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CORE and the Haldane Conjecture *

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ABSTRACT

In an earlier paper[1] I showed that the Contractor Renormalization group (CORE) method could be used to map a theory of free quarks, and quarks interacting with gluons, into a generalized frustrated Heisenberg antiferromagnet (HAF) and proposed using CORE methods to study these theories. Since generalizations of HAF's exhibit all sorts of subtle behavior which, from a continuum point of view, are related to topological properties of the theory, it is important to know that CORE can be used to extract this physics. In this paper I show that simple Contractor Renormalization group (CORE) computations provide a first principles understanding of the famous Haldane conjecture. Explicit range-2 computations for the spin-1/2 and spin-1 Heisenberg antiferromagnet reveal the differences between these theories and show that the mass gap in the spin-1 theory is intimately related to the structure of a more general theory with Hamiltonian $H = \sum_i [\vec{s}(i) \cdot \vec{s}(i+1) - \beta\,(\vec{s}(i) \cdot \vec{s}(i+1))^2]$ which has a valence bond ground state when $\beta = -1/3$. I then argue that the case of a general spin-S HAF works similarly. More specifically, for integer S the renormalized Hamiltonian is described by a polynomial in the operators $\vec{s}(i) \cdot \vec{s}(i+1)$ with coefficients which lie near the values for which the Hamiltonian would be of the type introduced by Affleck, Lieb, Kennedy and Tasaki (AKLT)[6], all of which have valence bond ground states.

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1 Introduction

In a preceding paper[1] I used the Contractor Renormalization Group (CORE) formalism to establish the equivalence of a theory of free quarks, or a theory of quarks interacting with gluons, to a *generalized* frustrated antiferromagnet. I plan to use the CORE formalism to study the physics of these theories and so it is necessary to understand whether or not CORE can systematically deal with subtle phenomena which appear when one generalizes the basic Heisenberg antiferromagnet. This paper is devoted to the question of what happens when the spin-1/2 Heisenberg antiferromagnet (HAF) Hamiltonian,

$$H_{\rm HAF} = \sum_j \vec{s}_j \cdot \vec{s}_{j+1}, \tag{1}$$

is modified by replacing the spin-1/2 degree of freedom associated with each lattice site with a spin-S.

A conjecture due to F.D.M. Haldane, Ref. [2] says that for a spin-S HAF the spectrum of $H_{\rm HAF}$ is massless whenever S is a half-integer and exhibits a mass-gap when S is an integer. While there is a good deal of numerical evidence indicating that this conjecture is true and theoretical arguments for its validity based upon relating the long wavelength physics of the finite S lattice system to a continuum Wess-Zumino-Witten model, I know no simple first principles argument which explains the origin of this gap. This paper presents a straightforward CORE computation which provides a non-trivial, nonperturbative insight into why the Haldane conjecture is true.

To clarify the differences between the spin-1/2 and spin-1 HAF I present range-2 CORE computations for each case. These computations show that important differences appear with the first CORE transformation and subsequent iterations of the CORE transformations allows one to directly extract the physics of each theory.

1.1 Summarizing The Main Results

In the spin-1/2 theory the simplest non-trivial CORE computation shows that the form of the Hamiltonian, $H_{\rm HAF}$ remains unchanged as one carries out successive renormalization group transformations. From this one is able to compute the groundstate energy density and show the mass gap vanishes. The same technique applied to the spin-1 theory leads to a renormalized Hamiltonian which has the more general form

$$H = \sum_j \vec{s}_j \cdot \vec{s}_j - \beta \sum_j (\vec{s}_j \cdot \vec{s}_j)^2. \tag{2}$$

and shows that for the relevant range of β this Hamiltonian describes a theory with a mass gap. The result that the mass gap is non-zero for $-1 < \beta < 1$ is a non-trivial result since Eq. 35 defines a class of theories about which, until now, very little was known. One reason that this result is important to the general program discussed in my earlier paper[1], is that terms of the form $(\vec{s} \cdot \vec{s})^2$ appear when one uses CORE to map the free fermion theory in $3+1$ dimensions into a generalized HAF.

2 Generalized Heisenberg Antiferromagnets

2.1 Basic Algorithm

CORE has two parts. The first is a theorem which defines the Hamiltonian analog of Wilson's exact renormalization group transformation; the second is a set of approximation procedures which render nonperturbative calculation of the *renormalized Hamiltonian* doable. A detailed review of the general method can be found in Ref. [1] and a detailed presentation of the CORE formalism can be found in Ref. [3]. In this section I limit myself to a review of the basic concepts for the special case of a general Heisenberg antiferromagnet.

CORE replaces the Lagrangian notion of integrating out degrees of freedom by that of throwing away Hilbert space states, or alternatively *retaining* only a subset of the full Hilbert space defined as the image of a projection operator, P, acting on the original space, $\mathcal{H}$. In other words, $\mathcal{H}_{\rm ret} = P\mathcal{H}$. In what follows, for both the spin-1/2 and spin-1 case, this set of retained states will be defined by diagonalizing the Hamiltonian restricted to either a two or three site block and defining P as the operator which projects onto the subspace spanned by a small number of its lowest energy eigenstates.

The formula relating the original Hamiltonian, H, to a *renormalized Hamiltonian* having the same low energy physics is

$$H^{\rm ren} = \lim_{t\to\infty} [[T(t)^2]]^{-1/2}\, [[T(t)HT(t)]]\, [[T(t)^2]]^{-1/2}, \tag{3}$$

where $T(t) = e^{-tH}$ and where $[[O]] = POP$ for any operator O which acts on $\mathcal{H}$. It is not generally possible to evaluate Eq. 3 exactly, however it is possible to nonperturbatively approximate the infinite lattice version of $H^{\rm ren}$ to any desired accuracy. This is because $H^{\rm ren}$, as defined in Eq. 3, is an extensive operator and has the general form

$$H^{\rm ren} = \sum_j \sum_{r=1}^{\infty} h^{\rm conn}(j,r) \tag{4}$$

where each term, $h^{\rm conn}(j,r)$, stands for a set of *range-r connected* operators based at site j, all of which can be evaluated to high accuracy using finite size lattices.

CORE is useful because it is typically not necessary to calculate all of the terms in $H^{\rm ren}$. Often one can obtain highly accurate results by approximating $H^{\rm ren}$ by its range-2 or range-3 terms. I will show that all one has to do to identify the major difference between the case of the spin-1/2 HAF and its spin-1 counterpart is do a simple range-2 CORE computation.

In general the range-1 connected term in the renormalized Hamiltonian is defined to be the matrix obtained by evaluating the $j^{\rm th}$ block Hamiltonian in the set of retained eigenstates,

$$h^{\rm conn}(j,1) = [[H_{\rm block}(j)]]. \tag{5}$$

The range-2 part of the renormalized Hamiltonian is evaluated as follows: first, restrict the full Hamiltonian to two adjacent (i.e., connected) blocks and define the two-block retained states as tensor products of the single block retained states; next, use these states to define a projection operator and evaluate Eq. 3, where $H = H(j, j+1)$ is the Hamiltonian restricted to blocks B_j and B_{j+1} to obtain

$$H_{2-block}(j, j+1) = \lim_{t\to\infty} [[T(t)^2]]^{-1/2} \, [[T(t)HT(t)]] \, [[T(t)^2]]^{-1/2}; \tag{6}$$

finally, construct the *connected range-2* contribution to the renormalized Hamiltonian by subtracting the two ways of embedding the one-block computation into the connected two-block computation as follows,

$$h^{\rm conn}(j, 2) = H_{2-block}(j, j+1) - h^{\rm conn}(j, 1) - h^{\rm conn}(j+1, 1). \tag{7}$$

It might appear to be difficult to take the $t \to \infty$ limit of Eq. 6, however it is easy to show that this limit can be evaluated as a product of the form

$$H_{2-block}(j, j+1) = R \, H_{\rm diag} \, R^\dagger \tag{8}$$

where R is an orthogonal transformation and $H_{\rm diag}$ is a diagonal matrix. $H_{\rm diag}$ is constructed by expanding the image under R of each of the tensor product states in a complete set of eigenstates of the two-block problem and putting the energy of the lowest lying eigenstate appearing in the expansion of each rotated state on the diagonal. R is constructed to guarantee that for each rotated state the lowest energy eigenstate of the two-block problem which appears in its expansion in a complete set of eigenstates is distinct from that appearing in the expansion of the other rotated states. As we will see in a moment, given the symmetries of the problem, constructing R is straightforward for both the spin-1/2 and spin-1 HAF.

