$). Nevertheless, as we now show, the coefficients c_i are uniquely defined. In fact, the c_i have a precise physical interpretation as the "conformal coefficients" relating a_R and a in a conformal theory defined by

$$\beta(a) \,=\, -\beta_0 a^2 \,-\, \beta_1 a^3 + \cdots \,=\, 0. \tag{51}$$

To go from real-world QCD to a situation where such a conformal theory exists one has to tune N_f: when N_f is set large enough (but still below $\frac{11}{2}N_c$, the point where asymptotic freedom is lost) β_1 is negative while β_0 is positive and small. Then the perturbative β function has a zero at $a_{\rm FP} \simeq -\beta_0/\beta_1$; i.e. there is a non-trivial infrared fixed-point [32]–[36]. The perturbative analysis is justified if β_0, and hence $a_{\rm FP}$, is small enough.

**In SU(N_c) the combination $\left(C_F - \frac{1}{2}C_A\right)$ is sub-leading in N_c compared to $C_A = N_C$ and $C_F = ({N_c}^2-1)/(2N_c)$.

Physically, the existence of an infrared fixed-point in QCD means that correlation functions are *scale invariant* at large distances. This contradicts confinement which requires a characteristic distance scale. In particular, when $\beta_0 \to 0$ the infrared coupling is vanishingly small. Then it is quite clear that a non-perturbative phenomenon such as confinement will not persist. The phase of the theory where the infrared physics is controlled by a fixed-point is called the conformal window. In this work we are not concerned with the physics in the conformal window[††]. We shall just use formal expansions which have a particular meaning in this phase.

The BLM coefficients c_i are by definition N_f-independent. Therefore the expansion of a_R according to eq. (18) is valid, with the same c_i's both in the real world QCD and in the conformal window. In the conformal window a generic coupling $a(k^2)$ flows in the infrared to a well-defined limit $a(k^2 = 0) \equiv a_{\rm FP}$. In particular, eq. (18) becomes

$$a_R^{\rm FP} = a_{\rm FP} + c_1 {a_{\rm FP}}^2 + c_2 {a_{\rm FP}}^3 + c_3 {a_{\rm FP}}^4 + \cdots \tag{52}$$

where we used the fact that the k_i's are proportional to Q, which follows from their definition $k_i^2 = Q^2 \exp(-t_i)$, together with the observation that the scale-shifts t_i in (23) at any finite order are just constants when $a(Q^2) \longrightarrow a_{\rm FP}$. Eq. (52) is simply the perturbative relation between the fixed-point values of the two couplings (or effective charges) a_R and a.

Note that in this discussion we ignored the complication raised at the end of the previous section, concerning the possibility of applying BLM scale-setting in the case of several skeletons contributing at the same order (cf. eq. (49)). In this case the argument above holds as well, while the conformal coefficients will be the sum of all BLM coefficients appearing at the corresponding order. For the example considered in the previous section (48), we would then have $c_1 = s_1 = s_1^F + s_1^A$.

According to the general argument above, the BLM coefficients (18) should coincide with the conformal coefficients in (52). In the next section we calculate conformal coefficients directly and check this statement explicitly in the first few orders.

5 Calculating conformal coefficients

Let us now investigate the relation between the conformal coefficients c_i appearing in (52) and the perturbative coefficients r_i.

For this purpose, it is useful to recall the Banks-Zaks expansion: solving the equation $\beta(a) = 0$ in (51) for such N_f where β_0 is small and positive and β_1 is negative, we obtain: $a_{\rm FP} \simeq -\beta_0/\beta_1 > 0$. If we now tune N_f towards the limit $\frac{11}{2}N_c$ from below, β_0 and therefore $a_{\rm FP}$ become vanishingly small, which justifies the perturbative analysis [28, 29]. In particular, it justifies neglecting higher orders in the β function as a first approximation. In order to take into account the higher orders in the β function, one can construct a power expansion solution of the equation

[††]In [36] this phase is investigated from the point of view of perturbation theory in both QCD and supersymmetric QCD.

$\beta(a)=0$, with the expansion parameter as the leading order solution,

$$a_0 \equiv -\frac{\beta_0}{\beta_1|_{\beta_0=0}} = \frac{\beta_0}{-\beta_{1,0}}. \tag{53}$$

In the last equality we defined $\beta_1 \equiv \beta_{1,0} + \beta_{1,1}\beta_0$ where $\beta_{i,j}$ are N_f-independent. Similarly, we define[‡‡] for later use

$$\beta_2 \equiv \beta_{2,0} + \beta_{2,1}\beta_0 + \beta_{2,2}\beta_0^2 + \beta_{2,3}\beta_0^3. \tag{54}$$

We shall assume that the coupling a has the following Banks-Zaks expansion

$$a_{\rm FP} = a_0 \,+\, v_1 a_0^2 \,+\, v_2 a_0^3 \,+\, v_3 a_0^4 + \cdots \tag{55}$$

where v_i depend on the coefficients of $\beta(a)$, see e.g. [35]. For instance, the first Banks-Zaks coefficient is

$$v_1 = \beta_{1,1} - \frac{\beta_{2,0}}{\beta_{1,0}}. \tag{56}$$

Suppose that the perturbative expansion of $a_R(Q^2)$ in terms of $a(Q^2)$ is given by

$$a_R(Q^2) = a(Q^2) + r_1 a(Q^2)^2 + r_2 a(Q^2)^3 + \cdots \tag{57}$$

Based on the fact that r_i are polynomials of order i in N_f, and that a_0 is linear in N_f, one can uniquely write a decomposition of r_i into polynomials in a_0 with N_f-independent coefficients

$$\begin{aligned} r_1 &= r_{1,0} + r_{1,1}a_0 \\ r_2 &= r_{2,0} + r_{2,1}a_0 + r_{2,2}{a_0}^2 \\ r_3 &= r_{3,0} + r_{3,1}a_0 + r_{3,2}{a_0}^2 + r_{3,3}{a_0}^3 \end{aligned} \tag{58}$$

and so on. For convenience we expand here in a_0 rather than in β_0. The relations with the "skeleton decomposition" of r_1 and r_2 in eqs. (16) and (21) (or in (45)) are the following

$$\begin{aligned} r_{1,0} &= r_1^{(0)} & \qquad r_{2,0} &= r_2^{(0)} + \beta_{1,0} r_1^{(1)} \\ r_{1,1} &= -\beta_{1,0} r_1^{(1)} & \qquad r_{2,1} &= -\beta_{1,0} r_2^{(1)} r_1^{(0)} - \beta_{1,0}\beta_{1,1} r_1^{(1)} \\ & & r_{2,2} &= \beta_{1,0}^2 r_2^{(2)}. \end{aligned} \tag{59}$$

For r_3 we have, based on (45),

$$r_{3,0} = r_3^{(0)} + r_2^{(1)} r_1^{(0)} \beta_{1,0} + r_1^{(1)} \beta_{2,0}. \tag{60}$$

Using eq. (57) at $Q^2=0$ with (58) and the Banks-Zaks expansion for $a_{\rm FP}$ (55), it is straightforward to obtain the Banks-Zaks expansion for $a_R^{\rm FP}$

$$a_R^{\rm FP} = a_0 \,+\, w_1 a_0^2 \,+\, w_2 a_0^3 \,+\, w_3 a_0^4 + \cdots \tag{61}$$

[‡‡]We recall that in the skeleton scheme $\bar{\beta}_{2,3}=0$.

with

$$
\begin{aligned}
w_1 &= v_1 + r_{1,0} \\
w_2 &= v_2 + 2r_{1,0}v_1 + r_{1,1} + r_{2,0} \\
w_3 &= v_3 + 2r_{1,0}v_2 + r_{1,0}{v_1}^2 + 2r_{1,1}v_1 + 3r_{2,0}v_1 + r_{2,1} + r_{3,0}
\end{aligned}
\tag{62}
$$

Having the two Banks-Zaks expansions, one can also construct the series which relates two effective charges $a_R^{\rm FP}$ and $a_{\rm FP}$ at the fixed-point. Inverting the series in (55) one obtains a_0 as a power series in $a_{\rm FP}$,

$$
a_0 = a_{\rm FP} \, + \, u_1 a_{\rm FP}^2 \, + \, u_2 a_{\rm FP}^3 \, + \, u_3 a_{\rm FP}^4 + \cdots \tag{63}
$$

with $u_1 = -v_1$ and $u_2 = v_1^2 - v_2$ etc. Substituting eq. (63) in (61) one obtains the "conformal expansion" of $a_R^{\rm FP}$ in terms of $a_{\rm FP}$ according to eq. (52) with

$$
\begin{aligned}
c_1 &= r_{1,0} \\
c_2 &= r_{1,1} + r_{2,0} \\
c_3 &= -r_{1,1}v_1 + r_{2,1} + r_{3,0} \\
c_4 &= 2r_{1,1}{v_1}^2 - r_{1,1}v_2 - r_{2,1}v_1 + r_{2,2} + r_{3,1} + r_{4,0}
\end{aligned}
\tag{64}
$$

Thus the coefficients v_i of the Banks-Zaks expansion (55) and the coefficients r_i of (57) are sufficient to determine the conformal coefficients c_i to any given order.

