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PERSPECTIVES OF LIGHT-FRONT QUANTIZED FIELD THEORY: *SOME NEW RESULTS*[*†]

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Abstract

A review of some basic topics in the light-front (LF) quantization of relativistic field theory is made. It is argued that the LF quantization is *equally appropriate* as the conventional one and that they lead, assuming the microcausality principle, to the same physical content. This is confirmed in the studies on the LF of the spontaneous symmetry breaking (SSB), of the degenerate vacua in Schwinger model (SM) and Chiral SM (CSM), of the chiral boson theory, and of the QCD in covariant gauges among others. The discussion on the LF is more economical and more transparent than that found in the conventional equal-time quantized theory. The removal of the constraints on the LF phase space by following the Dirac method, in fact, results in a substantially reduced number of independent dynamical variables. Consequently, the descriptions of the physical Hilbert space and the vacuum structure, for example, become more tractable. In the context of the Dyson-Wick perturbation theory the relevant popagators in the *front form* theory are causal. The Wick rotation can then be performed to employ the Euclidean space integrals in momentum space. The lack of manifest covariance becomes tractable, and still more so if we employ, as discussed in the text, the Fourier transform of the fermionic field based on a special construction of the LF spinor. The fact that the hyperplanes $x^{\pm} = 0$ constitute characteristic surfaces of the hyperbolic partial differential equation is found irrelevant in the quantized theory; it seems sufficient to quantize the theory on one of the charateristic hyperplanes.

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1 Introduction

Half a centuary ago, Dirac [1] discussed the unification, in a relativistic theory, of the principles of the quantization and the special relativity theory which were by then firmly established. The Light-Front (LF) quantization which studies the relativistic quantum dynamics of physical system on the hyperplanes : $x^0 + x^3 \equiv \sqrt{2}x^+ = const.$, called the *front form* theory, was also proposed and some of its advantages pointed out. The *instant form* or the conventional equal-time theory on the contrary uses the $x^0 = const.$ hyperplanes. The LF coordinates $x^\mu : (x^+, x^-, x^\perp)$, where $x^\pm = (x^0 \pm x^3)/\sqrt{2} = x_\mp$ and $x^\perp = (x^1, x^2) = (-x_1, -x_2)$, are convenient to use in the *front form* theory. They are *not related by a finite Lorentz transformation* to the coordinates $(x^0 \equiv t, x^1, x^2, x^3)$ usually employed in the *instant form* theory and as such the descriptions of the same physical content in a dynamical theory on the LF, which studies the evolution of the system in x^+ in place of x^0, may come out to be different from that given in the conventional treatment. This was found to be the case, for example, in the description of the spontaneous symmetry breaking (SSB) mechanism (Sec. 3) some time ago and in the recent study (Sec. 6) of some soluble two-dimensional gauge theory models, where it was also demonstrated that LF quantization is very economical in displaying the relevant degrees of freedom, leading directly to a physical Hilbert space. The LF quantized field theory may perhaps also be of some relevance in the understanding of the unification of the principles of the quantization with that of the general covariance [2].

We recall that the field theory at infinite momentum was employed in the context of the current algebra sum rules [3]. The Feynman rules adapted for infinite momentum frame (IMF), which were used by Weinberg [4], showed substantial simplifications in the context of the old fashioned perturbation theory computations. In the deep inelastic region with the IMF limit Bjorken [5] predicted the scaling of the deep inelastic structure functions. The parton model [6] of Feynman was also formulated in the IMF. At the same time the connection between the use of the LF variables and the IMF limit was being noticed by several authors [7], which prompted gradually the interest in the study of the *front form* theory as proposed by Dirac.

More recently, the interest in LF quantization has been revived [8, 9, 10] due to the difficulties encountered in the computation, in the conventional framework, of the nonperturbative effects in the context of QCD and the problem of the relativistic bound states of fermions [8, 9] in the presence of the complicated vacuum. Studies show [9, 8, 11] that the application of Light-front Tamm-Dancoff method may be feasible here. The technique of the regularization on the lattice has been quite successful for some problems but it cannot handle, for example, the bound states of light (chiral) fermions and has not been able yet to demonstrate, for example, the confinenment of quarks. The problem of reconciling the standard constituent quark model and the QCD to describe the hadrons is also not satisfactorily resolved. In the former we employ few valence quarks while in the latter the QCD vacuum state itself contains, in the conventional theory, an infinite sea of constituent quarks and gluons (partons) with the density of low momentum constituents getting very large in view of the infrared slavery. The *front form* dynamics may serve as a complementary tool to study such probelms, since we may possibly arrange to have a simple vacuum in it while transfering the complexity of the problem to the LF

Hamiltonian. In the case of the scalar field theory, for example, the corresponding LF Hamiltonian is, in fact, found [12] to be *nonlocal* due to the presence of *constraints* on the *LF phase space*.

The LF quantization of QCD in its Hamiltonian form provides an alternative approach to lattice gauge theory for the computation of nonperturbative quantities, such as [8, 9] the spectrum and the LF Fock state wavefunctions of relativistic bound states. LF variables have found natural applications in several contexts, for example, in the quantization of (super-) string theory and M-theory [13]. They have also been employed in the nonabelian bosonization [14] of the field theory of N free Majorana fermions. The vacuum structures [15, 16] in the LF quantized Schwinger model (SM) and the Chiral SM (CSM) have been recently studied in a transparent fashion. The LF quantized QCD in covariant gauges has also been studied [17] in the context of the Dyson-Wick perturbation theory, where it is shown that the lack of manifest covariance in the calculations becomes more tractable thanks to a useful form of the LF spinor introduced (Sec. 4). The relevant propagators are shown to be causal and the Wick rotation can be performed [18] to go over to the Euclidean space integrals allowing for the dimensional regularization to be used. The *front form* theory has also found applications in the nonperturbative sector of QCD in the context of the Bethe-Salpeter dynamics. The Covariant Instaneity ansatz (CIA) [19] introduced earlier, which invokes the Markov-Yukawa Transversality Principle, has been extended now to the covariant null plane (CNPA) [20, 21].

1.1 Light-Front Quantized Theory

We will make the *convention* to regard $x^+ \equiv \tau$ as the LF-time coordinate while x^- as the *longitudinal spatial* coordinate. We note that $[x^+, i\partial^-] = [x^-, i\partial^+] = -i$ where $\partial^\pm = \partial_\mp = (\partial^0 \pm \partial^3)/\sqrt{2}$ etc. so that the coordinates x^+ and x^- appear in a symmetric fashion. In terms of the null vector $n^\mu = (1,0,0,1)/\sqrt{2}$ and its dual $\tilde{n} = (1,0,0,-1)/\sqrt{2}$, with $n \cdot n = \tilde{n} \cdot \tilde{n} = 0$, $\tilde{n} \cdot n = 1$, they may be written also as $x^+ = n \cdot x$ and $x^- = \tilde{n} \cdot x$ (See also Sec. 5). The temporal evolution in x^0 or x^+ of the system is generated by the Hamiltonians which are different in the two *forms* of the theory.

Consider [16, 10] the invariant distance between two spacetime points : $(x-y)^2 = (x^0 - y^0)^2 - (\vec{x}-\vec{y})^2 = 2(x^+ - y^+)(x^- - y^-) - (x^\perp - y^\perp)^2$. On an equal $x^0 = y^0 = const.$ hyperplane the points have spacelike separation except for if they are *coincident* when it becomes lightlike one. On the LF with $x^+ = y^+ = const.$ the distance becomes *independent of* $(x^- - y^-)$ and the seperation is again spacelike; it becomes lightlike one when $x^\perp = y^\perp$ but with the difference that now the points need *not* necessarily be coincident along the longitudinal direction. The LF field theory hence *need not necessarily be local in* x^-, even if the corresponding *instant form* theory is formulated as a local one. For example, the commutator $[A(x^+, x^-, x^\perp), B(0,0,0^\perp)]_{x^+=0}$ of two scalar observables would vanish on the grounds of microcausality principle in relativistic field theory for $x^\perp \neq 0$ when $x^2|_{x^+=0}$ is spacelike. Its value would hence be proportional to $\delta^2(x^\perp)$ and a finite number of its derivatives, implying locality only in $x^\perp$ but not necessarily so in x^-. Similar arguments in the *instant form* theory lead to the locality in all the three spatial coordinates. In view of the microcausality both of the commutators $[A(x), B(0)]_{x^+=0}$ and $[A(x), B(0)]_{x^0=0}$ are nonvanishing only on the light-cone, $x^2 = 0$. The possibility of nonlocality in the longitudinal direction in the *front form* treatment seems to allow us to display in some cases the structures parallel to those found in string theory (Sec. 4.6).

We note that in the LF quantization we time order with respect to τ (which is a monotonic parameter as well) rather than t. The microcausality principle, however, ensures that the retarded commutators $[A(x), B(0)]\theta(x^0)$ and $[A(x), B(0)]\theta(x^+)$, which appear [22] in the S-matrix elements of relativistic field theory, do not lead to disagreements in the two formulations. In the regions $x^0 > 0, x^+ < 0$ and $x^0 < 0, x^+ > 0$, where the commutators seem different the x^2 is spacelike and both of them vanish. Hence, admitting [23] the microcausality principle to hold, the LF hyperplane seems *equally valid and appropriate* as the conventional one of the *instant form* theory for the canonical quantization. This is demonstrated to be so, for example, in the context of SSB, SM, CSM, and QCD in covariant gauges discussed in this article.

We note that the hyper planes $x^{\pm} = 0$ define the characteristic surfaces of hyperbolic partial differential equation. It is known from their mathematical theory [24] that a solution exists if we specify the (Cauchy) initial data on both of the hyperplanes. From the actual computations in the *front form* theory we come to conclusion [16] that (barring some massless theories) it is sufficient in the canonical quantization of the *front form* theory to select one of the hyperplanes. The information on the commutators on the other characteristic hyperplane seems to be already contained [15] in the quantized theory.

A distinguishing feature of the *front form* theory is that it gives rise to a constrained dynamical system [25]. After the elimination of the phase space constraints in the Hamiltonian formulation it leads to an appreciable reduction in the number of independent field operators which would describe the Hilbert space of the theory. The vacuum structure, for example, then becomes more tractable and the computation of physical quantities simpler. This is, for example, verified [15, 16, 17] in the studies of the LF quantized SM, CSM, and QCD in covariant gauges reviewed in Secs. (6, 7).

1.2 LF Poincaré and IMF Generators. LF Spin Operator

The structure of the *LF phase space* is different from that of the one in the conventional theory. A different description of the same physical content, compared to that found in the conventional treatment, may emerge in the *front form* theory. For example, the SSB gets a different description [32, 10] and the broken continuous symmetry is now inferred from the study of the residual unbroken symmetry of the LF Hamiltonian operator while the symmetry of the LF vacuum remains intact. However, the expression which counts the number of Goldstone bosons present in the *front form* theory, comes out to be the same as that found in the the discussion of equal-time quantized theory. A new proof of the Coleman's theorem [26] on the absence of the Goldstone bosons in two dimensional theory also emerges [32, 10]. The LF vacuum is generally found to be simpler and in many cases the interacting theory vacuum is seen to coincide with the perturbation theory one [27].

Another important advantge pointed out by Dirac of the *front form* theory is that here *seven* out of the ten Poincaré generators are *kinematical*, e.g., they leave the plane $x^+ = 0$ invariant [1]. In the standard notation, *viz.*, $K_i = -M^{0i}, J_i = -(1/2)\epsilon_{ijk}M^{kl}, i, j, k = 1, 2, 3$, they are : P^+, P^1, P^2, $M^{12} = -J_3$, $M^{+-} = M^{03} = -K_3$, $M^{1+} = (K_1 + J_2)/\sqrt{2}$, and $M^{+2} = (K_2 - J_1)/\sqrt{2}$. In the conventional theory on the other hand only six such ones, $\vec{P}$ and $M^{ij} = -M^{ij}$, leave the hyperplane $x^0 = 0$ invariant. Expressed otherwise, the generator K_3 is dynamical one in the *instant form* theory but it turns out to be kinematical on the LF in the sense that there it generates [15] simply the scale transformations of the

LF components of P^μ and $M^{\mu\nu}$, and x^μ, with $\mu,\nu = +,-,1,2$.

There is also an interesting correspondence of the LF components of the Poincaré generators with the generators in the IMF. Consider the inertial frame $\mathcal{S}'$ moving along the 3-axis with velocity $v/c = \tanh\eta$ relative to the inertial frame $\mathcal{S}$. From $(M'^{\mu\nu}, P'^{\mu}) = exp(-i\eta K_3)\ (M^{\mu\nu}, P^\mu)\ exp(i\eta K_3)$ we derive (Appendix A)

$$\begin{aligned} J'_1 &= J_1\cosh\eta + K_2\sinh\eta, \quad J'_2 = J_2\cosh\eta - K_1\sinh\eta, \quad J'_3 = J_3 \\ K'_1 &= K_1\cosh\eta - J_2\sinh\eta, \quad K'_2 = K_2\cosh\eta + J_1\sinh\eta, \quad K'_3 = K_3 \\ (P_0+P_3)' &= e^{\eta}(P_0+P_3) \qquad (P_0-P_3)' = e^{-\eta}(P_0-P_3) \qquad P'_1 = P_1 \quad P'_2 = P_2 \end{aligned} \tag{1}$$

When $v/c \to 1(-1)$ or $\eta \to \infty(-\infty)$ the Lorentz transformation becomes singular. However, we may define the *renormalized* generators, $J'_a/\cosh\eta$, $K'_a/\cosh\eta$, and $e^{\mp\eta}(P_0 \pm P_3)'$ which have well defined limits. The generators thus obtained coincide in the limit with the LF components of the Poincaré generators. We note also that to particle at rest in $\mathcal{S}$ corresponds the four-momentum p'^μ in the inertial frame $\mathcal{S}'$: $p'_\mu/(m_0\cosh\eta)$, which tends to a null vector.

It is also worth remarking that the $+$ component of the Pauli-Lubanski pseudo-vector W^μ is special in that it contains only the LF kinematical generators. We may define the *LF Spin operator* by $\mathcal{J}_3 = -W^+/P^+$. In the masssive case the other two components of $\vec{\mathcal{J}}$, generating together an $SU(2)$ algebra, are shown to be $\mathcal{J}_a = -(\mathcal{J}_3 P^a + W^a)/\sqrt{P^\mu P_\mu}$, $a = 1,2$, which, however, do carry in them also the LF *dynamical generators* P^-, M^{1-}, M^{2-}. The case of both the massive and massless fermions is discussed in detail in Sec. 4; the general case is considered in Appendix B.