2.2 The Spin-1/2 Case

The one-dimensional spin-1/2 HAF is defined by the Hamiltonian

$$H = \sum_{j=-\infty}^{\infty} \vec{s}_j \cdot \vec{s}_{j+1} \tag{9}$$

where the matrices $\vec{s_j}$ are, up to a factor of 1/2, the 2×2-Pauli σ matrices. I already noted that the simplest non-trivial truncation procedure for this theory requires working with three site blocks. To see why a two site blocking procedure won't work just diagonalize the two site Hamiltonian

$$\begin{aligned} H_{\rm block} &= \vec{s}_1 \cdot \vec{s}_2 = \frac{1}{2} \left(\vec{s}_1 + \vec{s}_2\right)^2 - \frac{3}{4}, \\ &= \frac{1}{2} S^2_{\rm TOT}(1, 2) - \frac{3}{4}, \end{aligned} \tag{10}$$

where the notation $S^2_{\rm TOT}(1,2)$ is used to represent the total spin operator for sites 1 and 2. Since the two site Hamiltonian $H_{\rm block}$ is just $S^2_{\rm TOT}$ minus a constant and the two spin-1/2 states can make either a total spin zero or spin one state, Eq. 10 shows that of the four states in the two site Hilbert space, there is one spin zero state of energy $E_0 = -3/4$ and three degenerate spin one states with energy $E_1 = 1/4$. The fact that the spin-0 state has the lowest energy means that an algorithm based upon keeping a subset of the lowest lying eigenstates of $H_{\rm block}$ requires either that we keep the single spin-0 state, or that we keep all four eigenstates of $H_{\rm block}$. Obviously the first choice, truncating to one state per block, produces a renormalized Hamiltonian which is a one-by-one matrix, which will only allow us to compute the energy density of the ground state. Equally obviously, keeping all of the states per two site block doesn't define a truncation at all.

Fortunately, working with three site blocks is as simple as working with two site blocks but it allows for a more interesting computation. In this case the Hamiltonian has the form

$$\begin{aligned} H_{\rm block} &= \vec{s}_1 \cdot \vec{s}_2 + \vec{s}_2 \cdot \vec{s}_3 && (11)\\ &= \vec{s}_2 \cdot (\vec{s}_1 + \vec{s}_3) && (12)\\ &= \frac{1}{2}\left(S^2_{\rm TOT}(1,2,3) - S^2_{\rm TOT}(1,3) - \frac{3}{4} \right) && (13) \end{aligned}$$

Note that one can make a two spin-1/2 and one spin-3/2 multiplet out of the product of three spin-1/2 states which can be distinguished from one another by the eigenvalue of $S^2_{\rm TOT}(1,3)$. The three possibilities are one spin-1/2 representation for which $S^2_{\rm TOT}(1,3) = 2$ and $S^2_{\rm TOT}(1,2,3) = 3/4$, one spin-1/2 representation for which $S^2_{\rm TOT}(1,3) = 0$ and $S^2_{\rm TOT}(1,2,3) = 3/4$ and finally, one spin-3/2 representation for which $S^2_{\rm TOT}(1,3) = 2$ and $S^2_{\rm TOT}(1,2,3) = 15/4$. Substituting these values into Eq. 11 shows that the lowest lying eigenstates are the two degenerate spin-1/2 states for which $S^2_{\rm TOT}(1,3) = 2$ and these have energy $E_0 = -1$. Keeping this spin-1/2 representation and using it to generate the space of retained states leads to a non-trivial truncation procedure. In this case the dimension of the space of retained states is 2^{N_B}, where N_B stands for the number of blocks on the thinned lattice. (Of course, if one starts with an infinite lattice $N_B = \infty$ and so, even after truncation, the space of the retained states remains infinite dimensional.)

If we label the two spin-1/2 states which we keep in block B_j as $|\uparrow_j\rangle$ and $|\downarrow_j\rangle$, then the projection operator is

$$\begin{aligned} P_j &= |\uparrow_j\rangle\langle\uparrow_j| + |\downarrow_j\rangle|\downarrow_j\rangle \\ P &= \prod_j P_j && (14) \end{aligned}$$

By definition the connected range-1 Hamiltonian is $P_j\, H_{\rm block}(j)\, P_j$ which, because the two retained states are degenerate, is simply a multiple of the identity matrix; i.e.,

$$h^{\rm conn}(j,1) = \mathbf{-1}. \qquad (15)$$

and so, to this range, the renormalized Hamiltonian is

$$H^{\rm ren} = \sum_j h^{\rm conn}(j,1) = -V_{\rm thin}\,\mathbf{1}; \tag{16}$$

i.e., every state in the space of retained states is an eigenstate of the renormalized Hamiltonian with eigenvalue $-V_{\rm thin}$, where $V_{\rm thin}$ is the volume of the thinned lattice. Note that $V_{\rm thin} = V/3$ and so the contribution to the energy density of the original theory is $-1/3$. Clearly, since all retained states are eigenstates of the range-1 part of the renormalized Hamiltonian, this term plays no role in the dynamics of the renormalized theory. To get a nontrivial renormalized Hamiltonian it is necessary to calculate $h^{\rm conn}(j,2)$.

The first step in computing $h^{\rm conn}(j,2)$ is to expand the retained states for the two-block problem in terms of the exact eigenstates of the two-block Hamiltonian. A brute force way to do this is to exactly diagonalize the full two-block, or six-site, Hamiltonian, find its eigenvalues and eigenstates and then carry out the expansion. This is not an intelligent use of computing resources, one can achieve the desired goal with less work since the spin-1/2 HAF has so much symmetry.

The three site truncation procedure is based upon keeping the two states of the lowest lying spin-1/2 representation of $SU(2)$ for each three site block. Thus, if we are working with blocks $\{B_j, B_{j+1}\}$, then the four-dimensional space of retained states is spanned by the four tensor product states

$$|\uparrow_j\rangle|\uparrow_{j+1}\rangle,\ |\uparrow_j\rangle|\downarrow_{j+1}\rangle,\ |\downarrow_j\rangle|\uparrow_{j+1}\rangle,\ |\downarrow_j\rangle|\downarrow_{j+1}\rangle\,. \tag{17}$$

As stated earlier, to find the matrix R it is necessary to find a set of orthonormal combinations of these states which contract onto unique eigenstates of the six-site problem. While in general this requires expanding the tensor product states in terms of eigenstates of the six-site problem, the symmetries of this problem make finding R an exercise in group-theory because the six-site Hamiltonian has the same $SU(2)$ symmetry of the full problem and its eigenstates also fall into irreducible representations of $SU(2)$.