Clearly, the Banks-Zaks expansions (55) and (61) and the conformal expansion of one fixed-point in terms of another (52) are closely related. Strictly speaking, both type of expansions are meaningful only in the conformal window. However, we saw that the coefficients of (52) coincide with the ones of the BLM series (18) which is useful in real world QCD. We recall that the general argument in the previous section does not depend on the specific BLM scale-setting prescription used, provided that the scales k_i are proportional to Q and the c_i's are N_f independent. Comparing explicitly c_1, c_2 and c_3 in eq. (64) with the BLM coefficients obtained in the previous section, namely $c_i = r_i^{(0)}$, we indeed find that they are equal (compare using eq. (59), (60) and (56)). In particular, the "skeleton decomposition" of eq. (45), which can be formally performed in any scheme, provides an alternative way to *compute* conformal coefficients.

¿From (64) it follows that the conformal coefficients relating the fixed-point values of a_R and a cannot be obtained just from the perturbative relation (57) between the two. Additional information related to the β function $\beta(a)$, which is encoded in the Banks-Zaks coefficients v_i, is essential beyond the next-to-next-to-leading order. On the other hand, as we show below, the conformal coefficients are obtainable from the perturbative expansion if the log-structure is explicit. Using for example, a multi-scale form

$$
a_R(Q^2) = a(k_0^2) + r_1 a(k_1^2)^2 + r_2 a(k_2^2)^3 + r_3 a(k_3^2)^4 + \cdots \tag{65}
$$

where $k_i^2 = Q^2 \exp(-t_i)$ are arbitrary and the r_i's are written as in eq. (58) we obtain

$$
\begin{aligned}
a_R \;&=\; a(Q^2) + \left[r_{1,0} + \left(r_{1,1} - \beta_{1,0}t_0\right)a_0\right] a(Q^2)^2 + \Big[\beta_{1,0}t_0 + r_{2,0} \\
&+\; \left(-\beta_{1,0}\beta_{1,1}t_0 - 2r_{1,0}\beta_{1,0}t_1 + r_{2,1}\right)a_0 + \left(\beta_{1,0}^2 t_0^2 + r_{2,2} - 2r_{1,1}\beta_{1,0}t_1\right)a_0^2\Big] a(Q^2)^3 + \cdots
\end{aligned}
\tag{66}
$$

where we used the expansion of the β function coefficients in terms of β_0 (and expressed it in terms of a_0 using eq. (53)). Now we take the limit $Q^2 \longrightarrow 0$ in (66), and express a_0 as a power series in $a_{\rm FP}$ using the inverse Banks-Zaks expansion (63). We obtain,

$$\begin{aligned} a_R^{\rm FP} &= a_{\rm FP} + r_{1,0} a_{\rm FP}^2 + (r_{1,1} + r_{2,0}) a_{\rm FP}^3 \\ &+ \left[r_{2,1} + r_{3,0} + r_{1,1} u_1 + (-\beta_{1,0}\beta_{1,1} + \beta_{2,0} - \beta_{1,0} u_1)\, t_0 \right] a_{\rm FP}^4 \\ &+ \left[r_{2,2} + r_{3,1} + r_{4,0} + r_{2,1} u_1 + r_{1,1} u_2 + (-\beta_{1,0}\beta_{2,1} + \beta_{3,0} - \beta_{1,0} u_2 - \beta_{1,0}\beta_{1,1} u_1)\, t_0 \right. \\ &+ \left. 2 r_{1,0} (-\beta_{1,0}\beta_{1,1} + \beta_{2,0} - \beta_{1,0} u_1)\, t_1 \right] a_{\rm FP}^5 + \cdots \end{aligned} \tag{67}$$

The coefficients u_i are fixed by requiring that the logs t_i do not appear in the final conformal expansion. Thus in this procedure we obtain at once the (inverse) Banks-Zaks expansion (63) and the conformal expansion of $a_R^{\rm FP}$ in terms of $a_{\rm FP}$. The latter simply corresponds to the terms free of t_i in (67). The vanishing of the terms linear in t_0 gives a new constraint on u_i at each order. These constraints allow one to calculate u_1 at order $a_{\rm FP}^4$, u_2 at order $a_{\rm FP}^5$, etc. Higher orders in t_0 which appear at order $a_{\rm FP}^6$ and beyond are proportional to previous constraints and thus vanish automatically upon the substitution of u_i. The same holds for all the terms which depend on t_i for $i \geq 1$.

In the next sections we shall investigate the large-order behavior of conformal expansions. We shall assume that the observable a_R has renormalons in its perturbative expansion (57), and investigate the consequences for the conformal expansion. As for the scheme coupling a we shall assume no renormalons and a simple two-loop β function. Whether these assumptions can be really justified remains an open question. We do believe that other choices of a truncated β function, namely other schemes, would not change the conclusions. Further simplification is achieved if we consider a hypothetical model of QCD in which β_1 is independent of N_f. In this case we can define a_0 as

$$a_0 \equiv -\frac{\beta_0}{\beta_1}, \tag{68}$$

rather than (53). The advantage is that a, which is assumed to obey a two-loop renormalization group equation, has a trivial Banks-Zaks expansion: $a_{\rm FP} = a_0$; i.e. $v_i = 0$ for any $i \geq 1$. It obviously follows that the Banks-Zaks expansion (61) and the conformal expansion of $a_R^{\rm FP}$ in terms of $a_{\rm FP}$ (52) coincide, with the coefficients

$$c_i = \sum_{k=0}^{[i/2]} r_{i-k,k} \tag{69}$$

for any i, where the square brackets indicate a (truncated) integer value. In this model then, c_i is simply the sum of all the possible $r_{j,k}$ coefficients such that $j+k=i$ and $j \geq k$.

6 Conformal coefficients with renormalons

In this section we demonstrate that conformal coefficients can in general diverge factorially due to the presence in the perturbative expansion of factorials of the kind characterizing renormalons.

We begin with the simplest example corresponding to a single simple pole in the Borel transform of the observable a_R

$$B(z) = \frac{1}{1-(z/z_p)} \tag{70}$$

where $z_p = p/\beta_0$ is the renormalon location. Note that in this example we choose the renormalon residue to be simple, but in fact in QCD it can be a generic N_f-dependent function. The inverse Borel transform is defined as

$$a_R(Q^2) = \int_0^\infty B(z) e^{-z/a} dz \tag{71}$$

and yields

$$a_R(Q^2) = -z_p \mathrm{Ei}(1, -z_p/a) e^{-z_p/a}. \tag{72}$$

The perturbative series of $a_R(Q^2)$ then has the following factorially increasing coefficients

$$r_i \;=\; i! \left(\frac{\beta_0}{p}\right)^i. \tag{73}$$

In our model where β_1 is negative and N_f-independent (i.e. eq. (68) applies) the decomposition of the coefficients of (73) in powers of a_0 according to (58) yields $r_{i,i} = (-\beta_1/p)^i \, i!$ and $r_{i,j} = 0$ for any $j \neq i$. The resulting conformal coefficients (69) in this model are therefore

$$c_i = \begin{cases} 0 & i \text{ odd} \\ (i/2)! \; (-\beta_1/p)^{i/2} & i \text{ even} \end{cases} \tag{74}$$

Thus the conformal coefficients do diverge factorially. In some sense the factorial divergence is slowed down: c_i contains just $(i/2)!$ rather than $i!$. Consequently we define $u = i/2$ and write the Banks-Zaks expansion as:

$$a_R^{\mathrm{FP}} \;=\; a_0 \sum_{u=0}^{\infty} u! \left(\frac{-\beta_1}{p}\right)^u {a_0}^{2u} = a_0 \sum_{u=0}^{\infty} u! (-\delta)^{-u} \tag{75}$$

where

$$\delta \equiv \frac{p\beta_1}{\beta_0^2}. \tag{76}$$

We found that in the simple Borel pole example the factorial divergence of the perturbative series does enter the conformal coefficients. However, this example is not completely self-consistent: on one hand it was assumed that a runs according to the two-loop β function (it has a fixed-point), but on the other hand we used the one-loop

β function form of the Borel singularity, namely a simple pole. In fact, it is known that a non-vanishing two-loop coefficient in the β function modifies the Borel singularity to be a branch point. For instance, for the leading infrared renormalon associated with the gluon condensate ($p = 2$) we have the following singularity structure in the Borel plane [2]

$$B(z) = \frac{1}{[1 - (z/z_p)]^{1+\delta}} \tag{77}$$

where δ is defined in (76). The corresponding perturbative coefficients are

$$r_i = \frac{\Gamma(1 + \delta + i)}{\Gamma(1 + \delta)} \left(\frac{\beta_0}{p}\right)^i. \tag{78}$$

The large-order behavior is $r_i \sim i!\, i^\delta\, (\beta_0/p)^i$, which is different from the previous example (73).