2 LF quantized scalar theory

2.1 Covariant Phase Space Factor on the LF

Some interesting insight on the *front form* quantized field theory may already be gained by considering the Lorentz invariant phase space-*LIPS* or *Covariant phase space* [28] factor, which is found relevant in the analysis of the physical processes, introduced first in the context of the covariant version of the statistical model of Fermi [29]. On the LF the dispersion relation associated with the free massive particle is $2p^+p^- = (p^\perp p^\perp + m^2) > 0$. It has no square root, like in the conventional one. The *signs*, for example, of p^+ and p^- are *correlated* since $p^+p^- > 0$ [30]. The *LISP* factor in the LF coordinates is thus defined as: $\int d^4p\ \ \theta(\pm p^+)\theta(\pm p^-)\delta(p^2-m^2) = \int d^2p^\perp dp^+ \int dp^-\ \ \theta(\pm p^+)\theta(\pm p^-)\,\delta(2p^+p^- - \left[m^2 + p^{\perp 2}\right]) = \int d^2p^\perp dp^+ \theta(p^+)/(2p^+)$, compared to the conventional one: $\int d^4p\,\theta(\pm p^0)\delta(p^2-m^2) = \int d^3\vec{p}/(2E_p)$ with $E_p = +\sqrt{\vec{p}^2+m^2} > 0$. A distinguishing feature in the case of the LF is thus the appearence of $\theta(p^+)/(2p^+)$ in the phase space factor. Such considerations are relevant, for example, in writing the Fourier transform of the fields and the discussion of chiral boson theory (Sec. 3.4).

2.2 LF Commutator

Consider, for example, a real massive free scalar field $\phi(\tau, x^-, x^\perp)$, satisfying $(\Box + m^2)\phi = 0$ where $\Box = (2\partial_+\partial_- - \partial_\perp\partial_\perp)$. For $p^+ > 0$, and consequently $p^- > 0$, the complete

set of plane wave solutions of the equation of motion are $e^{+ip\cdot x}$ and $e^{-ip\cdot x}$ where $p\cdot x = p^-x^+ + p^+x^- - p^\perp x^\perp$ and $\tau \equiv x^+$ indicates the LF-time coordinate. The Fourier transform of ϕ on the LF may clearly be written as,

$$\phi(x) = \frac{1}{\sqrt{(2\pi)^3}} \int d^2p^\perp \frac{dp^+}{\sqrt{2p^+}} \theta(p^+) \left[a(p^+,p^\perp)e^{-ip\cdot x} + a^\dagger(p^+,p^\perp)e^{ip\cdot x}\right] \tag{2.1}$$

where we have isolated $\sqrt{2p^+}$ only for latter convenience and $p^\perp$ as well as p^+ are to be integrated from $-\infty$ to ∞, which is very convenient when we deal with generalized functions like $\theta(p^+)$. In the quantized theory $a(p)$ and $a^\dagger(p)$ denote the creation and annihilation operators of the quantum excitations associated with the quantized field operator ϕ. They are assumed to satisfy the canonical commtation relations, with the nonvanishing one given by $\left[a(k), a^\dagger(p)\right] = \delta(k^+ - p^+)\delta^2(k^\perp - p^\perp) \equiv \delta^3(k-p)$. The Fock space is constructed employing these operators.

The equal-LF-time commutator of the field operator can be computed by employing its Fourier transform expression

$$\begin{aligned}
[\phi(x), \phi(y)]_\tau &= \frac{1}{(2\pi)^3} \int d^2p^\perp d^2k^\perp \frac{dp^+ dk^+ \theta(p^+)\theta(k^+)}{\sqrt{2p^+ 2k^+}} \\
&\quad \times \left[e^{-i(p.x-k.y)} - e^{i(p.x-k.y)}\right]_{x^+=y^+=\tau} \delta^3(p-k) \\
&= \frac{\delta^2(x^\perp - y^\perp)}{(2\pi)} \int \frac{dp^+}{2p^+} \left[\theta(p^+) + \theta(-p^+)\right] e^{-ip^+(x^- - y^-)} \\
&= -\frac{i}{4\pi}\epsilon(x^- - y^-)\delta^2(x^\perp - y^\perp).
\end{aligned} \tag{2.2}$$

Here we have used the free particle dispersion relations for k^μ and p^μ, made use of the delta function in the integrand, set $[\theta(p^+) + \theta(-p^+)] = 1$ (or rather the Cauchy principal value-CPV) in the sense of the distribution theory, and used the integral representation of the *sign* function $\epsilon(x) = 1$ or $= -1$ according as $x > 0$ or $x < 0$. The equal-τ commutator obtained here, often termed *the LF commutator*, is *nonlocal* along the longitudinal direction x^-, which as we argued before is not unexpected in the *front form* theory. It vanishes for the spacelike distances, and is nonvanishing only on the light-cone for $x^- \neq y^-$, when we *assume* $\epsilon(0) = 0$.

2.3 Length Dimensions $L_\perp$ and $L_\parallel$

It is natural and suggested also, for example, from the expression of the LF commutator to introduce [9] *two distinct units of length dimensions*, $L_\perp$ and $L_\parallel$ in the *front form* theory. Indicating the dimension of a quantity by [..] we write: $[x^\perp] = L_\perp$, $[x^-] = L_\parallel$, $[\partial_-] = 1/L_\parallel$. Requiring that $p^\perp \cdot x^\perp$ be dimensionless we find $[p^\perp] = [m] = 1/L_\perp$, if we recall the dispersion relation. Making similar arguments we find $[p^+] = 1/L_\parallel$, $[p^-] = L_\parallel/(L_\perp)^2$, $[x^+] = (L_\perp)^2/L_\parallel$ while $[H^{lf}] \equiv [P^-] = L_\parallel/(L_\perp)^2$ for the LF Hamiltonian and $[\mathcal{H}^{lf}] = 1/(L_\perp)^4$ for the Hamiltonian density. Similar considerations apply to the other composite operators like current densities and we remark that $\theta(x)$ and $\epsilon(x)$ do not carry any dimensions. The dimensional analysis is useful in finding [9] the possible counter terms required in the renormalization of the theory. From the LF commutator

(2.2) we conclude that $[\phi] = 1/L_\perp$, which is also found to be the case for the gauge field but for the fermionic field we have $[\psi_+] = 1/(L_\perp\sqrt{L_\parallel})$, where ψ_+ indicates the dynamical component of the fermion field.

2.4 LF Hamiltonian. Dirac Procedure

The free scalar theory is described by the Lagrangian density $\mathcal{L} = \partial_+\phi\partial_-\phi - (1/2)\partial_\perp\phi\partial_\perp\phi - m^2\phi^2/2$. It is first order in $\partial_+\phi$ and the canonical momenta defined as $\pi = \partial\mathcal{L}/\partial(\partial_+\phi) = \partial_-\phi$ describes a constraint on the phase space dynamics of the *front form* scalar theory. We have here a constrained dynamical system [25]. The canonical Hamiltonian density is found to be $\mathcal{H}_c = m^2\phi^2/2$. There is a systematic procedure[1] - called the Dirac method [25]- which allows us to construct the self-consistent Hamiltonian formulation, required to canonically quantize the theory with phase space constraints. The primary constraint above is written as

$$\chi \equiv (\pi - \partial_-\phi) \approx 0 \tag{2.3}$$

where $\approx$ stands for weak equality, meaning that it should not be employed inside the Poisson brackets, but only after they have been computed.

We define next an extended Hamiltonian density $\mathcal{H}_e = \mathcal{H}_c + u\chi$ where u is a Lagrange multiplier field. Hamiltons equations of motion employ $H_e \equiv \int d^2x^\perp dx^- \mathcal{H}_e$ and we require the persistency condition on the constraint, e.g., $\{\chi(\tau, x^-, x^\perp), H_e(\tau)\} \approx 0$. In the simple case under study we are led to a differential equation which would determine the multiplier field u. In the gauge theory considered below new secondary constriants may arise. We now include them also in the extended Hamiltonian and repeat the procedure, untill no more constraints show up. For the computational purposes we may initially start with the standard Poisson brackets at equal-LF-time τ, with the nonvanishing one defined by[2] $\{\pi(\tau, x), \phi(\tau, y)\} = -\delta^3(x - y) \equiv -\delta^2(x^\perp - y^\perp)\delta(x^- - y^-)$.

The nature of the set of constraints found in the theory is then analyzed. A constraint is first class if it has vanishing Poisson brackets with itself, with the the other constraints, and with the Hamiltonian; otherwise it is a second class one. Corresponding to a first class constraint we may be required to add in the theory some appropriate and accessible gauge-fixing external constraints [25]. In the present case their is one local constraint $\chi \approx 0$. From the constraint matrix

$$\{\chi(\tau, x^-, x^\perp), \chi(\tau, y^-, y^\perp)\} = -2\delta^2(x^\perp - y^\perp)\partial_-\delta(x^- - y^-). \tag{2.4}$$

we conclude that it is second class by itself, since the right hand side is nonvanishing. There is, in fact, also a first class constraint in the theory in the form of the zero-momentum-mode of χ; we will comment on it in Sec. (2.6).

We go over now from the Poisson to the modified Poisson brackets, called Dirac brackets, which have the property that we are allowed to set $\chi = 0$ as a strong equality relation, valid even inside the Dirac brackets. The Hamilton's equations employ [25] now the modified brackets. We construct first the *inverse* of the constraint matrix: $\{\chi(\tau, x^-, x^\perp),$

[1] See also Secs. 5,6, and Appendix C.

[2] In the context of the canonical quantization we mostly deal with equal-τ brackets and commutators. We will fequently suppress τ from writing and write occasionally x to indicate the set $(x^-, x^\perp)$.

$\chi(\tau, y^-, y^\perp)\}^{-1} = -\delta^2(x^\perp - y^\perp)\epsilon(x^- - y^-)/4$. The modified bracket is then defined by

$$\{f(x), g(y)\}_D = \{f(x), g(y)\} - \int\int d^3u d^3v \{f(x), \chi(u)\}\{\chi(u), \chi(v)\}^{-1}\{\chi(v), g(y)\} \quad (2.5)$$

In view of its very constructuion the Dirac bracket of any dynamical variable with χ is seen to vanish identically.

It is clear that in place of H_e we may then employ the *reduced Hamiltonian* obtained by setting $\chi \equiv (\pi - \partial_- \phi) = 0$ in it, which would also remove the Lagrange multiplier field, while π becomes now a dependent variable, e.g., removed from the theory. For the independent field ϕ which survives in the *front form* scalar theory here considered we find

$$\{\phi(\tau, x), \phi(\tau, y)\}_D = -\frac{1}{4}\epsilon(x^- - y^-)\delta^2(x^\perp - y^\perp) \quad (2.6)$$

The Hamilton's equation: $\dot{\phi}(\tau, x) = \{\phi(\tau, x), H_c\}_D$, where an overdot indicates the derivation with respect to τ, does recover also the Lagrange equation. The theory is canonically quantized by the correspondence $i\{f, g\}_D \to [f, g]$, the commutator of the corresponding quantized operators. The Hamiltons equations correspond to the equations of motion of field operators, e.g., $idf/d\tau = [f, H]$, in the Heisenberg picture. The commutator of the scalar field operators on the LF is thus given by

$$[\phi(\tau, x), \phi(\tau, y)] = \frac{-i}{4}\epsilon(x^- - y^-)\delta^2(x^\perp - y^\perp) \quad (2.7)$$

which is the same as found above by the simple arguments based on the Fourier expansion of the field in the *front form* theory. Employing this commutator we recover in the present case the Lagrange equation of motion for the field operator as well.

2.5 Scalar Field Propagator in momentum space

The Fourier expansion (2.1) may also be regarded as furnishing the momentum space realization of the commutator (2.7) and the propagator in momentum space is easily derived. The propagator in configuration space is defined by

$$\langle 0|\, T(\phi(x)\phi(0))\, |0\rangle = \theta(\tau)\, \langle 0|\, (\phi(x)\phi(0))\, |0\rangle + \theta(-\tau)\, \langle 0|\, (\phi(0)\phi(x))\, |0\rangle\,. \quad (2.8)$$

It follows that

$$\begin{aligned}
\langle 0|\, T(\phi(x)\phi(0))\, |0\rangle &= \frac{1}{(2\pi)^3}\int d^3p \frac{\theta(p^+)}{2p^+}\left[\theta(\tau)e^{-ip.x} + \theta(-\tau)e^{ip.x}\right] \\
&= \frac{i}{(2\pi)^4}\int d^3p d\lambda\, e^{-i(\lambda\tau + p^+x^- - p^\perp x^\perp)}\frac{[\theta(p^+) + \theta(-p^+)]}{(m^2 + p^\perp p^\perp - 2p^+\lambda - i\epsilon)} \\
&= \frac{i}{(2\pi)^4}\int d^4p \frac{e^{-ip.x}}{(p^2 - m^2 + i\epsilon)} \quad (2.9)
\end{aligned}$$

Here we have used the integral representations[3] of $\theta(\pm\tau)$ and performed the well known standard manipulations. The factor $[\theta(p^+) + \theta(-p^+)]$ in the integrand has been set to unity and the dummy integration variable λ has been renamed as p^- for convenience in

[3] $\theta(\tau)e^{-ip^-\tau} = 1/(2i\pi)\int d\lambda e^{-i\lambda\tau}/(p^- - \lambda - i\epsilon)$

the last line. The d^4p stands for $d^2p^\perp dp^+dp^-$, with the understanding, as is clear from the derivation above, that the integration over the p^- has to be performed first. The range of integration is from $-\infty$ to ∞ for all of these variables.

The momentum space representations of the energy-momentum tensor are also found easily and we check that $N(p) = a^\dagger(p)a(p)$ has the usual interpretation of a number operator. In fact,

$$\begin{aligned} H^{lf}{}_c &\equiv P^- = \int d^2x^\perp dx^- : \left[\frac{m^2}{2}\phi^2 + \frac{1}{2}\partial_\perp\phi\partial_\perp\phi\right] : \\ &= \frac{1}{2}\int d^2p^\perp\, dp^+\theta(p^+) : \left[a^\dagger(p)a(p) + a(p)a^\dagger(p)\right] : \frac{m^2 + p^\perp p^\perp}{2p^+} \\ &= \int d^2p^\perp\, dp^+\theta(p^+)\left[a^\dagger(p)a(p)\right]\; p^- \\ P^+ &= \int d^2x^\perp dx^- : (\partial_-\phi)^2 : \\ &= \int d^2p^\perp\, dp^+\theta(p^+)\;\left[a^\dagger(p)a(p)\right]\; p^+ \end{aligned} \tag{2.10}$$

2.6 First class constraint. Symmetry in x^+ and x^-

It is worth making an *important remark*. There is, in fact, present [31] in the scalar theory discussed above *still another constraint which is first class*. We easily show that the zero-longitudinal-momentum mode $\sqrt{2\pi}\tilde{\chi}(\tau, k^+ = 0) = \int dx^-\chi$, represents a first class constraint in the theory. For example, considering for simplicity the two dimensional theory, (2.4) reads in the momentum space as

$$\{\tilde{\chi}(\tau, k^+), \tilde{\chi}(\tau, p^+)\} = -2ik^+\delta(k^+ + p^+). \tag{2.11}$$

It clearly indicates the presence of the first class constraint $\tilde{\chi}(\tau, k^+ = 0) \approx 0$ in the theory. Such a constraint or symmetry requires us to introduce in the theory an external (gauge-fixing) constraint [25], such that the pair becomes a second class set. We will take advantage of this gauge freedom in order to decompose the scalar field into the *bosonic condensate* variable and the quantum fluctuation field. When combined with the *standard* Dirac procedure it allows us to build [32, 10] a description of the SSB mechanism on the LF.