The argument goes as follows. The space of retained states is generated from a tensor product of two spin-1/2 representations and it can be uniquely decomposed into a direct sum of one spin-0 and one spin-1 representation. Furthermore, the three spin-1 states can be uniquely identified by their total S_z eigenvalues, $1, 0, -1$. The linear combinations corresponding to these $|S, S_z\rangle$ eigenstates are

$$\begin{aligned}
|0,\ \ 0\rangle &= -\ \frac{1}{\sqrt{2}}\left(|\uparrow_j\rangle|\downarrow_{j+1}\rangle - |\downarrow_j\rangle|\uparrow_{j+1}\rangle\right)\\
|1,\ \ 1\rangle &= |\uparrow_j\rangle|\uparrow_{j+1}\rangle\\
|1,\ \ 0\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow_j\rangle|\downarrow_{j+1}\rangle + |\downarrow_j\rangle|\uparrow_{j+1}\rangle\right)\\
|1,-1\rangle &= |\downarrow_j\rangle|\downarrow_{j+1}\rangle
\end{aligned}$$

Since $SU(2)$ is an exact symmetry of the six-site problem only eigenstates of $H_{6\text{sites}}$ having the same S and S_z can appear in the expansion of each one of these states; thus it follows directly from Eq. 18 that all one need to find $h^{\text{conn}}(j,2)$ is to find the energy of the lowest lying spin-0 and lowest lying spin-1 multiplet for $H_{6-sites}$. This observation, coupled with the fact that the spin-0 states is odd under left-right interchange, whereas the spin-1 state is even, reduces the general problem of diagonalizing a 64×64-matrix to that of diagonalizing a couple of 3×3-matrices. As the states in the spin-1 multiplet are degenerate the result of this calculation is an H_{diag} of the form

$$H_{\text{diag}} = \begin{pmatrix} \epsilon_0 & 0 & 0 & 0 \\ 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{pmatrix} \tag{18}$$

Using Eq. 18 it is simple to compute $R^\dagger H_{\text{diag}} R$ acting on the original tensor product states. Fortunately, one can avoid doing even this amount of work. Due to the $SU(2)$ symmetry of the theory $R^\dagger H_{\text{diag}} R$ must have the form

$$R^\dagger H_{\text{diag}} R = \alpha_0 \mathbf{1} + \alpha_1 \vec{s}_j \cdot \vec{s}_{j+1} \tag{19}$$

To relate α_0 and α_1 to ϵ_0 and ϵ_1 use the usual trick and rewrite $R^\dagger H_{\text{diag}} R$ as

$$\begin{aligned} R^\dagger H_{\text{diag}} R &= \alpha_0 \mathbf{1} + \alpha_1 \vec{s}_j \cdot \vec{s}_{j+1} \\ &= \alpha_0 \, \mathbf{1} + \frac{\alpha_1}{2} \left((\vec{\mathbf{s}}_\mathbf{j} + \vec{\mathbf{s}}_{\mathbf{j+1}})^2 - \frac{\mathbf{3}}{\mathbf{2}} \right). \end{aligned} \tag{20}$$

Since $(\vec{s}_j + \vec{s}_{j+1})^2$ equals 0 for a spin-0 state and 2 for a spin-1 state, it follows

$$\begin{aligned} \epsilon_0 &= \alpha_0 - \frac{3\alpha_1}{4} \\ \epsilon_1 &= \alpha_0 + \frac{\alpha_1}{4} \end{aligned} \tag{21}$$

Solving for α_0 and α_1 in terms of ϵ_0 and ϵ_1

$$\begin{aligned} \alpha_0 &= \frac{3\epsilon_1 + \epsilon_0}{4} \\ \alpha_1 &= \epsilon_1 - \epsilon_0. \end{aligned} \tag{22}$$

A straightforward computation of the energies of the lowest spin-0 and spin-1 eigenstates of $H_{6-\text{sites}}$ gives

$$\begin{aligned} \epsilon_0 &= -2.493577 \\ \epsilon_1 &= -2.001995 \\ \alpha_0 &= -2.124891 \\ \alpha_1 &= 0.491582 \end{aligned} \tag{23}$$

To obtain $h^{\rm conn}(j,2)$ it is necessary to subtract $h^{\rm conn}(j,1)$ and $h^{\rm conn}(j+1,1)$ from $R^\dagger\, H_{\rm diag}\, R$ as follows

$$\begin{aligned} h^{\rm conn}(j,2) &= R^\dagger\, H_{\rm diag}\, R - h^{\rm conn}(j,1) - h^{\rm conn}(j+1,1) \\ &= (\alpha_0+2)\mathbf{1} + \alpha_1 \vec{s}_j \cdot \vec{s}_{j+1}. \end{aligned} \tag{24}$$

Finally, given $h^{\rm conn}(j,2)$, the range-2 renormalized Hamiltonian is

$$\begin{aligned} H^{\rm ren} &= \sum_j \left(h^{\rm conn}(j,1) + h^{\rm conn}(j,2)\right) \\ &= \sum_j \left(\left(\alpha_0+1\right)\mathbf{1} + \alpha_1\, \vec{\mathbf{s}}_{\mathbf{j}} \cdot \vec{\mathbf{s}}_{\mathbf{j+1}}\right) \\ &= V_{\rm thin}\left(\alpha_0+1\right)\mathbf{1} + \alpha_1 \sum_{\mathbf{j}} \vec{\mathbf{s}}_{\mathbf{j}} \cdot \vec{\mathbf{s}}_{\mathbf{j+1}} \end{aligned} \tag{25}$$

For an infinite lattice, the fact that the term $V\left(\alpha_0+1\right)\mathbf{1}$ only contributes a constant to the energy density of all states and plays no dynamical role means that the energy density of the thinned lattice is (α_0+1) plus α_1 times the energy density of the theory we started with. Since each site of the thinned lattice corresponds to three sites on the original lattice we have, according to the simple range-2 renormalization group approximation, that the energy density of the spin-1/2 HAF, $\mathcal{E}$, satisfies the following equation

$$\mathcal{E} = \frac{(\alpha_0+1)}{3} + \frac{\alpha_1}{3}\,\mathcal{E}. \tag{26}$$

or

$$\mathcal{E} = \frac{\alpha_0+1}{3\left(1-\alpha_1/3\right)}. \tag{27}$$

Substituting the values of α_0 and α_1 obtained from the two-block computation we find $\mathcal{E}_{\rm ren.grp.} = -0.4484$, which is to be compared to the exact result $\mathcal{E}_{\rm exact} = -0.4431$. Thus the error in this CORE result, obtained from an exceptionally simple first principles calculation, is a factor of two better than that obtained from the leading term in Anderson's[4] spin-wave approximation which assumes that the spin s is a large number and then continues the answer to $s = 1/2$.

Since the CORE equation says that the mass-gap of the renormalized theory should be the same as that of the original theory, the fact that $\alpha_1 < 1$ means that this gap must vanish. Specifically, since $\left(\alpha_0+1\right)\mathbf{1}$ plays no role in the dynamics of the renormalized theory the gap is determined by the range-2 term which is just $\alpha_1 \sum \vec{s}_j \cdot \vec{s}_{j+1}$. But this is just α_1 times the original Hamiltonian and so it follows that the mass gap of the theory must satisfy the equation

$$m = \alpha_1\, m. \tag{28}$$

Since $0 < \alpha_1 < 1$ this means $m = 0$.

This ends our demonstration of how CORE is applied to the spin-1/2 theory.

2.3 The Spin-1 Case

Unlike the spin-1/2 HAF the spin-1 theory admits a non-trivial two site truncation procedure; i.e., since the nine eigenstates of the two site problem fall into a spin-0, spin-1 and spin-2 representation of $SU(2)$ one can choose to truncate to the subspace spanned by the spin-0 and spin-1 multiplets alone. Implementing this truncation procedure leads to a renormalized theory which has four instead of three states per site and a more general Hamiltonian; however, subsequent truncations based upon the same algorithm preserve the form of the more general theory and give rise to RG-flows which are simple to compute.

Before discussing the two site blocking procedure in detail I wish to argue that the observation that it is important to keep both the lowest lying spin-0 and spin-1 states is robust in that while one can begin (as I did for the case of the spin-1/2 HAF) with a three site blocking procedure which keeps only the lowest lying $SU(2)$ representation, the dynamics of the renormalized Hamiltonian makes it impossible to continue to do this in subsequent iterations. The fact is that after the first iteration the renormalized Hamiltonian takes a generalized form which forces one to keep both the lowest lying spin-0 and spin-1 representations; reducing us to the prescription of the two site case.