As opposed to the previous example, the r_i are not polynomials in β_0, so starting with (78) we cannot obtain the form (58). To see this, let us examine the expansion of the Γ function in r_i

$$f_i(\delta) \equiv \frac{\Gamma(1 + \delta + i)}{\Gamma(1 + \delta)} = (\delta + i)(\delta + i - 1)(\delta + i - 2) \ldots (\delta + 1). \tag{79}$$

It is clear that $f_i(\delta)$ can be written as a sum

$$f_i(\delta) = \sum_{k=0}^{i} f_k^{(i)} \delta^k \tag{80}$$

where $f_k^{(i)}$ are numbers. Since δ is proportional to $1/{\beta_0}^2$, $f_i(\delta)$ contains all the even powers of $1/\beta_0$ from 0 up to $2i$. The additional positive power of β_0 in (78) finally leads to having half of the terms with positive power of β_0 and half with negative powers. The latter correspond to non-polynomial functions of N_f which are impossible to obtain in a perturbative calculation. This suggests that the current example is unrealistic.

The first possibility that comes to mind how to avoid having negative powers of β_0 in the coefficients is simply to truncate them and keep only the positive powers. This procedure can be seen as an intermediate step between the large β_0 limit (in which the β function is strictly one-loop) and the actual QCD situation. The truncation makes sense provided it does not alter the eventual large-order behavior of r_i, a point which we shall check explicitly. Note that there is some ambiguity in the truncation: one can in principle truncate (80) at different k values and still obtain the same asymptotic behavior. We shall choose the most natural possibility: truncate just the terms which lead to negative powers of β_0.

In order to proceed we should find the coefficients $f_k^{(i)}$. This can be done by writing a recursion relation using the property $f_i(\delta) = (\delta + i) f_{i-1}(\delta)$. This condition is equivalent to the following

$$f_k^{(i)} = \begin{cases} 1 & k = i \\ f_{k-1}^{(i-1)} + i f_k^{(i-1)} & 0 < k < i \\ i! & k = 0 \end{cases} \tag{81}$$

It is straightforward to use these recursion relations to obtain $f_k^{(i)}$ to arbitrarily high order.

After truncating the terms with negative powers of β_0, the coefficients (78) become

$$\tilde{r}_i = \left(\frac{\beta_0}{p}\right)^i \sum_{k=0}^{[\frac{i}{2}]} f_k^{(i)}\, \delta^k. \tag{82}$$

In order to make sure that the truncation of the high powers of δ does not affect the large-order behavior of the series we calculated the ratio

$$\tilde{r}_i/r_i = \left[\sum_{k=0}^{[i/2]} f_k^{(i)} \delta^k\right] \Big/ \left[\sum_{k=0}^{i} f_k^{(i)} \delta^k\right] \tag{83}$$

for various values of δ, as a function of the order i. It turns out that this ratio approaches 1 fast, indicating a common asymptotic behavior. For instance, for $\delta = 462/625$, corresponding to eq. (76) with $p = 1$ and the values of β_0 and β_1 in QCD with $N_f = 4$, we find $\tilde{r}_i/r_i \simeq 0.995$ for $i = 8$.

At the next step we write the decomposition of $\tilde{r}_i$ as a polynomial in a_0 according to (58),

$$\tilde{r}_{i,j} = f^{(i)}_{\frac{i-j}{2}} (-1)^j \left(\frac{\beta_1}{p}\right)^{\frac{i+j}{2}}, \tag{84}$$

where j is odd for odd i and even for even i (as always $j \leq i$). Finally, the conformal coefficients (69) in this example are

$$\tilde{c}_{2u} = \sum_{j=0}^{u} \tilde{r}_{2u-j,j} = \left[\sum_{k=u}^{2u} f^{(k)}_{k-u} (-1)^k\right] \left(\frac{\beta_1}{p}\right)^u \tag{85}$$

and the expansion is

$$a_R^{\mathrm{FP}} = a_0 \sum_{u=0}^{\infty} \left[\sum_{k=u}^{2u} f^{(k)}_{k-u} (-1)^{u+k}\right] (-\delta)^{-u}. \tag{86}$$

The square brackets should be compared with $u!$ (75) characterizing the simple Borel pole example. It turns out that the $\tilde{c}_{2u}$ increase *faster* than $u!$, but slower than $(2u)!$. Thus the factorial behavior of the conformal coefficients persists also in this example.

Another possible approach to analyze the Borel cut example (77) is the following: the large-order behavior of the coefficients is

$$r_i = \frac{1}{\Gamma(1+\delta)}\, i!\, i^\delta \left(\frac{\beta_0}{p}\right)^i. \tag{87}$$

Let us now ignore the $1/\Gamma(1+\delta)$ factor, which can be absorbed into the residue of the renormalon and expand $i^\delta \sim \exp(\delta \ln(i)) = 1 + \delta \ln(i) + \frac{1}{2}\delta^2 \ln^2(i) + \cdots$. Again we find that high powers of δ lead to non-polynomial dependence of the coefficients. As before we truncate these terms and write an approximation to r_i

$$\bar{r}_i \;=\; i! \sum_{k=0}^{[\frac{i}{2}]} \frac{1}{k!} \ln^k(i) \left(\frac{p\beta_1}{\beta_0^2}\right)^k \left(\frac{\beta_0}{p}\right)^i . \tag{88}$$

We checked numerically that the ratio $\bar{r}_i/r_i$ approaches 1 as i increases, so the asymptotic behavior is not altered by this truncation. We comment that the logarithmic factor in (88) which has the same asymptotic behavior as

$$\ln i \simeq \Psi(2i+1) + \gamma_E = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2i}, \tag{89}$$

can actually be understood diagrammatically, as explained in [3].

We proceed and write

$$\bar{r}_{i,j} \;=\; \frac{i!}{\frac{i-j}{2}!} (-1)^j \left(\ln i\right)^{\frac{i-j}{2}} \left(\frac{\beta_1}{p}\right)^{\frac{i+j}{2}} , \tag{90}$$

and finally, using (69), the conformal coefficients are

$$\bar{c}_{2u} = \sum_{j=0}^{u} \bar{r}_{2u-j,j} = \left[\sum_{k=u}^{2u} \frac{k!}{(k-u)!} (-1)^k (\ln k)^{k-u} \right] \left(\frac{\beta_1}{p}\right)^u . \tag{91}$$

The large-order behavior of $\bar{c}_{2u}$ turns out to be again between $u!$ and $2u!$. In fact, the two ways we used to construct the coefficients in this example lead to roughly the same asymptotic behavior of the conformal coefficients: the ratio between $\bar{c}_{2u}$ and $\tilde{c}_{2u}$ approaches a geometrical series at large orders. The reason for this is simply the fact that in both examples the dominant term in the sum is the one coming from the highest power of the coupling ($\tilde{r}_{2u,0}$ in (85) and $\bar{r}_{2u,0}$ in (91)). In fact, the contributions to the conformal coefficients from increasing orders in the coupling are monotonically increasing in both cases. We stress, however, that the decomposition of r_i into polynomials in β_0 ($r_{i,j}$) is not at all similar in the two cases.

7 Conformal coefficients without renormalons

In the previous section we saw that, in general, conformal coefficients can diverge factorially when the corresponding perturbative series has Borel singularities of the renormalon type. We tried to provide semi-realistic examples by requiring that the coefficients would be polynomials in N_f and that the large order behavior of the series (i.e. the nature of the cut in the Borel plane) would be consistent with a two-loop β function.

But is this enough to imitate the effect of real-world QCD renormalons? We saw that the large-order behavior of the perturbative coefficients r_i (the nature of the Borel singularity) by itself does not uniquely determine its decomposition into powers of β_0: several choices of $r_{i,j}$ can fit. Actually, there is no good reason to think that the decompositions we suggested in the previous section are realistic. In this section we shall consider the case where the $r_{i,j}$ coefficients are determined directly from a genuine renormalon structure, rather than guessed based on the large-order behavior of r_i.

In a theory where the coupling does not run ($\bar{a}(k^2) = \bar{a}$), the skeleton expansion (2) *coincides* with the perturbative expansion. The integrals R_i of (3), (4) etc. simply reduce to $\bar{a}^{i+1}$ (the normalization is one by construction). Since renormalons are understood *by definition* as the particular factorial increase that emerges upon expansion of the skeleton terms R_i (in case of a running coupling), the skeleton coefficients s_i in (2) do not contain renormalons.

In particular in QCD the assumed structure of the skeleton expansion (2) implies that *taking the infrared limit* we obtain a *trivial conformal expansion* for each skeleton, namely

$$R_i(Q^2 = 0) = \bar{a}_{\rm FP}^{i+1}. \tag{92}$$

To see this, consider first the leading term R_0. Changing variables in (3), $\epsilon \equiv k^2/Q^2$ we obtain

$$R_0(Q^2) = \int_0^\infty \bar{a}\left(\epsilon Q^2\right)\,\phi_0(\epsilon)\frac{d\epsilon}{\epsilon}. \tag{93}$$

Assuming a conformal fixed-point for $\bar{a}(k^2)$ we take the limit $Q^2 \longrightarrow 0$ inside the integral and obtain $R_0^{\rm FP} = \bar{a}_{\rm FP}$ where we used the assumed normalization of ϕ_0 (5). By a similar argument $R_1(Q^2)$ of eq. (4) obeys (92). Again, since renormalons are understood as the particular factorial increase from the expansion of the skeleton terms R_i, the coefficients s_i are renormalon-free. It follows that the conformal relation corresponding to (2),

$$a_R^{\rm FP} \equiv a_R(Q^2 = 0) = \bar{a}_{\rm FP} + s_1\bar{a}_{\rm FP}^2 + s_2\bar{a}_{\rm FP}^3 + \cdots \tag{94}$$

is renormalon-free.