We also note that the *front form* formulation of *relativistic* theory is inherently symmetrical with respect to x^+ and x^- and it is a matter of *convention* that we take the plus component as the LF-time while the other as a spatial coordinate. The theory quantized at $x^+ = const.$ hyperplanes seems already to incorporate [15] in it the information on the equal-x^- commutation relations. For example, we easily derive from (2.1) the following equal$-x^-$ commutator

$$\begin{aligned} &\left[\phi(x^+, x^-, x^\perp), \phi(y^+, x^-, y^\perp)\right] = \\ &\quad \frac{1}{(2\pi)^3}\int d^2p^\perp\, \frac{dp^+\theta(p^+)}{2p^+}\,\left[e^{-ip^-(x^+-y^+)+ip^\perp(x^\perp-y^\perp)} - e^{ip^-(x^+-y^+)-ip^\perp(x^\perp-y^\perp)}\right]. \end{aligned} \tag{2.12}$$

In view of the free particle dispersion relation we may replace the measure $dp^+\theta(p^+)/2p^+$ by $dp^-\theta(p^-)/2p^-$. The equal-x^- commutator is then given by $(-i/(4\pi))\epsilon(x^+-y^+)\delta^2(x^\perp-$

$y^{\perp}$). In two dimensional space-time it is customary to define the right and the left movers by $\phi(0,x^-) \equiv \phi^R(x^-)$, and $\phi(x^+,0) \equiv \phi^L(x^+)$. We find $\left[\phi^R(x^-),\phi^R(y^-)\right] = (-i/4)\epsilon(x^- - y^-)$ while $\left[\phi^L(x^+),\phi^L(y^+)\right] = (-i/4)\epsilon(x^+ - y^+)$. The symmetry under discussion seems responsible for appreciable simplifications in the *front form* quantized theory.

3 SSB Mechanism, Topological Kink Solution, and Chiral Boson theory on the LF

3.1 SSB in two dimensional scalar theory

The conventional *instant form* description of the tree level SSB is based on the space and time independent solutions of the Lagrange equation, $\phi_{class} \equiv \omega$, such that they also minimize the Hamiltonian functional; invoking the (external) physical considerations. We do not apparently have much physical intuition on the LF to avail of such arguments. The constrained dynamical system on the LF seems, however, to already contain in it the corresponding relevant constraints. For simplicity we consider first the two dimensional theory[4] with $\mathcal{L} = (\partial_+\phi)(\partial_-\phi) - V(\phi)$

This is probably the simplest example of a constrained dynamical system in the context of field theory. It is reasonable to expect that the well tested Dirac procedure, when applied to it, *must* result in a satisfactory description of SSB on the LF.

The Lagrange equation, $2\dot{\phi}' = -V'(\phi)$, is of first order in LF-time τ. The left hand side remains unaltered under $\phi \to \phi + c(\tau)$ and $\phi = const.$ are clearly possible solutions[5]. Integrating over the space variable and assuming appropriate boundary conditions we are led to the following constraint [32, 10] on the potential

$$\int dx^- \; \frac{\delta V(\phi)}{\delta \phi} = 0. \tag{3.1}$$

We show now that this constraint is also present on the phase space and in the quantized theory. The description of SSB then follows from the discussion on the structure of the Hilbert space.

In order to take care of the first class constraint $\int dx^- \chi \approx 0$ mentioned in Sec. (2.6) we make the following *separation* of the dynamical (collective) bosonic *condensate* variable $\omega(\tau)$ from the (quantum) fluctuation variable $\varphi(\tau,x)$

$$\phi(\tau,x) = \omega(\tau) + \varphi(\tau,x). \tag{3.2}$$

Here we also set $\int dx^- \varphi(\tau,x) = 0$ so that the fluctuation field carries no zero-longitudinal-momentum mode in it. The *separation thus corresponds in a sense to an external gauge-fixing constraint* which we must impose [25] in the theory. It was introduced [32] originally on physical considerations and ω was termed as the dynamical *bosonic condensate* variable.

[4]Here $\tau = (x^0 + x^1)/\sqrt{2}$, $x \equiv x^- = (x^0 - x^1)/\sqrt{2}$. An overdot indicates the LF-time derivative while a prime indicates derivative with respect to x^-; the generalization to $3+1$ dimensions is discussed in Sec. (3.2).

[5]The self-dual *kink* solution which depends on x^- as well is discussed in Sec. (3.3).

We apply now the standard Dirac procedure to construct LF Hamiltonian formulation. The canonically quantized theory results [10, 32] in the following commutators

$$[\varphi(x,\tau),\varphi(y,\tau)] \;=\; -\frac{i}{4}\epsilon(x-y), \tag{3.3}$$

$$[\omega(\tau),\varphi(x,\tau)] \;=\; 0. \tag{3.4}$$

and for $V(\phi)=(\lambda/4)(\phi^2-m^2/\lambda)^2$, with a negative sign for the mass term and $\lambda\geq 0$, $m\neq 0$, the LF Hamiltonian is given by

$$H^{lf}\equiv P^- = \int dx\,\Big[\omega(\lambda\omega^2-m^2)\varphi+\frac{1}{2}(3\lambda\omega^2-m^2)\varphi^2+\lambda\omega\varphi^3+\frac{\lambda}{4}\varphi^4\Big]. \tag{3.5}$$

We recover also the *constraint equation* (3.1) now as a second class constraint on the phase space:

$$\omega(\lambda\omega^2-m^2)+\quad\frac{1}{R}\int_{-R/2}^{R/2}dx\Big[\,(3\lambda\omega^2-m^2)\varphi+\lambda(3\omega\varphi^2+\varphi^3)\,\Big]=0 \tag{3.6}$$

where $R\;\rightarrow\;\infty$ and the Cauchy principle value of $\int_{-\infty}^{\infty}dx\,f(x)$ is defined by $lim_{R\rightarrow\infty}\int_{-R/2}^{R/2}dx\,f(x)$.

The commutation relations indicate that the operator ω is a c-number or a background field. Eliminating ω would lead to *LF Hamiltonian which is nonlocal* [32, 10] along the longitudinal coordinate x^- even though the scalar theory written in the conventional coordinates is local.

At the tree or classical level, φ are bounded ordinary functions in x^- and when $R\rightarrow\infty$ only the first term survives in the constraint equation leading to $\omega(\lambda\omega^2-m^2)=0$. This result is the same as that obtained in the conventional theory. There, however, it is essentially added to the theory, on physical considerations, which require the energy functional to attain its minimum (extremum) value. The *stability property*, say, of a particular constant solution may be inferred as usual from the analysis of the classical partial differential equation of motion. For example, $\omega=0$ is shown to be an unstable solution on the LF for the potential considered above, while the other two solutions with $\omega\neq 0$ give rise to the stable phases. A similar analysis, it is clear, of the corresponding *partial* differential equations in the conventional coordinates can also be made; the Fourier transform theory is convenient to use. Also the new ingredient in the form of the constraint equation on the LF does have its counterpart in the conventional *instant form* framework as is shown in [18]. It is *remarkable* that the *front form* theory seems to contain inside it all the necessary ingredients in order to describe the SSB, when we follow the Dirac procedure to handle the constrained LF dynamics of the scalar field.

We could have employed the DLCQ [33], including the condensate term also in it. The existence of the *continuum* limit of DLCQ theory adding to it also the dynamical condensate variable was demonstrated [32, 34], contradicting the then prevalent notion on the contrary[6]. The demonstration assures [35] us of the self-consistency of the *front form* theory itself. In the theory described in finite volume, the commutator of ω with φ is found nonvanishing and as such it is an *operator*; only when $R\rightarrow\infty$ does it becomes a classical background field.

[6]See, T. Maskawa and K. Yamawaki, Prog. Theo. Phys. 56 (1976) 270; K. Nakanishi and K. Yamawaki, Nucl. Phys. B122 (1977) 15. A history of the so called zero mode problem is traced [10] in hep-th/9312064.

It is worth stressing that in our discussion the condensate variable is introduced as a dynamical variable. The Dirac *procedure must decide* whether it comes out to be c- or q-number. In the discussions of the bosonized SM and CSM models the operator ω is not a background field, like in the scalar theory. It turns out to be an operator and plays an important role in describing the structure of the Hilbert space and the degenerate vacua in these gauge theories (Sec. 6).

The field commutator obtained above can be realized in momentum space through the Fourier transform of φ: $\varphi(x,\tau) = (1/\sqrt{2\pi})\int dk\ \theta(k)\ [a(k,\tau)\, e^{-ikx} + a^\dagger(k,\tau) e^{ikx}]/(\sqrt{2k})$, where $k^+ \equiv k$ and the operators $a(k,\tau)$ and $a^\dagger(k,\tau)$ satisfy the canonical equal-τ commutation relations, with the nonvanishing one given by $[a(k,\tau), a^\dagger(k',\tau)] = \delta(k-k')$.

The (perturbative) vacuum state is defined by $a(k,\tau)|vac\rangle = 0\,,\ k > 0$. The tree level description of the *SSB* is given as follows. The values of $\omega = \langle|\phi|\rangle_{vac}$ obtained from $V'(\omega) = 0$ characterize the different vacua in the theory. Distinct Fock spaces corresponding to different values of ω are built as usual by applying the creation operators on the corresponding vacuum state. The $\omega = 0$ corresponds to *symmetric phase* since the Hamiltonian operator is then symmetric under $\varphi \to -\varphi$. For $\omega \neq 0$ this symmetry is violated and the system is said to be in a *broken or asymmetric phase.*

The constraint equation (3.6) also shows that the value of ω would be altered from its tree level value in view of the quantum corrections, arising from the other terms. The renormalization of the two dimensional scalar theory was discussed [18] to one-loop order by employing the Dyson-Wick expansion based on LF-time ordering. It was found that it is convenient to derive [18] the renormalized constraint equation instead of solving the constraint equation first, which would require the difficult job of dealing with nonlocal and nonlinear Hamiltonian.

In the supernormalizable theory here the two renormalized equations, viz, the mass renormalization condition and the renormalized constraint equation, allow us to to study [18] the phase transition in the two dimensional scalar theory, which was conjectured long time ago by Simon and Griffiths [36].

3.2 Spontaneously broken continuous symmetry

The *extension to* $3+1$ *dimensions* and to the global *continuous symmetry* is straightforward [10]. Consider real scalar fields $\phi_a (a = 1, 2, ..N)$ which form an isovector of global internal symmetry group $O(N)$. We now write $\phi_a(x, x^\perp, \tau) = \omega_a + \varphi_a(x, x^\perp, \tau)$ and the Lagrangian density is $\mathcal{L} = [\dot{\varphi}_a \varphi'_a - (1/2)(\partial_\perp \varphi_a)(\partial_\perp \varphi_a) - V(\phi)]$. The Taylor series expansion of the constraint equations $\beta_a = 0$ gives a set of coupled equations $R\, V'_a(\omega) + V''_{ab}(\omega)\int dx \varphi_b + V'''_{abc}(\omega)\int dx \varphi_b \varphi_c/2 + ... = 0$. Its discussion at the tree level leads to the conventional theory results. The LF symmetry generators are found to be $G_\alpha(\tau) = -i\int d^2x^\perp dx \varphi'_c(t_\alpha)_{cd}\varphi_d = \int d^2k^\perp\, dk\, \theta(k)\, a_c(k,k^\perp)^\dagger (t_\alpha)_{cd} a_d(k,k^\perp)$ where $\alpha, \beta = 1, 2, .., N(N-1)/2$, are the group indices, t_α are hermitian and antisymmetric generators of $O(N)$, and $a_c(k,k^\perp)^\dagger$ $(a_c(k,k^\perp))$ is creation (destruction) operator, contained in the momentum space expansion of φ_c. These are to be contrasted with the generators in the equal-time theory, $Q_\alpha(x^0) = \int d^3x\, J^0 = -i\int d^3x (\partial_0 \varphi_a)(t_\alpha)_{ab}\varphi_b - i(t_\alpha \omega)_a \int d^3x (d\varphi_a/dx_0)$. All the symmetry generators thus annihilate the LF vacuum and the SSB is now seen in the broken symmetry of the quantized theory Hamiltonian. The expression which counts the number of Goldstone bosons in the *front form* theory is found to be identical to that in the conventional theory. In contrast, the first term on the right hand side of $Q_\alpha(x^0)$, which

is similar to the one on the LF, does annihilate the conventional theory vacuum but the second term gives now non-vanishing contributions for some of the (broken) generators. The symmetry of the conventional theory vacuum is thereby broken while the quantum Hamiltonian remains invariant. The *physical content* of SSB in the *instant form* and the *front form*, however, is the same though achieved by different descriptions. Alternative proof on the LF, in two dimensions, can be given of the Coleman's theorem related to the absence of Goldstone bosons; we are unable [10] to implement the second class constraints over the phase space. The tree level Higgs mechanism may also be discussed straightforwardly [10]. We remark that the simplicity of the LF vacuum is in a sense compensated by the involved nonlocal Hamiltonian. The latter, however, may be treatable using advance computational techniques. Also in connection with renormalization it may not be necessary [18]; we may instead obtain the renormalized constraint equations.