The three site Hamiltonian of the spin-1 HAF is given by Eq. 11. The main difference is that in the spin-1 case more values are allowed for $S^2_{\rm TOT}(1,2,3)$ and $S^2_{\rm TOT}(1,3)$. Direct substitution of these allowed values into Eq. 11 shows that the lowest lying $SU(2)$ multiplet for the three site Hamiltonian is the spin-1 representation obtained by coupling the spins on sites 1 and 3 to a spin-2 state and then coupling this state to the spin-1 on site 2 to get a state of total spin-1. Following the dictum of keeping the lowest lying irreducible representation of $SU(2)$ we obtain a truncation procedure based upon three site blocks which has the same spin content per site as in the original theory, paralleling the spin-1/2 calculation. There is one important difference however, although the number of states per site remains the same, the range-2 renormalized Hamiltonian takes the more general form

$$H^{\rm ren} = \sum_j C\,\mathbf{1} + \alpha\,\vec{s}(j)\cdot\vec{s}(j+1) - \beta\,(\vec{s}(j)\cdot\vec{s}(j+1))^2. \tag{29}$$

To derive this general form I begin by observing that, as in the spin-1/2 case, the range-1 connected Hamiltonian must be a multiple of the unit matrix, since we keep only a single representation of $SU(2)$ per site. As before, this means that the first non-trivial contribution to the renormalized Hamiltonian comes from the range-2 terms. The first contribution to the connected range-2 Hamiltonian comes from consideration of the two-block (or six-site) problem. Since the truncation retains one spin-1 multiplet per block, the retained states of the two-block problem (obtained by taking the tensor product of the retained spin-1 states for each block) fall into a spin-0, spin-1 and spin-2 representation of $SU(2)$. The general CORE rules tell us that the renormalized range-2 Hamiltonian will have these states as eigenstates, with

eigenvalues ϵ_0, ϵ_1 and ϵ_2, where these stand for the energies of the lowest lying spin-0, spin-1 and spin-2 states of the six-site problem. Obviously one can use a brute force approach to construct the transformation R and use it to derive the general form of the connected range-2 term in the original tensor product basis but, by using a little ingenuity, one can avoid this step completely.

The trick is to construct the projection operators $P_0(i,i+1)$, $P_1(i,i+1)$ and $P_2(i,i+1)$ for each pair of sites i and $i+1$ of the renormalized theory; i.e.,

$$\begin{aligned} P_0(i,i+1) &= \frac{1}{12}\left(S^2_{\rm TOT}(i,i+1)-2\right)\left(S^2_{\rm TOT}(i,i+1)-6\right) \\ P_1(i,i+1) &= -\frac{1}{8}\,S^2_{\rm TOT}(i,i+1)\left(S^2_{\rm TOT}(i,i+1)-6\right) \\ P_2(i,i+1) &= \frac{1}{24}\,S^2_{\rm TOT}(i,i+1)\left(S^2_{\rm TOT}(i,i+1)-2\right) \end{aligned} \tag{30}$$

where the operators $\vec{s}_i$ denote the spin operators acting on the retained states of the renormalized theory for site i and where I have defined

$$S^2_{TOT}(i,i+1) = \left(\vec{s}_i + \vec{s}_{i+1}\right)^2 = 2\,\vec{s}_i\cdot\vec{s}_{i+1} + 4. \tag{31}$$

Without actually computing anything one can now write

$$\lim_{t\to\infty}[[T(t)\,H\,T(t)]] = R^\dagger\,H_{diag}\,R^\dagger = \epsilon_0\,P_0 + \epsilon_1\,P_1 + \epsilon_2\,P_2 \tag{32}$$

which, using Eq.30, can be immediately rewritten in the form given in Eq. 29.

We now see that after a single renormalization group transformation one has a theory with the same spin content per site but with a more general Hamiltonian and so, in order to carry out the next renormalization group step, it is necessary to restudy the eigenvalue problem (for either two or three site blocks) for generic values of C, α and β. Of course, since the only important question from the point of view of a CORE computation is the ordering of eigenstates in the two or three block problem we can, without loss of generality, set $C=0$ and $\alpha=1$. Thus, as advertised in the overview, we see that in order to study the generic problem it is necessary to start from the Hamiltonian

$$H^{\rm ren} = \sum_j \vec{s}(j)\cdot\vec{s}(j+1) - \beta\left(\vec{s}(j)\cdot\vec{s}(j+1)\right)^2. \tag{33}$$

(Note, the value $\beta=0$ corresponds to the original spin-1 HAF.)

The result of diagonalizing the two site version of this Hamiltonian for $-1\le\beta\le 1$ is shown in Fig.1 and the results for the three site problem in Fig.2, where I have limited discussion to the range $-1\le\beta\le 1$ for reasons which will become apparent. Note that due to the different numbers of eigenstates, etc., these plots look quite different from one another, however they share several important common features. First, observe that the lowest lying spin-0 and spin-1 state become degenerate at $\beta=-1/3$ and then cross one another. This level crossing means, as I said earlier,

that any CORE computation which wishes to treat the region from $-1 \leq \beta \leq 1$ must keep both multiplets; i.e., in either the two or three site case, after the initial renormalization group step we arrive at a generalized Hamiltonian which forces us to adopt the two site prescription of keeping the lowest lying spin-0 and spin-1 states. Second, it is worth noting that something very special happens at the point $\beta = -1$. In the two site case we see that at this point the lowest lying multiplet is the three-dimensional spin-1 representation of $SU(2)$ and that the spin-0 and spin-2 states become degenerate and form a single six-dimensional subspace which in fact coincides with the six-dimensional representation of $SU(3)$. The degeneracy patterns shown here demonstrate that the Hamiltonian for $\beta = -1$ can be rewritten as

$$H_{\beta=-1} = \vec{Q}(i) \cdot \vec{Q}(i+1) \tag{34}$$

where the $\vec{Q}_i$'s stand for the generators of $SU(3)$. In this picture we see that the spin-1 representation can be identified as the triplet representation of $SU(3)$ and the degenerate multiplets of the two site problem can be understood to be the $\overline{3}$ and 6 representations of $SU(3)$ obtained from the tensor product of two 3's. A brief look at Fig.2 supports this picture. Here we see that at $\beta = -1$ the 27 states become one one-dimensional multiplet, two eight-dimensional multiplets and one ten-dimensional multiplet of degenerate states. This is, of course, completely consistent with what would be obtained from the product of three fundamental triplet representations of $SU(3)$ with the Hamiltonian given in Eq. 34. This explains my earlier statement that something interesting happens for $\beta = -1$ and shows that if one really wished to properly handle this point, it would be necessary to either adopt a truncation procedure which keeps more states, or one which goes beyond the range-2 cluster contribution in order to make up for the violence one is doing to the $SU(3)$ symmetry of the problem. Clearly, treating the full $SU(3)$ symmetry of the problem correctly would require us to eschew a two site blocking procedure, since in this case the only non-trivial truncation would be to a single state. If we adopted a three site blocking procedure then we could adopt a non-trivial truncation based upon keeping nine states, i.e., the lowest lying singlet and octet representations. Discussion of this problem goes beyond the scope of this paper. However I mention it to explain why one expects from the outset to have trouble using the four-state truncation algorithm which I will discuss for values of $\beta \leq -1$.