The absence of renormalons in the skeleton expansion implies also that the Banks-Zaks expansion is free of renormalons provided that the β function of the "skeleton coupling" $\bar{\beta}(\bar{a})$ does not contain renormalons. Consider for example the leading skeleton term R_0, for which we have $R_0(Q^2 = 0) = \bar{a}_{\rm FP}$. If the expansion of $\bar{a}_{\rm FP}$ in terms of a_0 is renormalon-free, it follows immediately that the Banks-Zaks expansion of R_0 has the same property. As in the previous section, for simplicity one can consider a model in which $\bar{a}$ obeys a two-loop renormalization group equation (as before β_1 is taken to be independent of N_f). Then we simply have $R_0(Q^2 = 0) = \bar{a}_{\rm FP} = a_0$.

We showed that the conformal expansion of a dressed skeleton R_i is trivial. In particular, it is free of renormalons in spite of the fact that the corresponding perturbative series (e.g. eq. (8) for R_0) does have renormalons. Having in mind the examples of the previous section, this conclusion seems surprising. As opposed to these examples, in case of a renormalon integral the conformal coefficients are built from the perturbative coefficients in such a way that the factorial increase is cancelled. It is interesting to examine how this cancelation comes about.

Let us consider the leading skeleton term (3), where we expect according to the general argument that $R_0(Q^2=0)=\bar{a}_{\rm FP}$. In order to analyze the conformal coefficients we restrict ourselves now to the contribution to R_0 from small k^2, which is the origin of infrared renormalons, and expand the momentum distribution function

$$\phi_0\left(k^2/Q^2\right)=\sum_n\left(\frac{k^2}{Q^2}\right)^n\,\Phi_0^n \tag{95}$$

where Φ_0^n are numbers. It is enough to consider a specific infrared renormalon with $n=p$, and so we choose our "observable" to be

$$\hat{a}_{R_0}(Q^2)\equiv p\int_0^{Q^2}\left(\frac{k^2}{Q^2}\right)^p\bar{a}(k^2)\frac{dk^2}{k^2} \tag{96}$$

where the upper integration limit is set for simplicity to be Q^2.

It was shown in [40, 41] that if $\bar{a}(k^2)$ satisfies the two-loop renormalization group equation, the Borel representation of $\hat{a}_{R_0}$ is

$$\hat{a}_{R_0}(Q^2)=\int_0^\infty e^{-\frac{\beta_1}{\beta_0}z}\frac{1}{\left[1-(z/z_p)\right]^{1+\delta}}\,e^{-z/\bar{a}}\,dz \tag{97}$$

where $\bar{a}\equiv\bar{a}(Q^2)$. This integral resums all those terms in eq. (8) which depend only on the first two coefficients of the β function. Note that eq. (96) is well defined thanks to the infrared fixed-point of the coupling $\bar{a}(k^2)$. On the other hand, eq. (97) is not well defined due to the infrared renormalon, and it differs [40, 41] from eq. (96) by an ambiguous power-correction. The equality between (96) and (97) should be therefore understood just as an equality of the (all-order) power series expansion of the two expressions.

To expand (97) we note that the Borel transform of $\hat{a}_{R_0}$ with respect to the *modified coupling* $\tilde{a}$,

$$\frac{1}{\tilde{a}}\equiv\frac{1}{\bar{a}}+\frac{\beta_1}{\beta_0}, \tag{98}$$

coincides with the Borel transform (77). Using the coefficients (78) we have

$$\hat{a}_{R_0}=\sum_{i=0}^\infty r_i\tilde{a}^{i+1}=\sum_{i=0}^\infty\frac{\Gamma(1+\delta+i)}{\Gamma(1+\delta)}\left(\frac{\beta_0}{p}\right)^i\tilde{a}^{i+1}. \tag{99}$$

Substituting

$$\tilde{a}^{i+1}=\left(\frac{\bar{a}}{1+\bar{a}\beta_1/\beta_0}\right)^{i+1}=\bar{a}^{i+1}\sum_{k=0}^\infty\frac{(i+k)!}{i!\,k!}\,\bar{a}^k\left(-\frac{\beta_1}{\beta_0}\right)^k \tag{100}$$

we obtain

$$\hat{a}_{R_0}=\bar{a}\sum_{i=0}^\infty\sum_{k=0}^\infty\frac{\Gamma(1+\delta+i)}{\Gamma(1+\delta)}\,\frac{(i+k)!}{i!\,k!}\,\left(\frac{\beta_0}{p}\right)^i\left(-\frac{\beta_1}{\beta_0}\right)^k\bar{a}^{i+k}. \tag{101}$$

Defining $n = k + i$ and performing first the summation over i we obtain

$$\hat{a}_{R_0} = \bar{a} \sum_{n=0}^{\infty} r_n \bar{a}^n \qquad (102)$$

with the perturbative coefficients r_n given by

$$r_n \;=\; \sum_{i=0}^{n} \frac{\Gamma(1+\delta+i)}{\Gamma(1+\delta)} \, \frac{n!}{i!\,(n-i)!} \left(\frac{\beta_0}{p}\right)^i \left(-\frac{\beta_1}{\beta_0}\right)^{n-i}. \qquad (103)$$

We now use (80) to expand the Γ function and write explicitly the dependence on β_0. Defining $j = 2i - 2k - n$, we obtain $r_n = \sum r_{n,j} a_0^j$ with

$$r_{n,j} \;=\; \sum_{i=\frac{n+j}{2}}^{n} \frac{n!}{(n-i)!\,i!} (-1)^i f^{(i)}_{i-\frac{n+j}{2}} \left(\frac{\beta_1}{p}\right)^{\frac{n+j}{2}}. \qquad (104)$$

A major difference between this example and the examples of the previous section should be noted: here the decomposition of r_n into powers of a_0 is straightforward and does not lead to any non-polynomial dependence. Clearly, truncation is not required.

Finally, using (69), the conformal coefficients corresponding to the leading skeleton are given by

$$c_{2u} = \sum_{j=0}^{u} r_{2u-j,j} = \sum_{k=u}^{2u} r_{k,2u-k} = \left[\sum_{k=u}^{2u} \sum_{i=u}^{k} \frac{k!}{(k-i)!\,i!} (-1)^i f^{(i)}_{i-u}\right] \left(\frac{\beta_1}{p}\right)^u = 0, \qquad (105)$$

where the last equality was checked explicitly. In other words, the final result is

$$\hat{a}_{R_0}(Q^2 = 0) = \bar{a}_{\mathrm{FP}} \qquad (106)$$

in accordance with our expectations. As explained before, the vanishing of the conformal coefficients in this case can be understood directly from the defining integral R_0. We note that contrary to the examples of the previous section, (85) and (91), in (105) the term originating from the highest power of the coupling does not dominate. This is crucial for the eventual cancelation.

8 Examples

The absence of renormalons in conformal relations strongly suggests that the effective convergence of the skeleton expansion or, alternatively the BLM series in the skeleton scheme (which, as we saw, coincides with the relevant conformal relation) is better than that of standard perturbative series in a standard scheme such as $\overline{\mathrm{MS}}$.

Can we check explicitly the absence of renormalons in conformal relations? The purpose of this section is to examine through available examples in QCD whether the expectation stated above is realized. Indeed, as we shall now recall, it has been noted by several authors (e.g. in [12, 31, 18, 34, 35]) that conformal coefficients and Banks-Zaks coefficients are typically small. We would like to interpret these observations based on the assumed skeleton expansion and relate them to the absence of renormalons. As concrete examples we shall concentrate on the following observables:

a) The Adler D-function,

$$D(Q^2) = Q^2 \frac{d\Pi(Q^2)}{dQ^2} \equiv N_c \sum_f e_f^2 \left[1 + a_D\right] \tag{107}$$

where a_D is normalized as an effective charge, and $\Pi(Q^2)$ is the electromagnetic vacuum polarization,

$$4\pi^2 i \int d^4x \, e^{iq\cdot x} \langle 0|T\left\{j^\mu(x), j^\nu(0)\right\}|0\rangle = \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \Pi(Q^2). \tag{108}$$

b) The polarized Bjorken sum-rule for electron nucleon deep-inelastic scattering,

$$\int_0^1 \left[g_1^p(x,Q^2) - g_1^n(x,Q^2)\right] dx \equiv \frac{g_A}{6}\left[1 - a_{g_1}\right]. \tag{109}$$

c) The non-polarized Bjorken sum-rule for neutrino nucleon deep-inelastic scattering,

$$\int_0^1 dx \left[F_1^{\bar{\nu}p}(x,Q^2) - F_1^{\nu n}(x,Q^2)\right] \equiv 1 - \frac{C_F}{2} a_{F_1}. \tag{110}$$

d) The static potential,

$$V(Q^2) \equiv -4\pi^2 C_F \frac{a_V}{Q^2}. \tag{111}$$

In all four cases perturbative calculations have been performed (refs. [42] through [45], respectively) up to the next-to-next-to-leading order r_2 in eq. (1).