3.3 Kink solution and Topological quantum number

The classical Lagrange equation of the two dimensional self-interacting theory, $2\partial_-\partial_+\phi = -V'(\phi)$, with the $V(\phi)$ given above, is known to have finite energy topological soliton solutions [37] called *kink* solutions. The theory has an internal symmetry, $\phi \to -\phi$. They can be recovered in the *front form* theory as well. The *kink* corresponds to the *self-dual* solution satisfying $\partial_-\phi = -\partial_+\phi$ and given by

$$\phi_{kink} = \pm\frac{m}{\sqrt{\lambda}}\tanh\left[\frac{m}{2}(x^+ - x^-)\right] \tag{3.7}$$

where the upper (lower) sign corresponds to the kink (anti-kink) solutions. The kink on the LF carries both the LF energy and longitudinal momentum such that $P^+ = P^-$ and[7] its mass is determined to be $M = \sqrt{2P^+P^-} = \sqrt{8}m^3/(3\lambda)$. The kink interpolates between the two vacuua of the theory: $\phi_{kink}(0, x^- = \infty) = -m/\sqrt{\lambda}$ and $\phi_{kink}(0, x^- = -\infty) = m/\sqrt{\lambda}$. The topological charge may be defined by $Q = \int dx^- j^+$ where $j^\mu = -(\sqrt{\lambda}/(2m))\epsilon^{\mu\nu}\partial_\nu\phi$, with $\epsilon^{+-} = -\epsilon^{-+} = \epsilon_{01} = 1$, is the conserved topological current density. The toplogical charges of kink, anti-kink and vacuum solutions are 1, -1, and 0 respectively. The Q is absolutely conserved prohibiting the decay of the kink into vaccum. Similar (topological) quantum numbers on the LF arise also, for example, in the context of the structure of the degenerate vacua in the canonical quantiztion of SM and CSM models discussed below.

3.4 Chiral Boson theory on the LF

The chiral boson (or self-dual scalar) field in $1+1$ dimensions plays an important role, for example, in the formulation of string theories [38], in the description [39] of boundary excitations of the quantum Hall state, and in a number of two- dimensional statistical systems which are related to the Coulomb-gas model.

We recall that the free massive theory with $\mathcal{L} = \partial^\mu\phi\partial_\mu\phi/2 - m^2\phi^2/2$ has the LF Hamiltonian $m^2\phi^2/2$. The dispersion relation $2p^+p^- = m^2 > 0$ governs the *correlation* between the signs of p^+ and p^-. In the massless theory, at the classical level, a chiral boson solution, $\partial_0\phi = \partial_1\phi$ (and an anti-chiral one, $\partial_0\phi = -\partial_1\phi$) is obtained. Several

[7] $P^- = \int dx^- V(\phi)$ and $P^+ = \int dx^- (\partial_-\phi)^2$.

quantized theory models [40, 41, 42] of chiral boson have been proposed. The *front form* theory of chiral boson looks more appropriate and transparent [43] when compared to the conventional one.

The Floreanini and Jackiw (FJ) model [41] is based on the following *manifestly non-covariant Lagrangian*

$$\begin{aligned} \mathcal{L} &= (\partial_0\phi - \partial_1\phi)\partial_1\phi \\ &= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial_0\phi - \partial_1\phi)^2. \end{aligned} \tag{3.8}$$

where ϕ is a real scalar field and $\eta^{00} = -\eta^{11} = 1,\ \eta^{01} = \eta^{10} = 0$.

In the *instant form* frame work it leads [41, 44] to the following equal-time commutator

$$\left[\phi(x^0, x^1), \phi(x^0, y^1)\right] = \frac{-i}{4}\epsilon(x^1 - y^1). \tag{3.9}$$

The commutator is nonvanishing, is nonlocal, and violates the microcausality principle, contrary to what we encounter usually in the conventional theory [23]. These *objections disappear* when we consider the theory quantized in the LF coordinates.

We will consider a *modified* FJ chiral boson model with the following Lagrangian density written in the LF coordinates

$$\begin{aligned} \mathcal{L} &= (\partial_+\phi - \frac{1}{\alpha}\partial_-\phi)\,\partial_-\phi \\ &= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{\alpha}(\partial_-\phi)^2, \end{aligned} \tag{3.10}$$

where $\eta^{+-} = \eta^{-+} = 1, \eta^{++} = \eta^{--} = 0$ and α is a fixed parameter. For $\alpha = 1$ it coincides with (3.8) in the conventional coordinates.

The LF quantization of the scalar theory with a potential term included in it has been discussed in Sec. (2.4). From (3.10) we derive

$$\begin{aligned} H^{lf} &= \int dx^-\ \frac{1}{\alpha}(\partial_-\phi)^2 \\ \left[\phi(\tau, x^-), \phi(\tau, y^-)\right] &= \frac{-i}{4}\epsilon(x^- - y^-) \end{aligned} \tag{3.11}$$

The LF commutator (3.11), which is nonlocal in x^- and nonvanishing only on the light-cone, does not conflict with the microcausality (Sec. 1.1 and [23]) unlike (3.9).

The Heisenberg equation of motion for the field operator is

$$\partial_+\phi = \frac{1}{i}\left[\phi, H^{lf}\right] = \frac{1}{\alpha}\partial_-\phi \tag{3.12}$$

and the Lagrange equation

$$\partial_-\left[\partial_+\phi - \frac{1}{\alpha}\partial_-\phi\right] = 0. \tag{3.13}$$

is recovered.

The commutator (3.11) can be realized in momentum space through the following Fourier transform ($x^+ \equiv \tau$)

$$\phi(x^+, x^-) = \frac{1}{\sqrt{2\pi}}\int dk^+\,\frac{\theta(k^+)}{\sqrt{2k^+}}\left[a(x^+, k^+)e^{-ik^+x^-} + a^\dagger(x^+, k^+)e^{ik^+x^-}\right], \tag{3.14}$$

if the operators a and $a^\dagger$ are assumed to satisfy the equal-τ canonical commutation relations, with the nonvanishing one given by $\left[a(x^+,k^+),a^\dagger(x^+,p^+)\right]=\delta(k^+-p^+)$. On using the equation of motion (3.12) we derive easily

$$a(x^+,k^+)=e^{-ik^-x^+}a(k^+),\qquad a^\dagger(x^+,k^+)=e^{ik^-x^+}a^\dagger(k^+). \tag{3.15}$$

where we set

$$k^-=\frac{1}{\alpha}k^+,\qquad \text{implying}\qquad 2k^+k^-=\frac{2}{\alpha}(k^+)^2. \tag{3.16}$$

The dispersion relation for the free FJ chiral boson is different from that for a free scalar particle with (finite k^+ but) vanishing mass, except for when $|\alpha|\to\infty$.

The Fourier transform now assumes the form

$$\phi(x^+,x^-)=\frac{1}{\sqrt{2\pi}}\int dk^+\,\frac{\theta(k^+)}{\sqrt{2k^+}}\left[a(k^+)e^{-ik\cdot x}+a^\dagger(k^+)e^{ik\cdot x}\right]. \tag{3.17}$$

where $k\cdot x\equiv k^-x^+ + k^+x^- = k^+(x^-+x^+/\alpha)$ and the nonvanishing commutator satisfies $\left[a(k^+),a^\dagger(p^+)\right]=\delta(k^+-p^+)$.

The components of the classical canonical energy-momentum tensor $T^{\mu\nu}$ following from the noncovariant Lagrangian density (3.10) are found to be

$$\begin{aligned} T^{+-}=-T^{-+}=\frac{1}{\alpha}T^{++} &= \frac{1}{\alpha}(\partial_-\phi)^2,\\ T^{--} &= (\partial_+\phi)^2-\frac{2}{\alpha}(\partial_+\phi)(\partial_-\phi). \end{aligned} \tag{3.18}$$

The on shell conservation equations

$$\partial_\mu T^{\mu\pm}=2(\partial_\mp\phi)\,\partial_-\left[\partial_+\phi-\frac{1}{\alpha}\partial_-\phi\right]=0 \tag{3.19}$$

may be easily checked. They allow us to define, if the surface integrals can be dropped, the conserved translation generators $P^\pm$

$$P^+=\int dx^-:T^{++}:=\int dx^-:(\partial_-\phi)^2:=\int dk^+\theta(k^+)\,N(k^+)\,(k^+) \tag{3.20}$$

and

$$P^-\equiv H^{lf}=\int dx^-:T^{+-}:=\frac{1}{\alpha}\,P^+ \tag{3.21}$$

where $N(k^+)=a^\dagger(k^+)a(k^+)$ is the number operator and $:\ :$ indicates the normal ordering.

From (3.18) and in virtue of $(T^{+-}+T^{-+})=0$ we may derive the following relation

$$\partial_+\left[x^-T^{++}+x^+T^{+-}\right]+\partial_-\left[x^-T^{-+}+x^+T^{--}\right]=0. \tag{3.22}$$

which is valid on shell. We may hence define another conserved generator

$$M=x^+P^-+\int dx^-\;x^-T^{++} \tag{3.23}$$

The generators M,P^+,P^- form a closed algebra: $[M,P^+]=-iP^+$, $[M,P^-]=-iP^-$, and $[P^+,P^-]=0$. The operator M thus generates the scale (boost) transformations on $P^\pm$ by the same amount which leaves P^+/P^- invariant. The mass operator

$2P^+P^-$, however, gets scaled and is not invariant under M. The usual (kinematical) Lorentz boost generator $M^{+-} \equiv -x^+P^- + \int dx^-\ x^-T^{++}$ has similar properties. It is, however, as seen from (3.19), is *not conserved* in the manifestly *noncovariant model* under consideration. The Lagrange equation is shown to be form invariant under the infinitesimal transformation [41, 45] $\phi \to \phi + \epsilon(x^- + x^+/\alpha)\partial_-\phi$ generated by M.

In the limit when $|\alpha| \to \infty$ we find $\phi \to \phi_R(x^-)$ while $H^{lf} \to 0$, which corresponds to the LF Hamiltonian of free massless scalar theory, Sec. (2.6). The field ϕ_R satisfies: $[\phi_R(x^-), \phi_R(y^-)] = -i\epsilon(x^- - y^-)/4$. The limiting case is thus seen to describe a right (moving) chiral boson theory with the Lagrangian density as given in (3.10).

An alternative form of the Lagrangian density may also be employed in our context. We recall that in the quantization of gauge theory it is found useful (Sec. 5) to introduce an auxiliary field $B(x)$ of canonical mass dimension two (in $3+1$ dimensions) and add $(B\partial_\mu A^\mu + \alpha B^2)$ as the gauge-fixing term to the Lagrangian density. In the two dimensional theory under consideration it is also possible to follow this procedure, since the corresponding $B(x)$ field here carries the canonical mass dimension one. The discussion parallel to the one given above may thus be based also on the following [46, 47] Lagrangian density

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \sqrt{2}B(x)(\partial_-\phi) + \frac{\alpha}{2}B(x)^2. \tag{3.24}$$

The elimination of the auxiliary field using its equation of motion leads to (3.10) and the conclusions reached are the same.

We make only brief comments on other models. Siegel's [40] theory which employs

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + B(x)(\partial_0\phi - \partial_1\phi)^2 \tag{3.25}$$

is afflicted by anomaly which is to be eliminated by the addition of a Wess-Zumino term. The resulting theory does not describe [48] pure chiral bosons since they are coupled to the gravity. In this model the auxiliary field carries vanishing canonical dimension and, for example, a B^2 term cannot be added without introducing the dimensionful parameters.

The model based on the idea of implementing the chiral constraint through a linear constraint [42, 46],

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + B_\mu(\eta^{\mu\nu} - \epsilon^{\mu\nu})\partial_\nu\phi, \tag{3.26}$$

where B_μ is Lagrange multiplier field, does not seem to exhibit physical excitations [49]. We note that the field B_μ carries dimension one and that this is the usual procedure in the classical theory which, however, seems to break down at the quantum level.

To summarize, the simple procedure of separating first the *condensate variable*, which in fact corresponds to a gauge-fixing condition needed on the phase space in the context of Dirac procedure, before applying the standard procedure itself, is found to be successful in describing [32, 10] the SSB, the phase transition in two dimensional scalar theory, the SSB of continuous symmetry in $3+1$ dimensional theory, in furnishing a new proof of the Coleman's theorem, and in the description of (the tree level) Higgs mechanism. It is also found successful in showing [15, 16] transparently and economically the vacuum structures in the SM and CSM models as will be reviewed in Sec. 5. The self-duality constraint in the interacting theory leads to the well known *kink* solution in the *front form* theory as well. The chiral boson theory discussion becomes transparent and the

LF commutator does not conflicts with the microcausality. A model of the chiral boson theory seems to emerge in the context of the *modified* FJ theory.

We will next review the essentials of the LF quantization of the Dirac and Maxwell fields.

4 LF quantized Dirac field

4.1 Anticommutators

On the LF there is a natural decomposition of the spinor space. The LF comonents [50] $\gamma^{\pm}$, where $\gamma^{\pm} = (\gamma^0 \pm \gamma^3)/\sqrt{2}$ have the properties $(\gamma^+)^2 = (\gamma^-)^2 = 0$, $\gamma^0\gamma^+ = \gamma^-\gamma^0$, ${\gamma^+}^\dagger = \gamma^-$, and $\gamma^+\gamma^- + \gamma^-\gamma^+ = 2I$. We may thus introduce the hermitian projection operators $\Lambda^{\pm}$

$$\Lambda^{\pm} = \frac{1}{2}\gamma^{\mp}\gamma^{\pm} = \frac{1}{\sqrt{2}}\gamma^0\gamma^{\pm}, \quad (\Lambda^{\pm})^2 = \Lambda^{\pm}, \quad \Lambda^+\Lambda^- = \Lambda^-\Lambda^+ = 0, \quad \gamma^0\Lambda^+ = \Lambda^-\gamma^0 \tag{4.1}$$

The corresponding $\pm$ projections of the LF Dirac spinor are $\psi_{\pm} = \Lambda^{\pm}\psi$ and $\psi = \psi_+ + \psi_-$, $\bar{\psi} = \psi^\dagger\gamma^0 = \bar{\psi}_+ + \bar{\psi}_-$, $\gamma^{\pm}\psi_{\mp} = 0$, $\Lambda^{\pm}\psi_{\pm} = \psi_{\pm}$ etc. The matrix $\Sigma_3 = {\Sigma^\dagger}_3 = i\gamma^1\gamma^2$, ${\Sigma_3}^2 = I$, which commutes with $\Lambda^{\pm}$ plays an important role on the LF and we note : $(\Lambda^+ + \Lambda^-) = I$, $(\Lambda^+ - \Lambda^-) = \Sigma_3\gamma_5$, $\gamma_5\psi = \Sigma_3(\psi_+ - \psi_-)$, and $\Sigma_3\gamma^{\perp}\Sigma_3 = -\gamma^{\perp}$.