2.4 The Calculation

Since I just finished arguing that generically, after a single renormalization group step, one will have to deal with a Hamiltonian of the form

$$H = \sum_i \vec{s}(i) \cdot \vec{s}(i+1) - \beta \left(\vec{s}(i) \cdot \vec{s}(i+1)\right)^2 \tag{35}$$

I will describe the two-block CORE procedure for this generalized spin-1 HAF. As I already indicated, since this Hamiltonian doesn't have a single-site term, the first step

of the CORE computation is to solve the two site problem exactly and truncate to the lowest spin-0 and spin-1 multiplets of the resulting nine state system (i.e., throw away the spin-2 multiplet). With this choice of projection operator the renormalized range-1 Hamiltonian is a diagonal 4×4 matrix of the general form

$$h^{\rm conn}(j,1) = H_{\rm diag} = \begin{pmatrix} \epsilon_0(\beta) & 0 & 0 & 0 \\ 0 & \epsilon_1(\beta) & 0 & 0 \\ 0 & 0 & \epsilon_1(\beta) & 0 \\ 0 & 0 & 0 & \epsilon_1(\beta) \end{pmatrix} \tag{36}$$

To obtain the range-2 term of the renormalized Hamiltonian we have to solve the two-block or four-site Hamiltonian exactly and use the information about the exact eigenvalues and eigenstates to construct R and $H_{\rm diag}$. While in principle R is a 16×16 matrix, in practice, as in the case of the spin-1/2 HAF, the $SU(2)$ symmetry of the problem greatly simplifies the job of finding R even though there aren't enough symmetries to render the problem trivial. More precisely, the single-block states fall into a spin-0 and spin-1 representation of $SU(2)$ so, taking tensor products, we see that the retained states for the two-block problem are two spin-0 representations, three spin-1 representations and one spin-2 representation of this group. Clearly, if we expand any one of the spin-2 states in eigenstates of the four-site problem only states with the same quantum numbers can appear. Hence, since each of the spin-2 states is distinguished by its third component of spin, each of the spin-2 states will contract onto a different eigenstate of the two-block or four-site problem but they will all have the same energy. This argument shows that the transformation R_1 which takes us from the original tensor product basis to the spin basis is all one has to do for the spin-2 states. Since there are two independent spin-0 representations contained in the tensor product of the single-block states we have to do a bit more work to fully construct R. To understand exactly what has to be done, let $|\Psi_1\rangle$ and $|\Psi_2\rangle$ denote the spin-0 states which can be formed from the $0 \otimes 0$ and $1 \otimes 1$ representations of $SU(2)$. These states can be expanded in terms of spin-0 eigenstates of the two-block problem as

$$\begin{aligned} |\Psi_0\rangle &= a_0|\phi_0\rangle + a_1|\phi_1\rangle + a_2|\phi_2\rangle + \ldots \\ |\Psi_1\rangle &= b_0|\phi_0\rangle + b_1|\phi_1\rangle + b_2|\phi_2\rangle + \ldots \end{aligned} \tag{37}$$

If, as will generally be the case, both a_0 and b_0 are non-vanishing, then both states will contract onto $|\phi_0\rangle$. One can always avoid this however by defining rotated states as follows

$$\begin{aligned} |\chi_0\rangle &= \cos(\theta)\,|\Psi_0\rangle + \sin(\theta)\,|\Psi_1\rangle \\ |\chi_1\rangle &= -\sin(\theta)\,|\Psi_0\rangle + \cos(\theta)\,|\Psi_1\rangle \end{aligned} \tag{38}$$

where $\cos(\theta) = a_0/\sqrt{a_0^2 + b_0^2}$ and $\sin(\theta) = b_0/\sqrt{a_0^2 + b_0^2}$. With this orthogonal change

of basis we have

$$\begin{aligned} |\chi_0\rangle &= \alpha_0|\phi_0\rangle + \alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle + \alpha_3|\phi_3\rangle + \ldots \\ |\chi_1\rangle &= \beta_1|\phi_1\rangle + \beta_2|\phi_2\rangle + \beta_3|\phi_3\rangle + \ldots \end{aligned} \tag{39}$$

With this definition $|\phi_0\rangle$ is the lowest lying eigenstate of the two-block Hamiltonian which appears in the expansion of $|\chi_0\rangle$ and $|\phi_1\rangle$ is the lowest lying eigenstate which appears in the expansion of $|\chi_1\rangle$; hence, if one applies e^{-tH} to the rotated states one sees that $|\chi_0\rangle$ contracts onto $|\phi_0\rangle$ and $|\chi_1\rangle$ contracts onto $|\phi_1\rangle$.

The situation is exactly the same for the spin-1 states since the spin-1 state made out of $1\otimes 0 - 0\otimes 1$ is even under a reflection about the middle of the two site block, whereas the spin-1 states made out of $1\otimes 0 + 0\otimes 1$ and $1\otimes 1$ are odd under the same reflection. This means that the expansion of the even spin-1 state cannot contain any eigenstates of the four-site problem in common with the expansion of the two odd spin-1 states. Thus, only the two odd spin-1 states need to be rotated into one another in order to guarantee that the lowest lying eigenstate appearing in the expansion of each state is unique, just as in the spin-0 case.

With this behind us, in the rotated basis, $H_{\rm diag}$ is a matrix whose diagonal entries are the eigenvalues of the lowest-lying eigenstates which appear in the expansion of the corresponding rotated state. Thus,

$$\begin{aligned} H_{2-block}(j,j+1) &= R H_{\rm diag} R^\dagger \\ h^{\rm conn}(j,2) &= H_{2-block}(j,j+1) - h^{\rm conn}(j,1) - h^{\rm conn}(j+1,1) \end{aligned} \tag{40}$$

Finally, given these results we have the renormalized Hamiltonian defined on the thinner lattice

$$H^{\rm ren} = \sum_j \left(h^{\rm conn}(j,1) + h^{\rm conn}(j,2) \right) \tag{41}$$

As with all renormalization group algorithms, one iterates this process until the sequence of renormalized Hamiltonians either runs to a fixed point, or until one arrives at a situation which can be handled by perturbation theory. The generic step of the recursion follows the pattern just described, except that now the two site Hamiltonian is defined to be

$$H_{2-site}(j,j+1) = h^{\rm conn}(j,1) + h^{\rm conn}(j+1,1) + h^{\rm conn}(j,2) \tag{42}$$

instead of Eq. 35. As before one diagonalizes $H_{2-site}(j,j+1)$ and retains the four lowest lying eigenstates which, if one starts out with $-1 < \beta < 1$, will be a spin-0 and spin-1 representation of $SU(2)$. From these states one constructs the new diagonal $h^{\rm conn}(j)$. Next, one constructs the new range-2 interaction by using these four states to construct the sixteen retained states for the two-block problem and expands them in terms of a complete set of eigenstates for the two-block Hamiltonian. From these expansions one determines R and $H_{\rm diag}$, from which one immediately constructs the

new $h^{\rm conn}(j,2)$. The results of running such iterations for starting values of $\beta = -1/3$ and $\beta = 2/3$ are shown in Table 1 and Table 2 respectively.

The point $\beta = -1/3$ is one of the special points for which the theory based upon the Hamiltonian, Eq. 33 is exactly solvable, so it is interesting to see how the sequence of renormalization group transformations works for this case. Table 1 shows the results of the first and tenth iterations for the case $\beta = -1/2$. What is tabulated for each iteration are the eigenvalues and total spins, $S^2 = S(S+1)$, for the eigenstates of the renormalized two site Hamiltonian. As we see, initially the sixteen states of the two site problem fall into irreducible representations of $SU(2)$ and while the states of each representation have the same energy, the different representations start out having distinct energies. This changes with increasing iterations until, as we see in the column for iteration ten, the system acquires a degenerate spin-0 and spin-1 multiplet and the remaining twelve states are all degenerate. This pattern reproduces itself unchanged for all succeeding iterations.