For later comparison with conformal relations, we quote some numerical values of the coefficients in the standard perturbative expansion in $a_{\overline{\rm MS}} \equiv a_{\overline{\rm MS}}(Q^2)$ for the vacuum polarization D-function (107)

$$\begin{array}{cccccccc}
a_D = a_{\overline{\rm MS}} & + & d_1 & a^2_{\overline{\rm MS}} & + & d_2 & a^3_{\overline{\rm MS}} + \cdots & \\
 & & 2.0 & & & 18.2 & & N_f = 0 \\
 & & 1.6 & & & 6.4 & & N_f = 3 \\
 & & 0.14 & & & -27.1 & & N_f = 16 \\
 & & 1.06 & & & 14.0 & & N_f = 0..16
\end{array} \tag{112}$$

and for the polarized Bjorken sum-rule (109)

$$\begin{array}{cccccccc}
a_{g_1} = a_{\overline{\rm MS}} & + & k_1 & a^2_{\overline{\rm MS}} & + & k_2 & a^3_{\overline{\rm MS}} + \cdots & \\
 & & 4.6 & & & 41.4 & & N_f = 0 \\
 & & 3.5 & & & 20.2 & & N_f = 3 \\
 & & -0.75 & & & -34.8 & & N_f = 16 \\
 & & 2.1 & & & 21.0 & & N_f = 0..16
\end{array} \tag{113}$$

where in the first three lines in (112) and (113) the coefficients are evaluated at given N_f values, while the last line corresponds to an average of $|r_i|$ in the range $N_f = 0$ through 16.

We see that the coefficients in a running coupling expansion in the $\overline{\mathrm{MS}}$ scheme increase fast already at the available next-to-next-to-leading order. This increase has been discussed in connection with renormalons, for example in [4]. A priori, it is hard to expect that the large-order behavior of the series will show up already in the first few leading orders. We mention, however, that in ref. [46] the Bjorken sum rule series (for $N_f = 3$) was analyzed in the Borel plane based on the three known coefficients, indicating that the first infrared renormalon at $p = 1$ does show up.

8.1 Conformal relations in the skeleton scheme

Let us consider first the conformal relation in the skeleton scheme (94). Since the skeleton coupling $\bar{a}$ has been identified only at the one-loop level (12), our information on the coefficients s_i is quite limited: by a direct calculation (using the next-to-leading order coefficient r_1 and either (16) or (64)) we can only determine s_1. For example, for the observables defined above it is

$$s_1 = r_1^{(0)} = \begin{cases} -(1/4)C_A - (1/8)C_F & = & -11/12 & D \\ -(1/4)C_A - (7/8)C_F & = & -23/12 & g_1 \\ -(1/4)C_A - (11/8)C_F & = & -31/12 & F_1 \\ -C_A & = & -3 & V \end{cases} \qquad (114)$$

Note the absence of a C_F term in the case of the static potential. This can be understood based on the Abelian limit, where it is known that this effective charge coincides with the skeleton coupling (there, the Gell-Mann Low effective charge) up to light-by-light type corrections. Therefore the momentum distribution function of the leading skeleton term ϕ_0 is just a δ-function, $\phi_0(k^2) = \delta(k^2)$, and in the Abelian limit there are strictly no (N_f-independent) sub-leading skeleton terms.

The higher-order coefficients s_i, for $i \geq 2$, depend on yet unknown characteristics of the skeleton coupling scheme. In particular, as we shall discuss in section 9, s_2 depends on the skeleton β function coefficient $\bar{\beta}_2$. However, as can be seen in eq. (145) below, the dependence on this coefficient cancels in the difference of s_2 between any two observables, which is therefore calculable.

It should be stressed that without a diagrammatic identification of the skeleton structure, one cannot isolate skeletons with fermion loops attached to three gluons, which may appear at the order considered. Therefore we shall just treat the entire N_f dependence (excluding Abelian light-by-light diagrams) as if it appears due to the running coupling, according to eq. (45) where s_2 is N_f independent. For the observables considered above we then find:

$$\begin{aligned} s_2^{g_1} - s_2^D &= \frac{3}{8}\,C_F\,C_A + \frac{3}{4}\,{C_F}^2 = 2.833 \\ s_2^{F_1} - s_2^D &= \left[\frac{43}{12} + \frac{85}{6}\,\zeta_3 - \frac{115}{6}\,\zeta_5\right]\,{C_A}^2 + \left[-34\,\zeta_3 - \frac{75}{8} + \frac{95}{2}\,\zeta_5\right]\,C_F\,C_A \\ &+ \left[\frac{21}{2} + \frac{47}{2}\,\zeta_3 - 35\,\zeta_5\right]\,{C_F}^2 = 7.045 \qquad (115) \\ s_2^V - s_2^D &= \left[\frac{1}{4}\,\pi^2 + \frac{43}{24} - \frac{1}{64}\,\pi^4\right]\,{C_A}^2 - \frac{25}{16}\,C_F\,C_A + \frac{23}{32}\,{C_F}^2 = 19.66 \end{aligned}$$

This gives some idea about the size of s_2 of these observables. While the vacuum polarization D-function and the polarized and non-polarized Bjorken sum-rules have rather small differences between their s_2 coefficients, the static potential s_2 differs significantly from the others. Thus if we assume for example that the skeleton scheme is such that s_2^D is small, then also $s_2^{g_1}$ and $s_2^{F_1}$ will be rather small, but not so s_2^V. Whereas in the Abelian limit the static potential effective charge, with light-by-light terms excluded, coincides with the skeleton effective charge (see the comment below eq. (114)), the two may eventually be quite distinct in the non-Abelian case (note the dominance of the ${C_A}^2$ piece in $s_2^V - s_2^D$).

Based on these differences one could conclude that in the non-Abelian theory the skeleton expansion does not always have good convergence properties. One should remember, however, that we mistreated here the N_f dependence which should be associated with the skeleton structure (fermion loops attached to three gluons). Eventually, this will have some impact on the magnitude of the skeleton coefficients s_2, which we cannot evaluate at present.

8.2 Direct relations between observables

There is a way to consider systematically conformal relations avoiding the use of the skeleton scheme. Having renormalon-free conformal expansions (94) for two QCD observables in terms of the skeleton effective charge $\bar{a}$, one can eliminate the latter to obtain a *direct* conformal relation between the two observables. The existence of a skeleton expansion (2) for the two observables implies that this conformal relation is free of renormalons.

Conformal coefficients of this type can be computed either from the Banks-Zaks expansion (64) or in the framework of BLM, as the coefficients in a commensurate scale relation [12]. The latter can be obtained by applying BLM directly to the perturbative relation between two observable effective charges (and so it does not require identification of the skeleton coupling). Such relations between several observables were investigated in [12] in the framework of multi-scale BLM, and were found to have typically smaller coefficients compared to the standard running coupling expansions, in accordance with the general expectation. The absence of renormalons in a commensurate scale relation between measurable quantities may have practical phenomenological implications allowing precision tests of perturbative QCD.

There is one example where a direct *all-order* conformal relation is known – this is the Crewther relation relating the vacuum polarization D-function effective charge a_D, defined by (107), with the polarized Bjorken sum-rule effective charge a_{g_1}, defined by (109). The Crewther relation is [30, 31, 18]

$$a_{g_1} - a_D + \frac{3}{4} C_F a_{g_1} a_D = -\beta(a) T(a) \tag{116}$$

where $T(a)$ is a power series in the coupling

$$T(a) = T_1 + T_2 a + T_3 a^2 + \cdots \tag{117}$$

and T_i are polynomials in N_f.

If a_D has a perturbative fixed-point $a_D^{\rm FP}$, then it is convenient [35] to write the r.h.s. of (116) in terms of a_D. $\beta(a_D^{\rm FP}) = 0$ and so the r.h.s. vanishes at $a_D = a_D^{\rm FP}$ corresponding to the infrared limit. Therefore a_{g_1} also freezes perturbatively, leading to the original conformal Crewther relation

$$a_{g_1}^{\rm FP} = \frac{a_D^{\rm FP}}{1 + \frac{3}{4} C_F a_D^{\rm FP}}. \tag{118}$$

Taking $N_c = 3$ we have $C_F = \frac{4}{3}$ and then the conformal coefficients are just *one* to any order in perturbation theory,

$$a_D^{\rm FP} = a_{g_1}^{\rm FP} + \left(a_{g_1}^{\rm FP}\right)^2 + \left(a_{g_1}^{\rm FP}\right)^3 + \cdots \tag{119}$$

Being a geometrical series this conformal relation provides a nice example of a perturbative relation free of renormalon divergence.