The action of the free Dirac field is [51]

$$\begin{aligned} S &= \int d^2x^{\perp}dx^- \mathcal{L} \qquad\qquad \text{where} \\ \mathcal{L} &= \bar{\psi}(i\{\gamma^+\partial_+ + \gamma^-\partial_- + \gamma^{\perp}\partial_{\perp}\} - m)\psi \\ &= i\sqrt{2}{\psi^\dagger}_+\partial_+\psi_+ + i\sqrt{2}{\psi^\dagger}_-\partial_-\psi_- \\ &\quad -{\psi^\dagger}_-(m + i\gamma^{\perp}\partial_{\perp})\gamma^0\psi_+ - {\psi^\dagger}_+(m + i\gamma^{\perp}\partial_{\perp})\gamma^0\psi_-. \end{aligned} \tag{4.2}$$

It shows that only the component ψ_+ carries kinetic term and the ψ_- component is nondynamical. The variation of the action with respect to ${\psi^\dagger}_-$ and ψ_- leads to the constraint equation

$$2i\partial_-\psi_- = (m + i\gamma^{\perp}\partial_{\perp})\gamma^+\psi_+ \tag{4.3}$$

and its conjugate, while for the dynamical component ψ_+ we obtain the equation of motion

$$4\partial_+\psi_+ = -(m + i\gamma^{\perp}\partial_{\perp})\gamma^-\frac{1}{\partial_-}(m + i\gamma^{\perp}\partial_{\perp})\gamma^+\psi_+, \tag{4.4}$$

after eliminating the dependent component ψ_-. Its right hand side may be simplified to $2(-m^2 + \partial^{\perp}\partial^{\perp})(1/\partial_-)\psi_+$. The canonical Hamiltonian density is easily seen to be ${\mathcal{H}_c}^{lf} = {\psi^\dagger}_+(m + i\gamma^{\perp}\partial_{\perp})\gamma^0\psi_-$ with ψ_- being a dependent field given by the constraint equation above. It is straightforward to verify that the equation of motion for the dynamical component ψ_+ in the quantized theory is recovered as an Heisenberg equation of motion if we postulate the following anticommutation relations, which are *local in all the spatial coordiantes*.

$$\{\psi_+(\tau,x^-,x^\perp),\psi_+{}^\dagger(\tau,y^-,y^\perp)\} = \frac{1}{\sqrt{2}}\Lambda^+\delta(x^- - y^-)\delta^2(x^\perp - y^\perp),$$
$$\{\psi_+(\tau,x^-,x^\perp),\psi_+(\tau,y^-,y^\perp)\} = 0, \qquad \{\psi_+^\dagger(\tau,x^-,x^\perp),\psi_+{}^\dagger(\tau,y^-,y^\perp)\} = 0. \quad (4.5)$$

The same result is also derived if we follow the straightforward Dirac procedure as in the case of the scalar theory. No first class constraint, however, arises in the present case. The scale dimension of ψ_+ is clearly $[\psi_+] = 1/(L_\perp\sqrt{L_\parallel})$. It follows from (4.3) that

$$\{\psi_-(\tau,x^-,x^\perp),\psi_+{}^\dagger(\tau,y^-,y^\perp)\} = \frac{1}{i4\sqrt{2}}(m + i\gamma^\perp\partial_\perp)\gamma^+\epsilon(x^- - y^-)\delta^2(x^\perp - y^\perp) \quad (4.6)$$

4.2 LF Spinor in momentum space and its properties

In order to write the Fourier transform we look for the complete set of linearly independent plane wave solutions of the free Dirac equation in the *front form* theory. For the massive field the signs of p^+ and p^- are correlated. Choosing, say, $p^+ > 0$ the independent set of the plane wave solutions are $u(p)e^{-ip\cdot x}$ and $v(p)e^{ip\cdot x}$ where the four-spinors $u(p)$ and $v(p)$ satisfy: $(m-\gamma^\mu p_\mu)u(p) = 0$ and $(m+\gamma^\mu p_\mu)v(p) = 0$. We will make the phase convention such that $v(p) = C\gamma^{0T}u(p)^*$, the charge conjugate of $u(p)$.

A very useful form [15, 10] of the free *LF four-spinor* is given by

$$u^{(r)}(p) = N(p)\left[\sqrt{2}p^+\Lambda^+ + (m+\gamma^\perp p_\perp)\Lambda^-\right]\tilde{u}^{(r)}, \quad (4.7)$$

where the normalization is chosen as $N(p) = 1/\left(\sqrt{2}\,p^+m\right)^{1/2}$, with $m > 0$ and $p^+ > 0$. The constant spinors $\tilde{u}^{(r)}$, which are also the spinors in the rest frame $\tilde{p} = (m/\sqrt{2}, m/\sqrt{2}, 0^\perp)$, satisfy $\gamma^0\tilde{u}^{(r)} = \tilde{u}^{(r)}$, $\Sigma_3\tilde{u}^{(r)} = r\,\tilde{u}^{(r)}$ with $r = \pm$. The charge conjugate rest frame spinors satisfy $\gamma^0\tilde{v}^r = -\tilde{v}^r$ and $\Sigma_3\tilde{v}^{(r)} = -r\tilde{v}^{(r)}$ while

$$v^{(r)}(p) = N(p)\left[\sqrt{2}p^+\Lambda^+ + (m-\gamma^\perp p_\perp)\Lambda^-\right]\tilde{v}^{(r)}. \quad (4.8)$$

We note that $\gamma_5 u^{(r)}(p;m) = r\,u^{(r)}(p;-m)$ and $\gamma_5 v^{(r)}(p;m) = -r\,v^{(r)}(p;-m)$ indicating the mass reversal property of γ_5 upto a phase factor. Also $\Sigma_3 u^{(r)}(p;m) = r\,u^{(r)}(p^+,-p^\perp;m)$ and $\Sigma_3 v^{(r)}(p;m) = -r\,v^{(r)}(p^+,-p^\perp;m)$. We do *not* introduce two spinors and work only with four-spinors and do not also employ any explicit matrix representation.

We recall that the LF *Spin operator* for the massive as well as massless particles is defined (Appendix B) by $\mathcal{J}_3 \equiv -W^+/P^+$ where W^μ is the Pauli-Lubanski four-vector. It contains solely the LF kinematical generators and the following useful identity can be demonstrated [15, 10]

$$\mathcal{J}_3(p) = e^{\,(-\frac{i}{p^+})(B_1p^1+B_2p^2)}\;J_3\;\;e^{(\frac{i}{p^+})(B_1p^1+B_2p^2)} = J_3 - \frac{1}{p^+}\,(p^1B_2 - p^2B_1) \quad (4.9)$$

where $\sqrt{2}B_1 = (K_1+J_2)$ and $\sqrt{2}B_2 = (K_2-J_1)$ are the kinemetical boost operators on the LF in the standard notation. Applying it to the spin 1/2 case[8] we derive ($J_3 = \Sigma_3/2$)

$$\mathcal{J}_3(p) = \frac{1}{2}\left[I + \frac{(\gamma^\perp p_\perp)\,\gamma^+}{p^+}\right]\Sigma_3$$

[8]For spin-1/2 case: $J_j = \Sigma_j/2$, $K_j = i\gamma^0\gamma^j/2$ where $j = 1,2,3$.

$$
\begin{aligned}
&= J_3 + \frac{(\gamma^\perp p_\perp)}{2p^+}\gamma^+\gamma_5 \\
\mathcal{J}_3(p)u^{(r)}(p) &= (r/2)u^{(r)}(p) \\
\mathcal{J}_3(p)v^{(r)}(p) &= -(r/2)v^{(r)}(p
\end{aligned}
\tag{4.10}
$$

where $r/2 = \pm(1/2)$ are the projections of $\vec{\mathcal{J}}(p)$ on the 3-axis in the rest frame and we used $i(\gamma^2 p^1 - \gamma^1 p^2) = (\gamma^\perp p_\perp)\Sigma_3$. The four-spinors are shown to satisfy the following orthogonality relations:

$$
\bar{u}^{(r)}(p)u^{(s)}(p) = \delta_{rs}, \qquad \bar{v}^{(r)}(p)v^{(s)}(p) = -\delta_{rs}, \qquad \bar{u}^{(r)}(p)v^{(s)}(p) = 0. \tag{4.11}
$$

and the following completeness relations follow easily

$$
\sum_{r=+,-} u^{(r)}(p)\bar{u}^{(r)}(p) = \frac{(\not{p} + m)}{2m}, \qquad \sum_{r=+,-} v^{(r)}(p)\bar{v}^{(r)}(p) = \frac{(\not{p} - m)}{2m} \tag{4.12}
$$

where $\not{p} = \gamma^\mu p_\mu$. We also have the useful relations: $m\bar{u}^{(r)}(p)\gamma^\mu u^{(s)}(p) = p^\mu \bar{u}^{(r)}(p)u^{(s)}(p)$ and $m\bar{v}^{(r)}(p)\gamma^\mu v^{(s)}(p) = -p^\mu \bar{v}^{(r)}(p)v^{(s)}(p)$.

4.3 Fermion propagator

The Fourier transform expansion of $\psi(x)$ over the complete set of linearly independent plane wave solutions constructed above may be written as

$$
\psi(x) = \frac{1}{\sqrt{(2\pi)^3}} \sum_{r=\pm} \int d^2p^\perp dp^+ \; \theta(p^+) \sqrt{\frac{m}{p^+}} \left[b^{(r)}(p)u^{(r)}(p)e^{-ip.x} + d^{\dagger(r)}(p)v^{(r)}(p)e^{ip.x} \right] \tag{4.13}
$$

where the $\theta(p^+)$ is necessarily present. For the dynamical component $\psi_+ \equiv \Lambda^+\psi$, it follows that

$$
\psi_+(x) = \frac{\sqrt{\sqrt{2}}}{\sqrt{(2\pi)^3}} \sum_{r=\pm} \int d^2p^\perp dp^+ \; \theta(p^+) \left[b^{(r)}(p)\tilde{u}_+^{(r)}e^{-ip\cdot x} + d^{\dagger(r)}(p)\tilde{v}_+^{(r)}e^{ip\cdot x} \right]. \tag{4.14}
$$

It is straightforward to verify that the anticommutation relations (4.5) for the independent field operator ψ_+ are in fact satisfied if we assume the standard canonical anticommutators, with the nonvanishing ones given by: $\{b^{(r)}(p), b^{\dagger(s)}(p')\} = \delta_{rs}\delta^2(p^+ - p'^+)\delta(p^\perp - p'^\perp)$ and $\{d^{(r)}(p), d^{\dagger(s)}(p')\} = \delta_{rs}\delta(p^+ - p'^+)\delta^2(p^\perp - p'^\perp)$.

The Λ^+ projections of our LF spinors are by construction very simple, ${u^{(r)}}_+(p) = (\sqrt{2}p^+/m)^{1/2}\Lambda^+\tilde{u}^{(r)}$; they are eigenstates of Σ_3 as well. This is very convenient since on the LF ψ_+ component is the independent dynamical degrees of freedom while ψ_- may be eliminated, even in the interacting theory, making use of the constraint equation. The simplified structure of ψ_+ gives rise to appreciable simplifications in the context of LF perturbation theory, compensating to some extent for the the nonlinearity of the interaction found along the longituidinal direction x^-. We have better control [17], say, over recovering the manifest rotational and even Lorentz covariance in the perturbation theory calculations if we use the LF four-spinor introduced above. The propagator for the spinor field ψ_+ also takes a very simple causal form on the LF, resembling the one of the scalar field.

The free propagator for the independent component ψ_+ in momentum space is easily derived using the above Fourier transform

$$\begin{aligned}\left\langle 0\right| T(\psi_{+A}(x)\psi^\dagger_{+B}(0))\left|0\right\rangle &= \\ &\left\langle 0\right| \left[\theta(\tau)\psi_{+A}(x)\psi^\dagger_{+B}(0) - \theta(-\tau)\psi^\dagger_{+B}(0)\psi_{+A}(x)\right]\left|0\right\rangle \\ = \quad & \frac{1}{\sqrt{2}}\frac{\Lambda^+_{AB}}{(2\pi)^3}\int d^2q^\perp dq^+\theta(q^+)\left[\theta(\tau)e^{-iqx} - \theta(-\tau)e^{iqx}\right] \end{aligned} \tag{4.15}$$

where $A, B = 1, 2, 3, 4$ label the spinor components. The only relevant differences, compared with the case of the scalar field, are, apart from the appearance of the projection operator, the absence of the factor $(1/2q^+)$ in the integrand, and the negative sign of the second term in the fermionic case. They, however, compensate and the standard manipulations to factor out the exponential give rise to the factor $[\theta(q^+) + \theta(-q^+)]$ which may be interpreted as unity in the distribution theory sense, parallel to what we find in the derivation of the scalar field propagator on the LF. Hence

$$< 0|T(\psi_+(x)\psi^\dagger{}_+(0))|0 >= \frac{i}{(2\pi)^4}\int d^4q\,\frac{\sqrt{2}q^+\,\Lambda^+}{(q^2 - m^2 + i\epsilon)}\,e^{-iq.x}. \tag{4.16}$$

It may also be derived by functional integral method; we do have to take care of the second class constraint in the measure. The fermionic propagator here contains no instantaneous term usually encountered when doing the old fashioned perturbation theory and the integrand factor may also be expressed as $\approx [\Lambda^+(\not{q} + m)\Lambda^-/(q^2 - m^2 + i\epsilon)]\,\gamma^0$. We verify that the propagator satisfies the equation for the Green's function corresponding to the equation of motion of ψ_+.

The momentum space representations of the currents and the components of the energy-momentum tensor are derived straightforwardly and they support the usual interpretation of $b^{\dagger(r)}(p)b^{(r)}(p)$ and $d^{\dagger(r)}(p)d^{(r)}(p)$ as the number operators. For example, for the canonical Hamiltonian we find

$$\begin{aligned} H^{lf}{}_c \quad &= \quad \frac{1}{\sqrt{2}}\int d^2x^\perp dx^- : \psi^\dagger{}_+(m^2 - \partial_\perp\partial_\perp)\frac{1}{i\partial_-}\psi_+ : \\ &= \quad \sum_{r,s}\int d^3pd^3k\theta(p^+)\theta(k^+) : \Big[b^{\dagger(r)}(p)b^{(s)}(k)\tilde{u}^{\dagger(r)}_+\tilde{u}^{(s)}_+ \\ &\quad -d^{(r)}(p)d^{\dagger(s)}(k)\tilde{v}^{\dagger(r)}_+\tilde{v}^{(s)}_+\Big] : \frac{(m^2 + p^\perp p^\perp)}{2p^+}\delta^3(p-k) \\ &= \quad \sum_r\int d^3p\theta(p^+)\left[b^{\dagger(r)}(p)b^{(r)}(p) + d^{\dagger(r)}(p)d^{(r)}(p)\right]\frac{(m^2 + p^\perp p^\perp)}{2p^+} \end{aligned} \tag{4.17}$$

where we use $\tilde{u}^{\dagger(r)}_+\tilde{u}^{(s)}_+ = \tilde{v}^{\dagger(r)}_+\tilde{v}^{(s)}_+ = \delta_{rs}/2$, $d^3p \equiv d^2p^\perp dp^+$, and : : indicates the normal ordering.