To understand what is happening in a simple way it is useful to rewrite this theory as a theory of spin-1/2 states. This can be easily done since each site of the lattice has both a spin-0 and spin-1 representation living on it and the product of two spin-1/2 representations contains exactly one spin-0 and one spin-1 representation, If we identify these representations with the four states per site of the original theory then we see that the Hilbert states of the original theory can be set in one-to-one correspondence with the states of a spin-1/2 theory on a lattice with twice as many sites. If we identify each two site block, $B(2j, 2j+1)$, with a single point of the original $\beta = -1/3$ theory, then the range-two reflection invariant Hamiltonian of the original theory must be equivalent to a generic range-four Hamiltonian of the form

$$\begin{aligned} H &= \sum_j [\alpha \mathbf{1} + A\vec{s}(2j)\cdot\vec{s}(2j+1) + B\vec{s}(2j+1)\cdot\vec{s}(2(j+1)) \\ &+ C\vec{s}(2j)\cdot\vec{s}(2(j+1)+1) + D\vec{s}(2j)\cdot\vec{s}(2(j+1)) \\ &+ D\vec{s}(2j+1)\cdot\vec{s}(2(j+1)+1))] \end{aligned} \tag{43}$$

Now, since for the case $\beta = -1/3$ the spin-0 and spin-1 states are degenerate it follows that $A = 0$, but at the starting level B, C and D do not vanish. Clearly one could obtain the exact values of these coefficients from the values of the level splittings in the first column of Table 1. The more interesting question is what values do these coefficients flow to as the number of iterations increase. Although one could do a brute force calculation of these results it is clear from the eigenvalues appearing in column two of Table 1 that the answer is that in this limit $A = C = D = 0$ and $B = .8359471\ldots$ and $\alpha = 3B/4$. With this choice of parameters we see that of the four spin-1/2 sites corresponding to the two site block of the original theory, only the inner two spins are coupled to one another: i.e., the Hamiltonian for the block is just

$$H = 3B/4\mathbf{1} + B\vec{s}(2j+1)\cdot\vec{s}(2(j+1)) = B/4 + B(S^{\rm tot}(2j+1, 2(j+1))/2 - 3/4) \tag{44}$$

From this we see that if the two inner spins are coupled to a spin-0 state then the two outer spins can be in any configuration (in particular either spin-0 or spin-1) producing four states of zero energy, which is what is seen. Furthermore, if the two inner spins are coupled to spin-1 then one gets $4 \times 3 = 12$ degenerate states with energy B, which is also what is seen. Turning to the full renormalized Hamiltonian on the infinite lattice we see that the Hamiltonian describes a fully dimerized spin-1/2 system in which there is no coupling between two spins in the same block and the block-block couplings only exist between adjacent spins. It follows that the ground state of the infinite volume theory is one in which each pair of neighboring spins is coupled to spin-0. Note that this is reminiscent of the exact solution of this model as a valence bond solid [5]. The lowest lying excited states are those for which any one pair of interacting spins couples to a spin-1 state and all the others couple to a spin-0 state. If one is not at the renormalization group fixed point where $A = C = D = 0$, but a small distance away, where these couplings are small but non-vanishing, then these degenerate states split into momentum bands. The interpretation of the fixed point gap is just the gap to all of the states which have arbitrarily small momentum in the infinite volume theory.

If we consider Table 2 we see quite a different picture, in that now the various multiplets are non-degenerate in the first iteration. Nevertheless, we see that after ten iterations the energy eigenvalues (to the accuracy shown) reproduce the same fixed point pattern as seen in the case $\beta = -1/3$ up to an overall constant. The only important difference between the case $\beta = -1/3$ and $\beta = 2/3$ is that the gap for $\beta = 2/3$ is smaller. Fig. 3 shows the result of carrying out renormalization group transformations for $-1 < \beta < 1.8$. Thus, the general picture emerging from this computation is that the spin-1 HAF in the region between $-1 < \beta < 1$ is controlled by the valence bond solid fixed point at $\beta = -1/3$ as one moves away from this point the mass goes down and at some point both above and below $\beta = 1/3$ the theory appears to become massless. Given the limitation of the CORE computation to range two terms in the renormalized Hamiltonian it is not surprising the location of the points where the theory actually becomes massless is not very accurate. The dashed curve in Fig. 3 is not meant to be taken seriously, it is drawn in to guide the eye and remind the reader that the points $\beta = \pm 1$ are known to be massless theories; one would hope that a computation going out to terms of range three or four would come closer to this picture. In any event, it seems clear from the picture that the point $\beta = 0$, which is the spin-1 HAF, lies close enough to the $\beta = -1/3$ theory that one can be confident that it corresponds to a massive theory as conjectured. This of course is what we set out to show.

A final point worth commenting upon is the fact that no CORE computations were done for $\beta \leq -1$. The reason for this is that the truncation scheme used was to keep only the lowest lying spin-0 and spin-1 states. One trouble with this is that the program I used to compute the CORE transformation simply selected the four lowest lying states, which for the nondegenerate system in which the spin-1 and spin-2 have different energies worked fine. Unfortunately, this scheme breaks down at β too near

-1 and one ends up selecting four states but not necessarily all from either the spin-0 or the spin-1 multiplet. In this case one gets spurious results. To do the full job correctly would have required a more carefully written program. Another problem which contributes to the lack of accuracy of the range-2 calculation in the vicinity of $\beta = -1$ is that the theory develops an $SU(3)$ symmetry at $\beta = -1$ and so a truncation scheme which keeps only the spin-0 and spin-1 multiplets isn't capable of manifestly preserving this symmetry. A scheme which did preserve the symmetry would need to keep full $SU(3)$ multiplets; i.e., the $SU(3)$ singlet state, which corresponds to the spin-0 state, and the full $SU(3)$ octet state, which corresponds to the sum of the spin-1 and spin-2 states. While CORE allows one to choose a truncation scheme which doesn't manifestly preserve the symmetries of the original theory and still obtain correct results, it does this at the expense of needing longer range couplings in the renormalized Hamiltonian in order to obtain high-accuracy.

2.5 General S

In the preceding section I discussed the application of CORE to the spin-1/2 and spin-1 HAF, where simple range-2 arguments sufficed to show that, in agreement with the Haldane conjecture, the spin-1/2 HAF is a massless theory and that the spin-1 HAF is massive. What I did not discuss is what this analysis has to say about the case of the spin-S HAF when S is greater than one. While a full analysis of the generic case would require doing a range-2 computation for all values of $S > 1$, which I do not know how to do, examination of the key difference between these two computations strongly suggests the physics which controls the general case.

To begin the discussion of the HAF for generic S consider the first CORE transformation for an arbitrary S HAF when one uses a three site blocking procedure. (The reason for using a three site algorithm is that I already showed that there is no two site blocking procedure which works for the spin-1/2 HAF.) For generic S the three site HAF Hamiltonian is given by Eq. 11 and the exact solution is as before, only the values for $S_{\rm TOT}(1,2,3)^2$ and $S_{\rm TOT}(1,3)^2$ change from case to case. It follows immediately that the lowest lying representation for the three site problem is always spin S and so, the state structure of the renormalized theory is the same as in the original theory, but as for the spin-1 HAF, the Hamiltonian changes. As always, truncating to the lowest lying representation yields a range-1 renormalized Hamiltonian which is simply a multiple of the unit matrix and so, the only real dynamics comes from computing the range-2 terms. In general, since the single-site retained states form a spin-S representation, the two site retained states decompose into a sum of representations going from $S^{'} = 0 \ldots 2S$. Therefore, the new Hamiltonian can be written as a sum of terms

$$H = \sum_j \sum_{S'=0}^{2S} \epsilon_{S'} P_{S'}(j, j+1) \tag{45}$$

where $P_{S'}(j, j+1)$ is the operator which projects the tensor product states onto the

spin-$S^{'}$ representation and $\epsilon_{S'}$ is the eigenvalue of the lowest lying spin-$S^{'}$ state appearing in the expansion of the projected tensor product state in terms of eigenstates of the two-block problem. Again, following the previous discussion, this Hamiltonian can always be rewritten as a polynomial in the operators $\vec{s}(j)\cdot\vec{s}(j+1)$. The important thing to notice at this point is that for integer S and $\epsilon_{S'} = 0$ for $S^{'} = 0 \ldots S$ and $\epsilon_{S'} > 0$, then the Hamiltonian is a theory of the form constructed by Affleck, Kennedy, Lieb and Tasaki (AKLT)[6] in order to exhibit theories having a valence-bond solid ground state. Thus, in the integer spin case any three site CORE transformation immediately maps the integer spin HAF into a theory which has a massive valence-bond solid theory nearby. While it would take doing a complete computation of the CORE flows for this theory in order to prove that the spin-S HAF lies in the basin of attraction of this theory, it is exactly what happened in the spin-1 case and it is not unreasonable to conjecture that this is the case for general S.