As noted in [9] (see also [31]) it is possible to write for two generic observables A and B, at two arbitrary scales Q_A and Q_B, the following decomposition of the perturbative series relating the two,

$$a_A = C_{AB}(a_B) + \beta(a_B) T_{AB}(a_B). \tag{120}$$

Here C_{AB} is the "conformal part" of the series, i.e.

$$C_{AB}(a_B) = a_B + c_1 a_B^2 + c_2 a_B^3 + \cdots \tag{121}$$

where c_i are the conformal coefficients appearing in the expansion of $a_A^{\rm FP}$ in terms of $a_B^{\rm FP}$, and $T_{AB}(a_B)$ is a perturbative series of the form (117). In other words the *non-conformal* part of the relation between the two observables is factorized [31] as $\beta(a_B) T_{AB}(a_B)$. Taking the limit $\beta \to 0$ then gives the conformal relation. In particular, one can write such a factorized relation between an observable effective charge and the skeleton coupling. Then the conformal coefficients c_i in (121) are the skeleton coefficients s_i. Explicitly, this can be shown based on the skeleton decomposition of the series (45),

$$\begin{aligned} a_R &= \left[\bar{a} + s_1 \bar{a}^2 + s_2 \bar{a}^3 + s_3 \bar{a}^4 + \cdots\right] + \left[\beta_0 \bar{a}^2 + \beta_1 \bar{a}^3 + \bar{\beta}_2 \bar{a}^4 + \cdots\right] \\ &\times \left[r_1^{(1)} + \left(s_1 r_2^{(1)} + r_2^{(2)} \beta_0\right) \bar{a} + \left(s_2 r_3^{(1)} + s_1 r_3^{(2)} \beta_0 + r_3^{(3)} \beta_0^2 + \frac{3}{2} r_2^{(2)} \beta_1\right) \bar{a}^2 + \cdots\right]. \end{aligned} \tag{122}$$

Coming back to the Crewther relation, it is natural to compare the fixed-point relation (119) with the corresponding running coupling expansion of $a_D \equiv a_D(Q^2)$ in terms of $a_{g_1} \equiv a_{g_1}(Q^2)$ at the same fixed scale Q^2,

$$\begin{array}{ccccccc} a_D = a_{g_1} & + & \tilde{d}_1 & a_{g_1}^2 & + & \tilde{d}_2 & a_{g_1}^3 + \cdots \\ & & -2.6 & & & 0.61 & \qquad N_f = 0 \\ & & -1.9 & & & 0.08 & \qquad N_f = 3 \\ & & 0.89 & & & 9.04 & \qquad N_f = 16 \\ & & 1.12 & & & 2.66 & \qquad N_f = 0..16\,. \end{array} \tag{123}$$

Whereas the next-to-next-to-leading order coefficient in (123) appears to be smaller than in the $\overline{\rm MS}$ scheme (eq. (112)), it still differs significantly from the conformal coefficient in (119) and it can be much larger, for some N_f values. Note that the difference between the conformal part of the next-to-next-to-leading order coefficient and the full coefficient for $N_f = 16$ is directly related to the term linear in β_1 in* eq. (45), since the terms which depend on β_0 are small (β_0 vanishes at $N_f = 16\frac{1}{2}$).

One may worry that the observed smallness of the coefficients in (119) is due to the special relation between the two specific effective charges, and thus it is not representative of conformal relations in general. Let us then examine another pair of effective charges, namely the relation between the non-polarized Bjorken sum-rule (110) and the vacuum polarization D-function. The conformal relation is

$$a_D^{\rm FP} = a_{F_1}^{\rm FP} + 1.67\left(a_{F_1}^{\rm FP}\right)^2 + 1.57\left(a_{F_1}^{\rm FP}\right)^3 + \cdots. \tag{124}$$

This can be compared with the running coupling expansion of $a_D \equiv a_D(Q^2)$ in terms of $a_{F_1} \equiv a_{F_1}(Q^2)$,

$$\begin{array}{lrlrll} a_D = a_{F_1} \;+ & d_1' & a_{F_1}^2 \;+ & d_2' & a_{F_1}^3 + \cdots & \\ & -3.76 & & 7.30 & & N_f = 0 \\ & -2.78 & & 3.01 & & N_f = 3 \\ & 1.50 & & 13.43 & & N_f = 16 \\ & 1.63 & & 4.78 & & N_f = 0..16 \end{array} \tag{125}$$

Again we see that the coefficients in the running coupling expansion are in general not as small as the conformal ones. We stress that the coefficients in running coupling relations between observables, such as (123) and (125), as opposed to conformal relations (119) and (124), are expected to increase factorially† due to renormalons.

The static potential is again an exception. Here the conformal relation with the vacuum polarization D-function is

$$a_D^{\rm FP} = a_V^{\rm FP} + 2.08\,(a_V^{\rm FP})^2 - 7.16\,(a_V^{\rm FP})^3 + \cdots. \tag{126}$$

This can be compared with the running coupling expansion of $a_D \equiv a_D(Q^2)$ in terms of $a_V \equiv a_V(Q^2)$,

$$\begin{array}{lrlrll} a_D = a_V \;+ & d_1'' & a_V^2 \;+ & d_2'' & a_V^3 + \cdots & \\ & 0.21 & & -7.22 & & N_f = 0 \\ & -0.11 & & -10.04 & & N_f = 3 \\ & 2.00 & & -1.63 & & N_f = 16 \\ & 0.87 & & 8.40 & & N_f = 0..16 \end{array} \tag{127}$$

In this relation the conformal coefficients are of the same order of magnitude as the running coupling coefficients.

*This equation can be viewed as a way to parameterize the non-conformal contribution in any scheme, in particular in physical schemes. The coefficients $r_i^{(j)}$ are not the moments of the distribution functions, but are still related to them.

†In case of the Crewther relation (116), using a renormalization scheme in which $\beta(a)$ is free of renormalons, the factorial increase should be entirely contained in $T(a)$.

8.3 The Banks-Zaks expansion

A further observation [34, 35] is that also the coefficients in the Banks-Zaks expansion are usually small. The Banks-Zaks expansion for the fixed-point value of the vacuum polarization D-function (107) is

$$a_{\mathrm{D}}^{\mathrm{FP}} = \quad a_0 \quad + \quad 1.22\, {a_0}^2 \quad + \quad 0.23\, {a_0}^3 \quad + \quad \cdots \tag{128}$$

whereas for the Bjorken sum-rule it is

$$a_{g_1}^{\mathrm{FP}} = \quad a_0 \quad + \quad 0.22\, {a_0}^2 \quad - \quad 1.21\, {a_0}^3 \quad + \quad \cdots. \tag{129}$$

Comparing (129) and (128) with the corresponding running coupling expansions in $\overline{\mathrm{MS}}$, namely (112) and (113), the difference in magnitude of the coefficients is quite remarkable. For the non-polarized Bjorken sum-rule defined by (130), the Banks-Zaks coefficients are even smaller

$$a_{\mathrm{F}_1}^{\mathrm{FP}} = a_0 \quad - \quad 0.45\, {a_0}^2 \quad + \quad 0.16\, {a_0}^3 \quad + \quad \cdots \tag{130}$$

and exhibit an impressive cancelation of numerical terms appearing in the running coupling coefficients [35]. As in sections 8.1 and 8.2, the static potential shows a different behavior. In this case the Banks-Zaks expansion [35, 36]

$$a_{\mathrm{V}}^{\mathrm{FP}} = a_0 \quad - \quad 0.86\, {a_0}^2 \quad + \quad 10.99\, {a_0}^3 \quad + \quad \cdots \tag{131}$$

has a significantly larger next-to-next-to-leading order coefficient. Note that this large Banks-Zaks coefficient (together with the small coefficient in (128)) explains the large conformal coefficient in the direct conformal relation (126) between the D-function and the static potential. If we assume that the Banks-Zaks expansion of the skeleton effective charge,

$$\bar{a}_{\mathrm{FP}} = a_0 + \bar{v}_1 {a_0}^2 + \bar{v}_2 {a_0}^3 + \cdots \tag{132}$$

does not involve large coefficients ($\bar{v}_1$ is known $\bar{v}_1 = 2.14$, see e.g. eq. (56) with $\bar{\beta}_{2,0}$ from eq. (147) below), it would follow from (128) that s_2^D is not large. In this case the large difference $s_2^V - s_2^D$ in (115) would be clearly attributed to a large s_2^V.

Another physical quantity for which the Banks-Zaks coefficients are relatively large is the critical exponent $\hat{\gamma}$ [29, 34, 35, 36]

$$\hat{\gamma} = \frac{1}{\beta_0} \left. \frac{d\beta(a)}{da} \right|_{a=a_{\mathrm{FP}}} \tag{133}$$

where

$$\hat{\gamma} = \quad a_0 \quad + \quad 4.75\, {a_0}^2 \quad - \quad 8.89\, {a_0}^3 \quad + \quad \cdots \tag{134}$$

Since this quantity does not depend on Q^2, there is no direct comparison between a running coupling expansion and the Banks-Zaks expansion (or a conformal expansion).

8.4 Expansions in $\overline{\mathrm{MS}}$

Finally, it is interesting to return to the expansion in $\overline{\mathrm{MS}}$ we examined in the beginning of the section.