4.4 Γ_5 Symmetry. Chirality transformation on the LF

The γ_5 transformation [52], $\psi \to \gamma_5\psi$ on the spinor field is associated with the mass reversal in the Dirac equation. It leaves the Dirac equation form invariant only when the

mass is vanishing. On the LF we can construct a generalized Γ_5 transformation which restores the form invariance even for the massive field.

Consider the covariant vector and axial current densities defined by $j^\mu = \bar{\psi}\gamma^\mu\psi$ and ${j_5}^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ respectively. The corresponding charge densities are defined on the LF by the + components of the currents

$$\begin{aligned} j^+ &= \bar{\psi}\gamma^+\psi = \sqrt{2}{\psi_+}^\dagger\psi_+ \\ {j_5}^+ &= \bar{\psi}\gamma^+\gamma_5\psi = \sqrt{2}{\psi_+}^\dagger\Sigma_3\psi_+. \end{aligned} \tag{4.18}$$

The momentum space representations of the charges are easily derived

$$\begin{aligned} Q &= \int d^2x^\perp dx^- : j^+ := \sum_r \int d^3p\theta(p^+)\left[b^{\dagger(r)}(p)b^{(r)}(p) - d^{\dagger(r)}(p)d^{(r)}(p)\right] \\ Q_5 &= \int d^2x^\perp dx^- : {j_5}^+ := \sum_r \int d^3p\theta(p^+)\,(r)\left[b^{\dagger(r)}(p)b^{(r)}(p) + d^{\dagger(r)}(p)d^{(r)}(p)\right] \end{aligned} \tag{4.19}$$

The charges Q and Q_5 commute with the LF Hamiltonian and are thus constants of motion. The former counts the fermionic number while the latter the twice the projection along the 3-axis of the LF spin operator $\mathcal{J}_3(p)$ discussed above.

From the commutation relations of the field ψ_+ we derive [9]

$$\begin{aligned} \{\psi_+, Q\} &= \psi_+, \\ \{\psi_-, Q\} &= \psi_-, \\ \{\psi_+, Q_5\} &= \gamma_5\psi_+ = \Lambda^+\gamma_5\psi_+, \\ \{\psi_-, Q_5\} &= \Lambda^-\frac{1}{2i\partial_-}(i\gamma^\perp\partial_\perp + m)\gamma^+(\gamma_5\psi_+) \neq \gamma_5\psi_-. \end{aligned} \tag{4.20}$$

The action of the infinitesimal generators on ψ is

$$\begin{aligned} \delta_Q\psi = \{\psi, i\epsilon Q\} &= i\epsilon\,\psi, \\ \delta_{Q_5}\psi = \{\psi, i\epsilon Q_5\} &= i\epsilon\,\gamma_5\left[I - \frac{m}{i\partial_-}\gamma^+\right]\psi, \end{aligned} \tag{4.21}$$

where we use (4.3) and (4.4). It is well known that the infinitesimal transformation with respect to Q is associated with the *form invariance* of the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$ and its conjugate under the global phase transformations. This symmetry gives rise to the on shell conserved Noether vector current j^μ.

The Dirac equation is form invariant under the γ_5 (or chiral transformations) only for the massless theory, when the axial current is also conserved at the classical level. Our discussion on the LF in the Hamiltonian formulation indicates that the Dirac equation is also form invaraint under the following nonlocal Γ_5 transformation, defined by Γ_5

$$\begin{aligned} \psi &\to \Gamma_5\psi, \\ \Gamma_5 &= \gamma_5\left[I - \frac{m}{i\partial_-}\gamma^+\right]. \end{aligned} \tag{4.22}$$

This can be demonstrated, say, if we use of the (on shell) identity

$$(i\gamma^\mu\partial_\mu - m)\gamma_5\left[I - \frac{m}{i\partial_-}\gamma^+\right] = -\gamma_5\left[I + \frac{m}{i\partial_-}\gamma^+\right](i\gamma^\mu\partial_\mu - m). \tag{4.23}$$

The on shell conserved current associated with the Γ_5 symmetry, which holds for both the massive and massless fermions, is hence given by

$$\begin{aligned} J_5{}^\mu &= \bar{\psi}\gamma^\mu\Gamma_5\psi = j_5{}^\mu - m\bar{\psi}\gamma^\mu\gamma_5\gamma^+\frac{1}{i\partial_-}\psi, \\ \partial_\mu J^\mu &\stackrel{o}{=} 0, \\ J_5{}^+ &= j_5{}^+. \end{aligned} \tag{4.24}$$

The chiral charge associated with the Γ_5 symmetry coincides with Q_5 and the generalized chiral transformation is $\psi \to e^{i\alpha\Gamma_5}\psi$.

4.5 Helicity Operator, LF Majorana and Weyl fermions

The Fourier transform of the self-charge conjugate *Majorana spinor field* satisfying, $\psi_M(x) = \psi_{Mc}(x)$, follows easily from (4.13)

$$\begin{aligned} \psi_M(x) &= \frac{1}{\sqrt{2}}(\psi(x) + \psi_c) \\ &= \frac{1}{\sqrt{(2\pi)^3}}\sum_{r=\pm}\int d^2p^\perp dp^+\ \theta(p^+)\sqrt{\frac{m}{p^+}}\left[b_M{}^{(r)}(p)u^{(r)}(p)e^{-ip.x}\right. \\ &\quad \left. + b_M{}^{\dagger(r)}(p)v^{(r)}(p)e^{ip.x}\right] \end{aligned} \tag{4.25}$$

where $b_M{}^{(r)}(p) = (b^{(r)}(p) + d^{(r)}(p))/\sqrt{2}$ and the nonvanishing anti-commutator is given by $\{b_M{}^{(r)}(p), b^\dagger{}_M{}^{(s)}(k)\} = \delta^{rs}\delta^3(p-k)$.

The *chiral* or γ_5-*projections* of the LF spinor are shown to satisfy the following properties ($r\,\gamma_5 u^{(r)}(p;m) = u^{(r)}(p;-m)$ and $-r\,\gamma_5 v^{(r)}(p;m) = v^{(r)}(p;-m)$)

$$\begin{aligned} \frac{(I + r\gamma_5)}{2}u^{(r)}(p) &= N(p)\left[\sqrt{2}p^+\Lambda^+ + (\gamma^\perp p_\perp)\Lambda^-\right]\tilde{u}^{(r)} \\ \frac{(I - r\gamma_5)}{2}u^{(r)}(p) &= N(p)\ m\,\Lambda^-\tilde{u}^{(r)} \to 0 \qquad \text{for } m \to 0 \\ \frac{(I - r\gamma_5)}{2}v^{(r)}(p) &= N(p)\left[\sqrt{2}p^+\Lambda^+ - (\gamma^\perp p_\perp)\Lambda^-\right]\tilde{v}^{(r)} \\ \frac{(I + r\gamma_5)}{2}v^{(r)}(p) &= N(p)\ m\,\Lambda^-\tilde{v}^{(r)} \to 0 \qquad \text{for } m \to 0 \end{aligned} \tag{4.26}$$

along with

$$\begin{aligned} \gamma^\mu p_\mu \quad \left[\frac{(I + r\gamma_5)}{2}u^{(r)}(p)\right] &= N(p)\ m^2\,\Lambda^-\tilde{u}^{(r)} \to 0 \qquad \text{for } m \to 0 \\ \gamma_5 \quad \left[\frac{(I + r\gamma_5)}{2}u^{(r)}(p)\right] &= r\ \left[\frac{(I + r\gamma_5)}{2}u^{(r)}(p)\right] \end{aligned} \tag{4.27}$$

etc., and we note that $[\mathcal{J}_3(p), \gamma_5] = 0$. In the *massless limit*, $m \to 0$, the projections $(I \mp \gamma_5)u^{(\pm)}(p)$, $(I \pm \gamma_5)v^{(\pm)}(p)$ vanish. Also, for example, the nonvanishing one

$$\frac{(I + \gamma_5)}{2}u^{(+)}(p) \tag{4.28}$$

is an eigenstate of γ_5 and $\mathcal{J}_3(p)$ with the eigenvalues 1 and 1/2 respectively, while the other one

$$\frac{(I-\gamma_5)}{2}u^{(-)}(p) \tag{4.29}$$

has the corresponding eigenvalues given by -1 and $-1/2$. The explicit discussion here shows that on the LF the definition of the spin operator for the massive and massless cases gets unified.

The *Helicity operator* $\hat{h}$ is defined by

$$\hat{h} \;=\; \frac{1}{2}\frac{\vec{\Sigma}\cdot\hat{\vec{P}}}{|\hat{\vec{P}}|} \;=\frac{\left[\Sigma_3\hat{P}^3+\gamma^\perp\hat{P}_\perp\gamma^0\gamma_5\right]}{2|\hat{\vec{P}}|} \tag{4.30}$$

which is *not* the same as the LF spin operator.

For *massless fermions* it is easily shown that

$$\begin{aligned}\hat{h}(p)\;u^{(r)}(p) &= (\frac{r}{2})\;u^{(r)}(p)\\ \hat{h}(p)\;v^{(r)}(p) &= -(\frac{r}{2})\;v^{(r)}(p)\end{aligned} \tag{4.31}$$

Experimental observations show that only the negative chirality, $(I-\gamma_5)u^{(-)}(p)/2$, neutrinos exist. Neutrinos have helicity $-1/2$, antineutrinos helicity $1/2$. There is no charge conjugation invariance if neutrinos have a definite chirality. The CP transformations of these spinors can be discussed as usual (Appendix B). The normalization factor in the massless case has to be redefined. The massive particle does not have Lorentz invariant helicity; in the rest frame of the particle there is no preffered direction in what to measure spin.

4.6 Bilocal operators

From the anticommutators in Sec. 4.1 we may derive the (free theory) equal-τ current commutation relations, for example, $[j^+(x),j^+(y)]_\tau = 0$. The commutators among the other components are derived straightforwardly. They involve bilocal operators [53] of the form $\bar{\psi}(x)\Gamma\psi(y)$, with the nonlocality *only* along the longitudinal direction. In the context of the deep inelastic scattering limit they are found relevant in the hadron tensor $W^{\mu\nu}$ and the explanation of the Bjorken scaling and the introduction of the parton model of Feynman. Similar bilocal operators appear also in bosonic theories, for example, in the LF quantization [54] of Chern-Simons systems. We recall (Sec. 1) that on the LF nonlocality in the x^- direction does not conflict with the microcausality principle. The bilocals have also been shown useful recently, for example, in the context [55] of the dynamics of hadrons in two dimensions and in revealing the string like structure in QCD_2.

5 LF quantization of Gauge theory

In perturbative QCD we employ, in the interaction representation, the free abelian gauge theory propagator. It is customary on the LF to adopt the light-cone gauge[9] $A_- = 0$

[9]See the discussion below on the LF quantized two dimensional SM where this gauge is not convenient to employ if we are seeking for nonperturbative effects in the theory.

which results in a simplified interaction Hamiltonian. The noncovariant gauge, however, introduces in the theory undesireable features. The rotational invariance becomes very difficult to track down making the comparasion with the conventional theory results sometimes extremely difficult. In the frequently employed old fashioned perturbation theory computations it is sometimes not easy to see if the conventional and the *front form* theories are really in agreement [56]. The LF quantized QCD was recently studied [17] in covariant gauges in the context of the Dyson-Wick perturbation theory expansion based on the LF-time ordered Wick products. Here all the relevant propagators become causal and the rotational invariance is easily recovered, when the LF spinor (4.7) introduced in the Sec. 4 is employed. The loop integrals can also be converted [18] to the Euclidean space integrals and the dimensional regularization may be used.

The Lagrangian density for the Abelian gauge theory written in LF coordinates is

$$\frac{1}{2}\left[(F_{+-})^2-(F_{12})^2+2F_{+\perp}F_{-\perp}\right]+B(\partial_+A_-+\partial_-A_++\partial_\perp A^\perp)+\frac{\xi}{2}B^2, \qquad (5.1)$$

where $F_{\mu\nu}\equiv(\partial_\mu A_\nu-\partial_\nu A_\mu)$. The covariant gauge-fixing is introduced by adding to the Lagrangian the linear gauge-fixing term $B\partial_\mu A^\mu+(\xi/2)B^2$ where B is the Nakanishi-Lautrup auxiliary field and ξ is a parameter. The canonical momenta are $\pi^+=0$, $\pi_B=0$, $\pi^\perp=F_{-\perp}$, $\pi^-=F_{+-}+B$ and the canonical Hamiltonian density is found to be

$$\mathcal{H}_c=\frac{1}{2}(\pi^-)^2+\frac{1}{2}(F_{12})^2-A_+(\partial_-\pi^-+\partial_\perp\pi^\perp-2\partial_-B)-B(\pi^-+\partial_\perp A^\perp)+\frac{1}{2}(1-\xi)B^2 \quad (5.2)$$

Following the Dirac procedure, the primary constraints are $\pi^+\approx 0$, $\pi_B\approx 0$ and $\eta\equiv\pi^\perp-\partial_-A_\perp+\partial_\perp A_-\approx 0$, where $\perp=1,2$ and $\approx$ stands for *weak equality* relation. We now require the persistency in τ of these constraints employing the preliminary Hamiltonian, which is obtained by adding to the canonical Hamiltonian the primary constraints multiplied by the Lagrange multiplier fields. We assume the standard Poisson brackets for the dynamical variables in the computation for obtaining the Hamilton's equations of motion. We are led to the following two secondary constraints

$$\begin{aligned}\Phi\equiv\partial_-\pi^-+\partial_\perp\pi^\perp-2\partial_-B &\approx 0,\\ \Psi\equiv\pi^-+2\partial_-A_++\partial_\perp A^\perp-(1-\xi)B &\approx 0.\end{aligned} \qquad (5.3)$$

The Hamiltonian is next enlarged by including these additional constraints as well. The procedure is repeated. No more constraints are seen to arise. we now go over from the standard Poisson brackets to the Dirac brackets, such that inside them we are able to substitute the above constraints as *strong* equality. The equal-τ Dirac bracket $\{f(x),g(y)\}_D$ which carries this property is constructed straightforwardly. Hamilton's equations now employ the Dirac brackets and the phase space constraints $\pi^+=0$, $\pi_B=0$, $\eta=0$, $\Phi=0$, and $\Psi=0$ then effectively reduce the (extended) Hamiltonian. In the covariant *Feynman gauge* with $\xi=1$ the free Hamiltonian takes the simple form

$$H_0{}^{LF}=-\frac{1}{2}\int d^2x^\perp dx^-\; g^{\mu\nu}A_\mu\,\partial^\perp\partial_\perp\,A_\nu. \qquad (5.4)$$

The theory is canonically quantized through the correspondence $i\{f(x),g(y)\}_D\;\rightarrow\;[f(x),g(y)]$, the commutator among the corresponding operators.