The situation is quite different for theories with half-integer S. In such cases any three site renormalization group transformation will map the theory into a sum of half-integral spin representations of $SU(2)$ with Hamiltonians of the form given in Eq. 45 and it is a theorem that an AKLT Hamiltonian for half-integral S can't have a valence-bond solid ground state. Generically, this result will coincide with what is found in a CORE computation, since for a half-integer spin a three site truncation will always require that one keeps at least one irreducible representation per site which will perforce have dimension two or greater and these CORE calculations will generally iterate in a manner similar to the spin-1/2 theory; i.e., they will predict a massless theory, which is consistent with the Haldane conjecture.

To summarize, while a full discussion of the generic case would require explicitly computing the CORE transformations and proving that the integer spin theories all lie in the basin of attraction of the theory to which the nearby AKLT model iterates, general arguments constrain the general aspects of the calculations and suffice to show several important things: first, for all $S > 1/2$ the first CORE computation generates a new Hamiltonian having the general structure of an AKLT model but with more general values of $\epsilon_{S'}$ and that these terms play an important role in subsequent iterations; second, that for integer S the corresponding AKLT Hamiltonian has a valence-bond solid ground state and is a massive theory; third, that for half-integer S the corresponding AKLT model does not admit a valence-bond solid ground state and the structure of the CORE flows is generically similar to the spin-1/2 case. While I do not know a way to explicitly compute the CORE transformations for arbitrary S, it seems safe to conjecture that the preceding discussion is how things would work for any S.

3 Conclusion

In the preceding sections of this paper I exhibited explicit, first principles, CORE computations for the spin-1/2 and spin-1 HAF which showed that even the simple

range-2 approximation to a full CORE computation agreed with the predictions of the Haldane conjecture. I then argued that these computations lead directly to a very attractive picture of how things can be expected to work for general S. There are two comments I would like to make about this argument. The first is that, as I stated at the outset, the spin-1/2 and spin-1 results show that CORE is more than capable of providing a simple explanation of phenomena which from other points of view appear quite subtle. This, of course, buttresses the hope that CORE can fruitfully be applied to the study of the complicated spin theories which are obtained from free fermion theories and theories of fermions interacting with gauge-fields. The second point I would like to make is that this same chain of argument shows that although CORE does have a strong dependence upon the ability to do numerical computations, it is inherently different from Monte Carlo computations in that the strong focus on the short distance Hamiltonian physics and computation of renormalization group flows allows one to directly extract a physical picture of what is going on.

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Table 1: CORE flow for case $\beta = -1/3$

Iteration 1		**Iteration 10**	
Levels	$\mathbf{S^2}$	**Levels**	$\mathbf{S^2}$
0	0	0	0
0	2	0	2
0	2	0	2
0	2	0	2
0.89791173	6	0.83159471	6
0.89791173	6	0.83159471	6
0.89791173	6	0.83159471	6
0.89791173	6	0.83159471	6
0.89791173	6	0.83159471	6
0.94191045	2	0.83159471	2
0.94191045	2	0.83159471	2
0.94191045	2	0.83159471	2
1.1835034	2	0.83159471	2
1.1835034	2	0.83159471	2
1.1835034	2	0.83159471	2
1.8944584	0	0.83159471	0

Table 2: CORE flow for case $\beta = 2/3$

Iteration 1		**Iteration 10**	
Levels	$\mathbf{S}^2$	**Levels**	$\mathbf{S}^2$
-0.75395437	0	-1.6479538	0
1.1561163	2	-1.6479538	2
1.1561163	2	-1.6479538	2
1.1561163	2	-1.6479538	2
2.7471518	6	-1.1820317	6
2.7471518	6	-1.1820317	6
2.7471518	6	-1.1820317	6
2.7471518	6	-1.1820317	6
2.7471518	6	-1.1820317	6
3.520943	2	-1.1820317	2
3.520943	2	-1.1820317	2
3.520943	2	-1.1820317	2
4.6626764	0	-1.1820317	0
5.6297153	2	-1.1820317	2
5.6297153	2	-1.1820317	2
5.6297153	2	-1.1820317	2

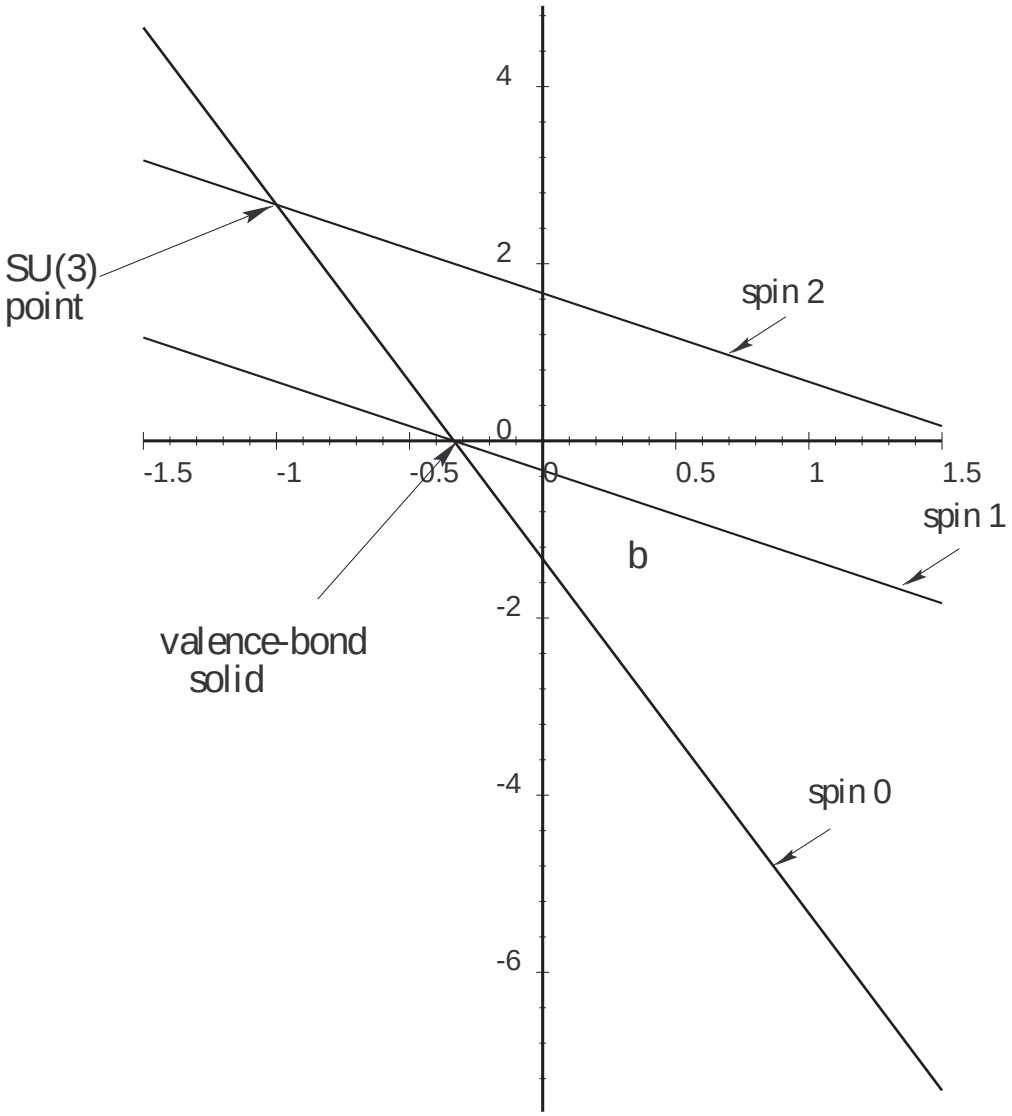

Figure 1: Energy levels for a two site block for the Hamiltonian given by Eq. 33 for $-1 \leq \beta \leq 1$.