The first observation is that conformal relations of observables with the $\overline{\mathrm{MS}}$ coupling tend to have large coefficients. For example, the conformal expansions

$$a_D^{\mathrm{FP}} = a_{\overline{\mathrm{MS}}}^{\mathrm{FP}} - 0.083\,(a_{\overline{\mathrm{MS}}}^{\mathrm{FP}})^2 - 23.22\,(a_{\overline{\mathrm{MS}}}^{\mathrm{FP}})^3 + \cdots \tag{135}$$

and

$$a_{g_1}^{\mathrm{FP}} = a_{\overline{\mathrm{MS}}}^{\mathrm{FP}} - 0.917\,(a_{\overline{\mathrm{MS}}}^{\mathrm{FP}})^2 - 22.39\,(a_{\overline{\mathrm{MS}}}^{\mathrm{FP}})^3 + \cdots \tag{136}$$

have large next-to-next-to-leading order coefficients, in a striking contrast with the conformal relation (119) between a_D^{FP} and $a_{g_1}^{\mathrm{FP}}$. Note that these large conformal coefficients do not provide an explanation of the large coefficients in (112) and (113). The former are by assumption independent of N_f, at the difference of the latter. For small β_0 (e.g. $N_f = 16$) the negative sign (and eventually also the magnitude) of the full coefficient can presumably be attributed to the conformal part. However, for larger values of β_0, relevant to real world QCD, the non-conformal part clearly dominates making the full next-to-next-to-leading order coefficients positive.

These large conformal coefficients in (135) and (136) are due to an intrinsic property of the $\overline{\mathrm{MS}}$ coupling, since they appear already at the level of the Banks-Zaks expansion [35, 36],

$$a_{\overline{\mathrm{MS}}}^{\mathrm{FP}} = a_0 + 1.1366\,{a_0}^2 + 23.2656\,{a_0}^3 + \cdots. \tag{137}$$

Note that $a_{\overline{\mathrm{MS}}}^{\mathrm{FP}}$ has, by far, a larger next-to-next-to-leading order Banks-Zaks coefficient compared to any known physical effective charge.

We stress that the large next-to-next-to-leading order coefficients in (135), (136) and (137) are probably not associated with renormalons. The $\overline{\mathrm{MS}}$ β function, being defined through an ultraviolet regularization procedure, should not be sensitive to the infrared. Therefore infrared renormalons are not expected. It is harder to conclude firmly concerning the absence of ultraviolet renormalons. Since there seems to be no reason to assume a skeleton structure or any other representation in the form of an integral over a running coupling, we suspect that ultraviolet renormalons do not exist there as well.

9 The skeleton expansion and the effective charge approach

A priori, the skeleton expansion approach, which relies on the assumption of a universal skeleton coupling, seems antagonist to the effective charge approach [5] which treats all effective charges independently and in a symmetric manner. Remarkably, we find that the two approaches are very simply related, at least at the level of the leading skeleton.

To see this, consider the effective charge defined by the leading skeleton term R_0. The corresponding effective charge β function is $\beta_{R_0}(R_0) \equiv dR_0/\ln Q^2$. Applying the general relation between effective charges [5] to R_0 and the skeleton coupling $\bar{a}$, we have

$$\beta_2^{R_0} = \bar{\beta}_2 + \beta_0 \left(r_2 - r_1^2 \right) - \beta_1 r_1, \tag{138}$$

where $\bar{\beta}_2$ and $\beta_2^{R_0}$ are the three-loop β function coefficients of the skeleton coupling and of R_0, respectively.

Using now r_1 and r_2 of eq. (8) we obtain

$$\beta_2^{R_0} = \bar{\beta}_2 + \left[r_2^{(2)} - \left(r_1^{(1)} \right)^2 \right] \beta_0^3. \tag{139}$$

This means that for any momentum distribution ϕ_0, $\beta_2^{R_0}$ is simply a sum of a universal piece $\bar{\beta}_2$, which characterizes the skeleton coupling, and an observable-dependent piece, namely the width of ϕ_0 (see section 3) multiplied by β_0^3. We recall that the next-to-next-to-leading order β function coefficient in the skeleton scheme $\bar{\beta}_2$ is a polynomial of order β_0^2 in β_0 (see the footnote following eq. (9)): $\bar{\beta}_2 = \bar{\beta}_{2,0} + \bar{\beta}_{2,1}\beta_0 + \bar{\beta}_{2,2}\beta_0^2$. As noted above, in the large β_0 limit $\beta_2^{R_0}$ is proportional to the width of the distribution ϕ_0. This last statement remains correct also for the β function of the full effective charge a_R of eq. (2) since adding sub-leading skeleton terms would not modify the leading $\mathcal{O}\left(\beta_0^3\right)$ term (see eq. (140) below).

Recall [14, 17] that the accuracy of the leading order BLM approximation is controlled by the width of the momentum distribution function $\left[r_2^{(2)} - \left(r_1^{(1)} \right)^2 \right]$. On the other hand, the accuracy of the effective charge approach at this order is controlled [5] by the magnitude of the three-loop coefficient of the effective charge β function $\beta_2^{R_0}$. However, as we just saw (139), in the large β_0 limit $\beta_2^{R_0}$ is proportional to the width of ϕ_0, and thus the criteria for the accuracy of the two approaches agree! Away from the large β_0 limit, we learn from eq. (139) that a small width implies proximity of $\beta_2^{R_0}$ and $\bar{\beta}_2$. If we assume in addition that the universal $\bar{\beta}_2$ is not large, a small width implies smallness of $\beta_2^{R_0}$, i.e. good convergence of the effective charge approach applied to R_0.

It is natural now to consider the possibility that R_0 is a good approximation to the full observable a_R of eq. (2). In the effective charge approach at the next-to-next-to-leading order, this can be realized if $\beta_2^{R_0}$ is a good approximation to β_2^R. Comparing the two we have

$$\begin{aligned} \beta_2^{R_0} &= \bar{\beta}_{2,0} + \bar{\beta}_{2,1}\beta_0 + \bar{\beta}_{2,2}\beta_0^2 + \left[r_2^{(2)} - \left(r_1^{(1)} \right)^2 \right] \beta_0^3 \\ \beta_2^{R} &= \beta_{2,0}^R + \beta_{2,1}^R\beta_0 + \beta_{2,2}^R\beta_0^2 + \left[r_2^{(2)} - \left(r_1^{(1)} \right)^2 \right] \beta_0^3, \end{aligned} \tag{140}$$

where we exhibited the fact that the term leading in β_0 is the same in $\beta_2^{R_0}$ and β_2^R. Thus, in the large β_0 limit we automatically have $\beta_2^{R_0} = \beta_2^R$. For the four examples considered in the previous section, this parameter is given in table 1.

$\bar\beta_{2,3}$	$\beta^D_{2,3}$	$\beta^{g_1}_{2,3}$	$\beta^{F_1}_{2,3}$	$\beta^V_{2,3}$
0	2.625	2.389	1.500	0

Table 1: Comparison of effective charge β function coefficients in the large β_0 approximation given by the width of ϕ_0, $\beta_{2,3} = r_2^{(2)} - \left(r_1^{(1)}\right)^2$.

Beyond the large β_0 limit one can ask whether

$$\beta^R_{2,0} + \beta^R_{2,1}\beta_0 + \beta^R_{2,2}\beta_0^2 \;\simeq\; \bar\beta_2 \equiv \bar\beta_{2,0} + \bar\beta_{2,1}\beta_0 + \bar\beta_{2,2}\beta_0^2, \tag{141}$$

namely whether $\beta^R_{2,0} + \beta^R_{2,1}\beta_0 + \beta^R_{2,2}\beta_0^2$ for a generic observable which admits a skeleton expansion is approximately universal and close to the three-loop skeleton coupling β function coefficient $\bar\beta_2$. If this holds for arbitrary β_0 then

$$\beta^R_{2,i} \simeq \bar\beta_{2,i} \tag{142}$$

for $i = 0, 1, 2$. The violation of the equalities in (141) and (142) is, of course, due to sub-leading terms in the skeleton expansion R_1 and R_2. This can be seen explicitly by substituting r_i of eq. (45) in the general relation

$$\beta^R_2 = \bar\beta_2 + \beta_0\left(r_2 - r_1^2\right) - \beta_1 r_1 \tag{143}$$

to obtain the "skeleton decomposition" of β^R_2,

$$\beta^R_2 = \bar\beta_2 + \left(s_2 - s_1^2\right)\beta_0 + s_1\left(r_2^{(1)} - 2r_1^{(1)}\right)\beta_0^2 + \left[r_2^{(2)} - \left(r_1^{(1)}\right)^2\right]\beta_0^3 - s_1\beta_1. \tag{144}$$

Finally, decomposing $\bar\beta_2$ and β_1 in terms of β_0, we obtain[‡]

$$\begin{aligned}\beta^R_2 \;&=\; \left[\bar\beta_{2,0} - \beta_{1,0}s_1\right] + \left[\bar\beta_{2,1} - \beta_{1,1}s_1 + \left(s_2 - s_1^2\right)\right]\beta_0 \\ &+\; \left[\bar\beta_{2,2} + \left(r_2^{(1)} - 2r_1^{(1)}\right)s_1\right]\beta_0^2 + \left[r_2^{(2)} - \left(r_1^{(1)}\right)^2\right]\beta_0^3.\end{aligned} \tag{145}$$

Clearly, if for a given observable the skeleton coefficients determining the normalization of the sub-leading skeleton terms (s_i) are small, then even away from the large β_0 limit β_2^R will be close to $\beta_2^{R_0}$.