The equal-τ commutators of the gauge field are found to be

$$[A_\mu(x), A_\nu(y)]_{x^+=y^+=\tau} = -ig_{\mu\nu}K(x,y) \tag{5.5}$$

where $K(x,y) = -(1/4)\epsilon(x^- - y^-)\delta^2(x^\perp - y^\perp)$ is nonlocal in the longitudinal coordinate. The transverse components of the gauge field have the physical LF commutators $[A_\perp(x), A_{\perp'}(y)]_\tau = i\delta_{\perp,\perp'}K(x,y)$, while for the $\pm$ components we have only the mixed commutator nonvanishing $[A_+(x), A_-(y)]_\tau = -iK(x,y)$, it has a negative sign which indicates the presence of unphysical degrees of freedom in Feynman gauge. For $\xi \neq 1$ the commutator, for example, of $A_\pm$ with $A_\perp$ is found to be nonvanishing. We note that the dimension of the gauge field is $[A_\mu] = 1/L_\perp$.

From the discussion analogous to that given in Sec. (2.6) for the scalar field it is clear, from the primary constraints, e.g., $\chi^\perp \approx 0$, in the discussion here, that there are also first class constraints present in the gauge theory. They may be taken care of like in the case of the scalar theory. In the context of perturbation theory we may possibly ignore the zero-longitudinal-mode of the components of the gauge field. However, when dealing with nonperturbative effects they may not be ignored. For example, in the discussion of the (nonperturbative) vacuum structure of the completely soluble QED_2 (SM) theory the zero-momentum mode of A_- plays a crucial role together with the bosonic condensate variable (Sec. 6).

The Heisenberg equations of motion lead to $\Box A_\mu = 0$ for all the components, and consequently the Fourier transform of the free gauge field over the complete set of plane wave solutions takes the following form on the LF

$$A^\mu(x) = \frac{1}{\sqrt{(2\pi)^3}} \int d^2k^\perp dk^+ \, \frac{\theta(k^+)}{\sqrt{2k^+}} \, e^{\mu(\lambda)}(k) \left[a_{(\lambda)}(k^+, k^\perp) e^{-ik.x} + a^\dagger_{(\lambda)}(k^+, k^\perp) e^{ik.x} \right] \tag{5.6}$$

where $e^{\mu(\lambda)}(k)$, $\lambda = -, +, 1, 2$ label the set of four linearly independent polarization four-vectors.

In the *front form* theory the two transverse (physical) polarization vector are spacelike as usual while[10] *the other two are null four-vectors.* For a fixed $k^\mu = (k^0, \vec{k})$, where $k^0 = |\vec{k}|$, we may construct them as follows

$$e^{(+)} = (1, \vec{k}/k^0)/\sqrt{2}, \quad e^{(-)} = (1, -\vec{k}/k^0)/\sqrt{2}, \quad e^{(1)} = (0, \vec{\epsilon}(k;1)), \quad e^{(2)} = (0, \vec{\epsilon}(k;2)). \tag{5.7}$$

Here $(0,1,2,3)$ components are specified for convenience while $\vec{\epsilon}(k;1)$, $\vec{\epsilon}(k;2)$ and $\vec{k}/|\vec{k}|$ constitute the usual orthonormal set of 3-vectors with the associated completeness relation. The polarization vectors are orthonormal: $g_{\mu\nu}e^{(\lambda)\mu}(k)e^{(\sigma)\nu}(k) = g^{\lambda\sigma}$ and satisfy the completeness relation: $g_{\lambda\sigma}e^{(\lambda)}{}_\mu(k)e^{(\sigma)}{}_\nu(k) = g_{\mu\nu}$.

The field commutation relations for the gauge field found above are shown to be satisfied if we assume, parallel to the discussion in the fermionic case, the canonical commutation relations: $\left[a_{(\lambda)}(k^+, k^\perp), a^\dagger{}_{(\sigma)}(k'^+, k'^\perp)\right] = -g_{\lambda\sigma}\ \delta(k^+ - k'^+)\ \delta^2(k^\perp - k'^\perp)$. We note that the operators $a_{(0)} = (a_{(+)} + a_{(-)})/\sqrt{2}$ and $a_{(3)} = (a_{(+)} - a_{(-)})/\sqrt{2}$ obey the usual canonical commutation relations except that in the case of $a_{(0)}$ a negative sign

[10] $e^{(-)}(k)$ is called the dual of $e^{(+)}(k)$. Such a pair of null vectors is employed also in the well known ML prescription in the light-cone gauge and in the context of CNPA [20, 21].

is obtained. The discussion of the Gupta-Bleuler consistency condition then becomes parallel to that in the conventional equal-time treatment of the theory.

The Feynman gauge free gauge field propagator on the LF can be derived straightforwardly

$$< 0|T(A_\mu(x)A_\nu(0))|0 >= \frac{i}{(2\pi)^4}\int d^4k\; e^{-ik.x}\;\frac{-g_{\mu\nu}}{k^2+i\epsilon} \tag{5.8}$$

The momentum space representations of the components of the energy-momentum tensor are straightforward to derive as in the fermionic case. The canonical Hamiltonian, for example, gets contributions from the physical transversly polarized photons as well as from the longitudinally polarized ones. The Gupta-Bleuler consistency condition is required [17] to be imposed in order to define the physical Hilbert space.

The computations done [17], employing the covariant gauge on the LF, for the electron self-energy, electron-muon scattering, and the Compton scattering demonstrate complete agreement with the results known in the conventional equal-time theory. We find that on the LF the tree level *seagull* term dominates the (classical) Thomas formula for the scattering at vanishingly small photon energies. It is suggestive that on the LF the (conventional theory) semi-classical approximation may reveal itself already at the tree level (after having removed the constraints). We will consider the LF quantized QCD after the study in the *front form* theory of the nonperturbative vacuum structures in some two dimensional completely solvable gauge theories.

6 Vacuum Structures in Schwinger and Chiral Schwinger Models

It is pertinent to study two dimensional gauge theories on the LF. The models like SM and CSM can be solved completely. They may give clues, for example, on the accessibility or not, in the fully interacting theory, of certain gauge-fixing condition, found practical in the context of perturbation theory. The study [15] of the SM, for example, shows that the light-cone gauge, $A_- = 0$, is *not* convenient on the LF; it would subtract out the gauge invariant information from the theory itself, which is needed for describing the nonperturbative vacuum structure in the theory.

The models mentioned above are known to have non-trivial vacuum structure, a non-perturbative effect, from the studies [57] in the conventional framework. Their study would indicate as to how to look for such and other nonperturbative effects in the LF quantized QCD in $3+1$ dimensions.

The massless QED_2 or SM is describe by

$$\mathcal{L} = \bar{\psi}\, i\gamma^\mu\partial_\mu\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e\bar{\psi}\,\gamma^\mu\psi A_\mu. \tag{6.1}$$

Its exact solvability [**?**] derives from the remarkable property of one-dimensional fermion systems, viz, that they can equivalently be described in terms of canonical one-dimensional boson fields. Some of the correspondences in the abelian bosonization are $\bar{\psi}\psi = K : \cos 2\sqrt{\pi}\,\phi :$, $\bar{\psi}\gamma_5\psi = K : \sin 2\sqrt{\pi}\,\phi :$, $\bar{\psi}\gamma_5\gamma_\mu\psi = \partial_\mu\phi/\sqrt{\pi}$, $\bar{\psi}\gamma_\mu\psi = \epsilon_{\mu\nu}\partial^\nu\phi/\sqrt{\pi}$, $\bar{\psi}\, i\gamma.\partial\psi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ where ϕ is a bosonic scalar field and K is a constant. The fermionic condensate $< \bar{\psi}\psi >_0$, for example, may then be expressed in terms of the value of the bosonic condensate. The bosonized theory can also be constructed with the use of the functional

integral method. The original fermionic and the bosonized theories are *equivalent* in the sense that they have the same current commutation relations and the energy-momentum tensor is the same when expressed in terms of the currents. For studying nonperturbative vacuum structure the bosonized theory is convenient to use. The *bosonized* version of QED_2 is found to be

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - gA_\mu\epsilon^{\mu\nu}\partial_\nu\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{6.2}$$

where $g = e/\sqrt{\pi}$. It carries in it all the symmetries of the original fermionic model including the information on the dynamical mass generation [**?**] for the gauge field. Under the $U(1)$ gauge field transforamtion the scalar field is invariant (or neutral) while under the chiral transformations, $U_5(1)$, in view of the correspondences above, the field suffers a translation by a constant.

Following the procedure of Secs. 2 and 3 we make the *separation*, of the condensate variable in the scalar field : $\phi(\tau, x^-) = \omega(\tau) + \varphi(\tau, x^-)$. The *chiral transformation* is now defined by: $\omega \to \omega + const.$, $\varphi \to \varphi$, and $A_\mu \to A_\mu$ so that the *boundary conditions* at infinity on the quantum fluctuation field φ are kept unaltered under these transformations and the mathematical framework be considered *well posed*. The bosonized Lagrangian written in the LF coordinates reads as is rewritten as

$$L = \int dx^- \left[\dot{\varphi}\varphi' + g(A_+\varphi' - A_-\dot{\varphi}) + \frac{1}{2}(\dot{A}_- - A_+{}')^2\right] - g\dot{\omega}h(\tau) \tag{6.3}$$

where $h(\tau) = \int dx^- A_-(\tau, x^-)$, an overdot (a prime) indicates the partial derivative with respect to τ (x^-). We work in the *continuum* and require (on physical considerations) that the relevant fields satisfy the necessary conditions such that their Fourier transforms with respect to the spatial longitudinal coordinate x^- exist.

The last term in the Lagrangian density shows that the *light-cone gauge*, $A_- = 0$, employed often in perturbation theory computations, *may not be appropriate to use in the fully interacting theory*[11], if we are seeking to study also the nonperturbative effects in the theory. Also the zero-momentum mode of A_- is a gauge invariant quantity under the boundary conditions assumed. We may, of course, impose different boundary conditions on the fields or add new ingredients in the theory so as to compensate for the elimination of the physical dynamical variable $h(\tau)$. A convenient alternative is the local gauge-fixing condition $\partial_- A_- = 0$, which is accessible on the phase space. We remove only the nonzero modes of A_-.

Following the Dirac method to eliminate the constraints in the *front form* theory only the three linearly independent operators survive: the *condensate* ω, $h(\tau)$, the zero-momentum-mode of A_- and canonically conjugate to ω as well, and φ which satisfies the LF commutator while it commutes with the others. The H^{lf} contains in it only the field φ. The Hilbert space can thus be described in two different fashions. Selecting φ and h as forming the complete set of mutually commuting operators leads to the *chiral vacua* while selecting φ together with ω leads to the description built on the *condensate* or θ-vacua. In the QED_2 the ω is *not* a background field rather it is shown [15] to be an operator and its eigenvalues, with continuous spectrum, label the *condensate* vacua of the theory. The cluster decomposition property requirement [23] indicates the preference in favor of the *condensate* vacua.

[11] Similar considerations are clearly pertinent to $3+1$ dimensional QCD as well.

The other related gauge theory model is the *chiral* QED_2 or CSM described by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}_R\, i\gamma^\mu\partial_\mu\psi_R + \bar{\psi}_L\,\gamma^\mu(i\partial_\mu + 2e\sqrt{\pi}A_\mu)\psi_L, \tag{6.4}$$

where[12] $\psi = \psi_R + \psi_L$ is a two-component spinor field and A_μ is the abelian gauge field, $\gamma_5\psi_L = -\psi_L$, and $\gamma_5\psi_R = \psi_R$. The classical Lagrangian is invariant under the local $U(1)$ gauge transformations $A_\mu \to A_\mu + \partial_\mu\alpha/(2\sqrt{\pi}e)$, $\psi \to [P_R + e^{i\alpha}P_L]\psi$ and under the global $U(1)_5$ chiral transformations $\psi \to exp(i\gamma_5\alpha)\,\psi$.

The bosonized version is convenient to study the vacuum structure; it is shown to be given by

$$S = \int d^2x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + eA_\nu(\eta^{\mu\nu} - \epsilon^{\mu\nu})\partial_\mu\phi + \frac{1}{2}ae^2A_\mu A^\mu\right] \tag{6.5}$$

Here the explicit mass term for the gauge field parametrized by the constant parameter a represents a regularization ambiguity [58] and the breakdown of $U(1)$ gauge symmetry. The model has received much attention since Jackiw and Rajaraman [58] pointed out that, despite the gauge anomaly the theory can be shown to be unitary and consistently quantized. In the LF coordinates it reads as

$$S = \int d^2x\ \left[\dot{\varphi}\varphi' + \frac{1}{2}(\dot{A}_- - A'_+)^2 + ae^2[A_+ + \frac{2}{ae}(\dot{\omega} + \dot{\varphi})]A_-\right]. \tag{6.6}$$

We note now that A_+ appears in the action as an *auxiliary* field, without a kinetic term. It is clear that the condensate variable may thus be subtracted out from the theory using the frequently adopted procedure of *field redefinition* [16] on it: $A_+ \to A_+ - 2\dot{\omega}/(ae)$, obtaining thereby

$$\mathcal{L}_{CSM} = \dot{\varphi}\varphi' + \frac{1}{2}(\dot{A}_- - A'_+)^2 + 2e\dot{\varphi}A_- + ae^2A_+A_-, \tag{6.7}$$

which signals the emergence of a *different structure* of the Hilbert space compared to that of the SM.

The Lagrange equations in the CSM follow to be

$$\begin{aligned} \partial_+\partial_-\varphi &= -e\partial_+A_-, \\ \partial_+\partial_+A_- - \partial_+\partial_-A_+ &= ae^2A_+ + 2e\partial_+\varphi, \\ \partial_-\partial_-A_+ - \partial_+\partial_-A_- &= ae^2A_-. \end{aligned} \tag{6.8}$$

and for $a \neq 1$ they lead to:

$$\begin{aligned} \Box G(\tau,x) &= 0 \\ \left[\Box + \frac{e^2a^2}{(a-1)}\right]E(\tau,x) &= 0, \end{aligned} \tag{6.9}$$

[12] In two dimensions the $\pm$ projections of the spinor coincide with the chiral or γ_5 projections. We define $\gamma^0 = \sigma_1$, $\gamma^1 = i\sigma_2$, $\gamma_5 = \gamma^0\gamma^1 = -\sigma_3$, $\Lambda^- = \gamma^0\gamma^-/\sqrt{2} = (1-\gamma_5)/2 \equiv P_L$, $\Lambda^+ = \gamma^0\gamma^+/\sqrt{2} = (1+\gamma_5)/2 \equiv P_R$, $x^\mu : (x^+ \equiv \tau, x^- \equiv x)$ with $\sqrt{2}x^\pm = \sqrt{2}x_\mp = (x^0 \pm x^1)$, $A^\pm = A_\mp = (A^0 \pm A^1)/\sqrt{2}$, $\psi_{L,R} = P_{L,R}\,\psi$, $\bar{\psi} = \psi^\dagger\gamma^0$.

where $E = (\partial_+ A_- - \partial_- A_+)$ and $G = (E - ae\varphi)$. Both the massive and massless scalar excitations are present in the theory and the tachyons would be absent in the specrtum if $a > 1$; the case considered in this paper. We will confirm in the Hamiltonian framework below that the E and G represent, in fact, the two independent field operators on the LF phase space.