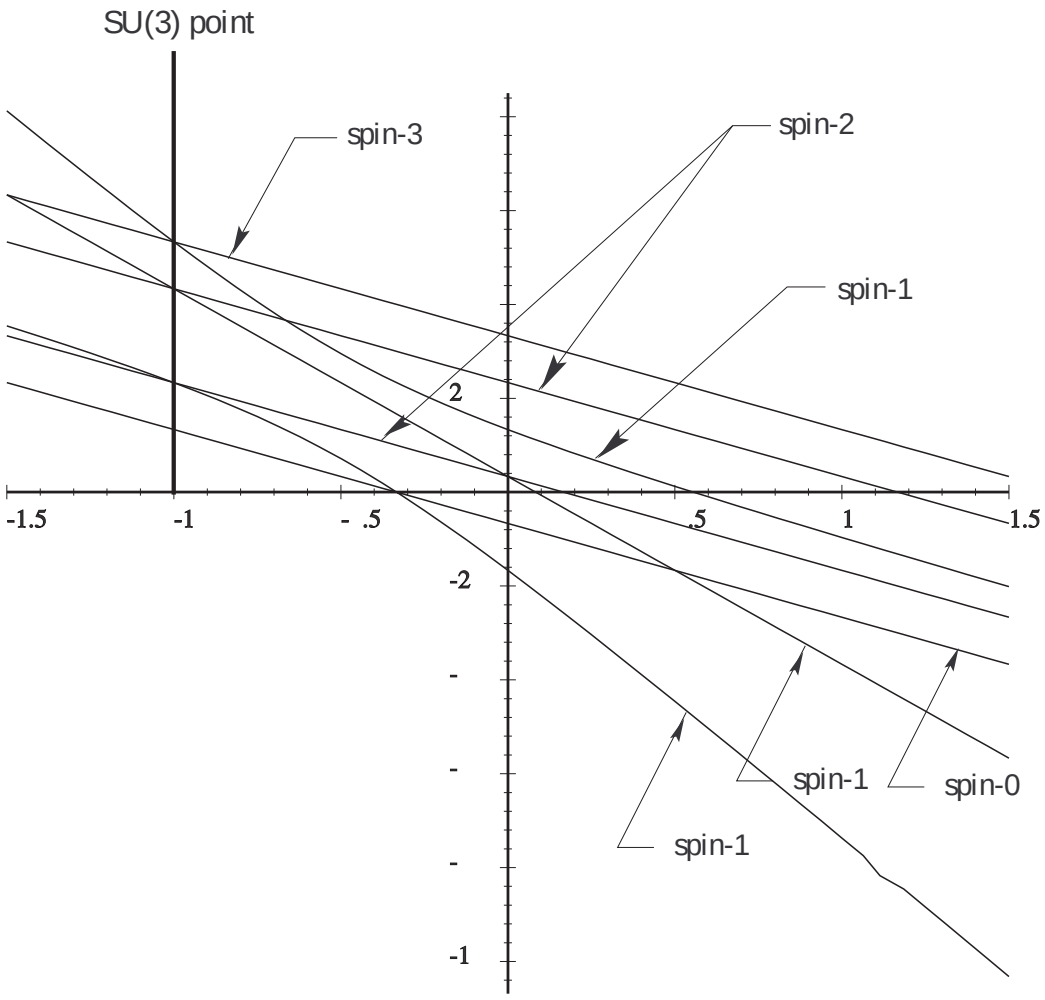

Figure 2: Energy levels for a three site block for the Hamiltonian given by Eq. 33 for $-1 \leq \beta \leq 1$.

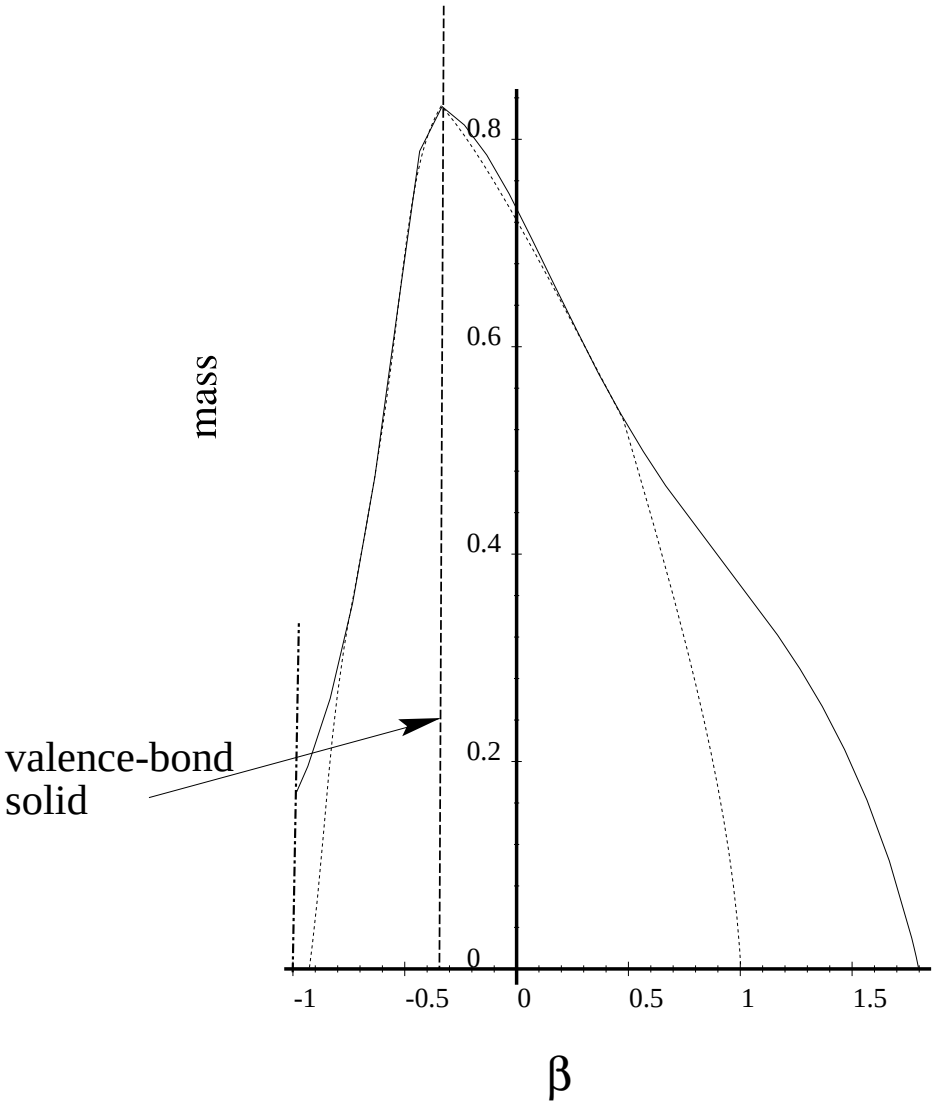

Figure 3: CORE predicted mass gap for $-1 \leq \beta \leq 1$.