In order to check (142) explicitly for a given observable, one needs to calculate the β function coefficients of both the observable effective charge $\beta^R_{2,i}$ and the skeleton effective charge $\bar\beta_{2,i}$. For the latter we currently know only $\bar\beta_{2,0}$ (see below) and so the examination of (142) for $\bar\beta_{2,1}$ and $\bar\beta_{2,2}$ cannot yet be accomplished.

[‡]The scheme of the skeleton coupling can be parameterized at the three-loop order [5] by the next-to-leading order coefficient (s_1 and $r_1^{(1)}$) and by $\bar\beta_2$ i.e. $\bar\beta_{2,i}$ for $i = 0, 1, 2$. Eq. (145) then shows explicitly that the effective charge β function coefficient β_2^R determines uniquely the remaining coefficients of the "skeleton decomposition" (45) namely, s_2, $r_2^{(1)}$ and $r_2^{(2)}$. This reflects the observation in section 3 that formally, the "skeleton decomposition" can be performed in any scheme.

To obtain $\bar{\beta}_{2,0}$ we can use the general result [47] or, alternatively use eq. (145), which is valid for a generic effective charge which admits a skeleton expansion. The latter yields,

$$\bar{\beta}_{2,0} = \beta_{2,0}^R + \beta_{1,0} s_1. \tag{146}$$

Using this relation for various effective charges, e.g. the vacuum polarization D-function (107) or the Bjorken sum-rule (109), in the skeleton coupling scheme (12) defined through the pinch technique, we obtain

$$\bar{\beta}_{2,0} = \frac{C_A}{512}\left(44C_F^2 - 88C_AC_F - 301C_A^2\right), \tag{147}$$

and for $N_c = 3$,

$$\bar{\beta}_{2,0} = -\frac{26845}{1536} \simeq -17.477. \tag{148}$$

Finally we check to what extent the suggested universality of the effective charge β function coefficients (142) holds for the four effective charges examined in the previous section, namely the effective charges related to the vacuum polarization D-function (107) and the polarized (109) and non-polarized (110) Bjorken sum-rules, as well as the static potential. The known coefficients are listed in the following table.

i	$\bar{\beta}_{2,i}$	$\beta_{2,i}^D$	$\beta_{2,i}^{g_1}$	$\beta_{2,i}^{F_1}$	$\beta_{2,i}^V$
0	-17.477	-23.607	-30.294	-34.753	-37.54
1	?	-16.032	-11.282	-6.903	5.366
2	?	8.210	8.057	8.783	11.740

Table 2: Comparison of effective charge β function coefficients.

Although the coefficients $\beta_{2,i}^R$ for these observables have some common trend (e.g. for a given i the signs are the same, with the exception of $\beta_{2,i}^V$ for $i = 1$) it turns out that the fluctuations in their magnitude are rather large. In particular, in case of $\beta_{2,0}^R$ for which we know the value of the universal piece characterizing the skeleton coupling $\bar{\beta}_{2,0}$, the latter and the contribution of the sub-leading skeleton R_1 (through s_1 in eq. (145)) are of the same order of magnitude. The fluctuations between different observables are moderate only for $\beta_{2,2}^R$.

In [35] it has been observed that β_2^R for the observables considered above (the static potential excluded) exhibit very close numerical proximity, especially for $N_f = 0$ through 7. The extent to which universality of the sort examined here (142) holds is not enough to explain this finding of [35].

The proximity of $\beta_{2,2}^R$ for the various effective charges implies that applying multi-scale BLM scale-setting for one observable in terms of another, the second scale-shift $t_{1,0}$ would be close to the leading skeleton scale-shift $t_{0,0}$. In this case the single scale setting procedure [8, 18] could give similar results. The same holds in the skeleton scheme, if $\bar{\beta}_{2,2}$ is close to $\beta_{2,2}^R$. This can be deduced from eq. (145) which gives,

$$\beta_{2,2}^R - \bar{\beta}_{2,2} = s_1\left(r_2^{(1)} - 2r_1^{(1)}\right) = 2s_1\left(t_{1,0} - t_{0,0}\right), \tag{149}$$

where in the last step we used the leading order results for the scale-shifts in eq. (26) and (29). In this respect it is interesting to note that applying multi-scale BLM in $\overline{\mathrm{MS}}$, one in general obtains large values for the $t_{1,0}$ scale-shift since $\beta_{2,2}^{\overline{\mathrm{MS}}} = 3.385$ is not close to $\beta_{2,2}$ of the physical effective charges. For example, when applying BLM to $a_D(a_{\overline{\mathrm{MS}}})$ one obtains $k_{0,0} = 0.707Q$ and $k_{1,0} = 0.366\ 10^{-6}Q$. This can be contrasted, for instance, with the BLM scales for $a_D(a_V)$: $k_{0,0} = 1.628Q$ and $k_{1,0} = 2.487Q$.

10 Conclusions

The existence of an Abelian-like skeleton expansion in QCD would make it possible to separate in a unique way running coupling effects from the conformal part of the perturbative expansion of a generic physical quantity. Running coupling effects could then be resummed to all orders in a renormalization-scheme invariant manner by renormalon integrals, up to an uncertainty which is related to infrared renormalons. This uncertainty can be resolved only at the non-perturbative level.

A skeleton expansion also justifies the BLM scale-setting method and implies a specific procedure to set the BLM scales, such that there is a one-to-one correspondence between the terms in the BLM series and the skeletons, provided that BLM scale setting is performed in the skeleton scheme.

We have shown that the N_f-independent coefficients of the postulated skeleton expansion and of the BLM series have a precise interpretation when a perturbative infrared fixed-point is present: they are the conformal coefficients in the series relating the fixed-point value of the observable under consideration with that of the skeleton effective charge. The perturbative infrared fixed-point appearing in multi-flavor QCD allows one to calculate these conformal coefficients through the Banks-Zaks expansion.

We have analyzed the large-order behavior of conformal coefficients in models for the perturbative coefficients which are dominated by the factorial divergence characteristic of renormalons. In general, factorially increasing perturbative coefficients can lead to factorially increasing conformal coefficients. However, we have shown that when the factorial divergence genuinely originates in a renormalon integral, it does not affect the conformal coefficients. The assumed skeleton structure thus implies that the conformal relation between the fixed-point value of a generic observable and that of the skeleton effective charge is renormalon-free. Therefore, upon eliminating the skeleton effective charge, conformal coefficients in commensurate scale relations between observables are also renormalon-free.

In order to argue that also the Banks-Zaks expansion is free of renormalons, it is necessary to assume that the β function of the skeleton coupling is itself renormalon-free. However, since we do not have a precise identification of the skeleton coupling at large orders, this remains an open question.

We conclude that BLM (conformal) coefficients do not diverge factorially due to renormalons, provided there is an underlying skeleton structure. Of course, there can be other effects which could make these coefficients diverge such as combinatorial factors related to the multiplicity of diagrams. Since in QCD this type of divergence is much softer than that of renormalons, we expect the BLM and possibly also the Banks-Zaks expansions to be "better behaved". This expectation is supported to some extent (section 8) by previous observations concerning the smallness of the first few known BLM coefficients [12] and the Banks-Zaks coefficients [34, 35, 36].

The assumed skeleton expansion has two ingredients: the conformal template, based on the bare skeleton diagrams, and running-coupling effects corresponding to dressing each skeleton. In this paper we addressed mainly the conformal coefficients. We saw that through the skeleton expansion, conformal relations which have a natural, maximally convergent, form (like the conformal Crewther relation) become relevant for real-world QCD predictions. Resummation of running coupling effects can be achieved in practice either by BLM scale setting, or - by evaluating the renormalons integral, as in [27], dealing with the infrared renormalon ambiguity through some well-defined regularization prescription, like principle-value or a cut-off. The advantage of this procedure is that once the infrared renormalon ambiguity is identified, it can be used for the parameterization of the related power suppressed effects.

The uniqueness of the skeleton coupling in QED, which is identified as the Gell-Mann Low effective charge, is an essential ingredient of the dressed skeleton expansion. We emphasize that it is an open question whether an Abelian-like skeleton expansion exists in QCD and whether there are constraints which would determine uniquely the skeleton coupling. The pinch technique may provide the answer [23, 24, 25] once it is systematically carried out to higher orders. The skeleton coupling is not constrained from the considerations raised in this paper: the only requirement following from the large N_f limit is that $\bar{\beta}_i$ in this scheme does not contain an N_f^{i+1} term. Since the decomposition of the coefficients (45) can be performed in any scheme yielding the moments $r_i^{(j)}$ to arbitrary high order, the corresponding functions ϕ_i can be formally constructed, up to the limitations discussed in section 3.4. It thus seems that one can formally associate a "skeleton expansion" to any given coupling. The absence of renormalons in the conformal coefficients in a specific scheme implies that there are other schemes which share the same property: it is straightforward to see from the definition of the skeleton terms R_i that an N_f-independent re-scaling of the argument of the coupling leaves the conformal coefficients unchanged. More generally, any "renormalon free" transformation of the skeleton coupling would leave the "skeleton coefficients" free of renormalons. It is certainly interesting to find further constraints on the identity of the skeleton effective charge in QCD.

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