The Dirac procedure as applied to the very simple action (6.7) of the CSM is straightforward. The canonical momenta are $\pi^+ \approx 0, \pi^- \equiv E = \dot{A}_- - A'_+, \pi_\varphi = \varphi' + 2eA_-$ which result in two primary weak constraints $\pi^+ \approx 0$ and $\Omega_1 \equiv (\pi_\varphi - \varphi' - 2eA_-) \approx 0$. A secondary constraint $\Omega_2 \equiv \partial_- E + ae^2 A_- \approx 0$ is shown to emerge when we require the τ independence (persistency) of $\pi^+ \approx 0$ employing the preliminary Hamiltonian

$$H' = {H_c}^{lf} + \int dx\ u_+ \pi^+ + \int dx\ u_1 \Omega_1, \tag{6.10}$$

where u_+ and u_1 are the Lagrange multiplier fields and ${H_c}^{lf}$ is the canonical Hamiltonian

$${H_c}^{lf} = \int\ dx\ \left[\ \frac{1}{2}E^2 + EA'_+ - ae^2 A_+ A_-\right]. \tag{6.11}$$

and we assume initially the standard equal-τ Poisson brackets : $\{E^\mu(\tau, x^-), A_\nu(\tau, y^-)\} = -\delta^\mu_\nu \delta(x^- - y^-)$, $\{\pi_\varphi(\tau, x^-), \varphi(\tau, y^-)\} = -\delta(x^- - y^-)$ etc.. The persistency requirement for Ω_1 results in an equation for determining u_1. The procedure is repeated with the following extended Hamiltonian which includes in it also the secondary constraint

$${H_e}^{lf} = {H_c}^{lf} + \int dx\ u_+ \pi^+ + \int dx\ u_1 \Omega_1 + \int dx\ u_2 \Omega_2. \tag{6.12}$$

No more secondary constraints are seen to arise; we are left with the persistency conditions which determine the multiplier fields u_1 and u_2 while u_+ remains undetermined. We also find[13] $(C)_{ij} = \{\Omega_i, \Omega_j\} = D_{ij}\ \ (-2\partial_x \delta(x-y))$ where $i, j = 1, 2$ and $D_{11} = 1,\ D_{22} = ae^2,\ D_{12} = D_{21} = -e$ and that π^+ has vanishing brackets with $\Omega_{1,2}$. The $\pi^+ \approx 0$ is first class weak constraint while Ω_1 and Ω_2, which does not depend on A_+ or π^+, are second class ones.

We go over from the Poisson bracket to the Dirac bracket $\{,\}_D$ constructed in relation to the pair, $\Omega_1 \approx 0$ and $\Omega_2 \approx 0$

$$\{f(x), g(y)\}_D = \{f(x), g(y)\} - \int\int dudv\ \{f(x), \Omega_i(u)\}(C^{-1}(u,v))_{ij}\{\Omega_j(v), g(y)\}. \tag{6.13}$$

Here C^{-1} is the inverse of C and we find $(C^{-1}(x,y))_{ij} = B_{ij}\ K(x,y)$ with $B_{11} = a/(a-1)$, $B_{22} = 1/[(a-1)e^2]$, $B_{12} = B_{21} = 1/[(a-1)e]$, and $K(x,y) = -\epsilon(x-y)/4$. Some of the Dirac brackets are $\{\varphi, \varphi\}_D = B_{11}\ K(x,y)$; $\{\varphi, E\}_D = eB_{11}\ K(x,y)$; $\{E, E\}_D = ae^2 B_{11}\ K(x,y)$; $\{\varphi, A_-\}_D = -B_{12}\ \delta(x-y)/2$; $\{A_-, E\}_D = B_{11}\ \delta(x-y)/2$; $\{A_-, A_-\}_D = B_{12}\partial_x\ \delta(x-y)/2$ and the only nonvanishing one involving A_+ or π^+ is $\{A_+, \pi^+\}_D = \delta(x-y)$.

The equations of motion employ now the Dirac brackets and inside them, in view of their very construction, we may set $\Omega_1 = 0$ and $\Omega_2 = 0$ as strong relations. The

[13] We make the convention that the first variable in an equal- τ bracket refers to the longitudinal coordinate $x^- \equiv x$ while the second one to $y^- \equiv y$ while τ is suppressed.

Hamiltonian is therefore effectively given by H_e with the terms involving the multipliers u_1 and u_2 dropped. The multiplier u_+ is not determined since the constraint $\pi^+ \approx 0$ continues to be first class even when the above Dirac bracket is employed. The variables π_φ and A_- are then removed from the theory leaving behind φ, E, A_+, and π^+ as the remaining independent variables. The canonical Hamiltonian density reduces to $\mathcal{H}_c^{lf} = E^2/2 + \partial_-(A_+E)$ while $\dot{A}_+ = \{A_+, H_e^{lf}\}_D = u_+$. The surface term in the canonical LF Hamiltonian may be ignored if, say, $E(= F_{+-})$ vanishes at infinity. The variables π^+ and A_+ are then seen to describe a decoupled (from φ and E) free theory and we may hence drop these variables as well. The effective LF Hamiltonian thus takes the simple form

$$H_{CSM}^{lf} = \frac{1}{2}\int dx\ E^2, \tag{6.14}$$

which is to be contrasted with the one found in the conventional treatment [57]. E and G (or E and φ) are now the independent variables on the phase space and the Lagrange equations are verified to be recovered for them, which assures us of the selfconsistency [25]. We stress that in our discussion we do *not* employ any gauge-fixing. The same result for the Hamiltonian could be alternatively obtained[14], however, if we did introduce the gauge-fixing constraint $A_+ \approx 0$ and made further modification on $\{,\}_D$ in order to implement $A_+ \approx 0, \pi^+ \approx 0$ as well. That it is accessible on the phase space to take care of the remaining first class constraint, but not in the bosonized Lagrangian, follows from the Hamiltons eqns. of motion. We recall [15] that in the SM φ, ω, and $\pi_\omega = (e/\sqrt{\pi})\int dx A_-$ were shown to be the independent operators and that the matter field φ appeared instead in the LF Hamiltonian.

The *canonical quantization* is peformed via the correspondence $i\{f,g\}_D \to [f,g]$ and we find the following equal-τ commutators

$$\begin{aligned}
[E(x), E(y)] &= iK(x,y)a^2e^2/(a-1), \\
[G(x), E(y)] &= 0, \\
[G(x), G(y)] &= ia^2e^2K(x,y).
\end{aligned} \tag{6.15}$$

For $a > 1$, when the tachyons are absent as seen from (6), these commutators are also physical and the independent field operators E and G generate the Hilbert space with a tensor product structure of the Fock spaces F_E and F_G of these fields with the positive definite metric.

The commutators obtained can be realized in the momentum space through the following Fourier transforms

$$\begin{aligned}
E(x,\tau) &= \frac{ae}{\sqrt{(a-1)}\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \frac{\theta(k)}{\sqrt{2k}}\left[d(k,\tau)e^{-ikx} + d^\dagger(k,\tau)e^{ikx}\right], \\
G(x,\tau) &= \frac{ae}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \frac{\theta(k)}{\sqrt{2k}}\left[g(k,\tau)e^{-ikx} + g^\dagger(k,\tau)e^{ikx}\right],
\end{aligned} \tag{6.16}$$

if the operators $(d, g, d^\dagger, g^\dagger)$ satisfy the well known canonical commutation relations of two independent harmonic oscillators; the well known set of Schwinger's bosonic oscillators, often employed in the angular momentum theory. The expression for the Hamiltonian

[14] A similar discussion is encountered also in the LF quantized Chern-Simons-Higgs system [54].

becomes

$$H^{lf}_{CSM} = \delta(0)\frac{a^2e^2}{2(a-1)} \int \frac{dk}{2k}\,\theta(k)\;\; N_d(k,\tau) \tag{6.17}$$

where we have dropped the infinite zero-point energy term and note that [22] $\left[d^\dagger(k,\tau), d(l,\tau)\right] = -\delta(k-l)$, $d^\dagger(k,\tau)d(k,\tau) = \delta(0)N_d(k,\tau)$ etc. with similar expressions for the independent g-oscillators. We verify that $[N_d(k,\tau), N_d(l,\tau)] = 0$, $[N_d(k,\tau), N_g(l,\tau)] = 0$, $\left[N_d(k,\tau), d^\dagger(k,\tau)\right] = d^\dagger(k,\tau)$ etc..

The Fock space can hence be built on a basis of eigenstates of the hermitian number operators N_d and N_g. The ground state of CSM is degenerate and described by $|0> = |E=0)\otimes|G\}$ and it carries vanishing LF energy in agreement with the conventional theory discusion [57]. For a fixed k these states, $|E=0)\otimes(g^\dagger(k,\tau)^n/\sqrt{n!})|0\}$, are labelled by the integers $n = 0, 1, 2, \cdots$. The θ-vacua are absent in the CSM. However, we recall [15] that in the SM the degenerate *chiral vacua* are also labelled by such integers. We remark also that on the LF we work in the Minkowski space and that in our discussion we do *not* make use of the Euclidean space theory action, where the (classical) vacuum configurations of the Euclidean theory gauge field, belonging to the distinct topological sectors, are useful, for example, in the functional integral quantization of the gauge theory.

On the LF both the bosonized SM and CSM are described in terms of a minimum number of dynamical variables, which survive after the elimination of the phase space constraints. We recall that the introduction of the *bosonic condensate* variable $\omega(\tau)$ (or in general $\omega(\tau, x^\perp)$) corresponds to the gauge-fixing required in order to deal with the first class constraint $\int(\pi - \partial_-\phi) \approx 0$. On the other hand we have the gauge invariant zero-momentum mode $h(\tau)$ of the gauge field A_-, apart from the quantum fluctuation field φ. They are in a sense the minimal set of operators which survive in the *front form* theory. With their help the vacuum structures of both the SM and CSM are described in a very economical and transparent fashion on the LF, which agree with the conventional theory conclusions. In the latter, however, we have to go through quite an elaborate and extensive discussion [57]. Finally, if we did adopt the light-cone gauge we must compensate for the loss of the gauge invariant information by some other ingredient, say, by imposing more complicated boundary conditions on the fields involved or by introducing new fields.

7 QCD in Covariant gauges

We describe briefly the recent study [17] done on the *front form* QCD in covariant gauges. The Lagrangian density corresponding to the quantum action of QCD is described in standard notation by

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^{a\mu\nu}F^a{}_{\mu\nu} + B^a\partial_\mu A^{a\mu} + \frac{\xi}{2}B^aB^a + i\partial^\mu\chi_1{}^a\mathcal{D}^{ac}{}_\mu\chi_2{}^c + \bar\psi^i(i\gamma^\mu D^{ij}{}_\mu - m\delta^{ij})\psi^j \tag{7.1}$$

Here ψ^j is the quark field with color index $j = 1..N_c$ for an $SU(N_c)$ color group, $A^a{}_\mu$ the gluon field, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a{}_\mu + g f^{abc}A^b{}_\mu A^c{}_\nu$ the field strength, $\mathcal{D}^{ac}{}_\mu = (\delta^{ac}\partial_\mu + g f^{abc}A^b{}_\mu)$, $D^{ij}{}_\mu\psi^j = (\delta^{ij}\partial_\mu - igA^a{}_\mu(\lambda^a/2)^{ij})\psi^j$, $a = 1..({N_c}^2 - 1)$ the gauge group index etc. The covariant gauge-fixing is introduced by adding to the Lagrangian the linear gauge-fixing term $(B^a\partial_\mu A^{a\mu} + \frac{\xi}{2}B^aB^a)$ where B^a is the Nakanishi-Lautrup auxiliary field and ξ is a parameter. The $\chi_1{}^a$ and $\chi_2{}^a$ are the (hermitian) anticommuting Faddeev-Popov ghost fields, and the action is invariant under the BRS transformation.

The quark field term in LF coordinates reads

$$\begin{aligned}\bar{\psi}^i(i\gamma^\mu D^{ij}{}_\mu - m\delta^{ij})\psi^j &= i\sqrt{2}\bar{\psi}^i_+\gamma^0 D^{ij}{}_+\psi_+{}^j + \bar{\psi}^i_+(i\gamma^\perp D^{ij}{}_\perp - m\delta^{ij})\psi_-{}^j \\ &+ \bar{\psi}^i_- \left[i\sqrt{2}\gamma^0 D^{ij}{}_-\psi_-{}^j + (i\gamma^\perp D^{ij}{}_\perp - m\delta^{ij})\psi_+{}^j\right] \qquad (7.2)\end{aligned}$$

where $D^{ij}{}_\pm = (\delta^{ij}\partial_\pm - igA^a{}_\pm(\lambda^a/2)^{ij})$. It shows that the minus components $\psi_-{}^j$ are in fact nondynamical (Lagrange multiplier) fields without kinetic terms. The variation of the action with respect to $\bar{\psi}^j{}_-$ and $\psi^j{}_-$ leads to the following gauge covariant constraint equation

$$i\sqrt{2}D^{ij}{}_-\psi_-{}^j = -(i\gamma^0\gamma^\perp D^{ij}{}_\perp - m\gamma^0\delta^{ij})\psi_+{}^j, \qquad (7.3)$$

and its conjugate. The $\psi^j{}_-$ components may thus be eliminated in favor of the dynamical components ψ^j_+

$$\psi_-{}^j(x) = \frac{i}{\sqrt{2}}\left[U^{-1}(x|A_-)\frac{1}{\partial_-}U(x|A_-)\right]^{jk}(i\gamma^0\gamma^\perp D^{kl}{}_\perp - m\gamma^0\delta^{kl})\psi_+{}^l(x). \qquad (7.4)$$

Here, for a fixed τ, $U \equiv U(x|A_-)$ is an $N_c \times N_c$ gauge matrix in the fundamental representation of $SU(N_c)$ and it satisfies

$$\partial_- U(x|A_-) = -ig\, U(x|A_-)\, A_-(x) \qquad (7.5)$$

with $A_- = A^a{}_-\lambda^a/2$. It has the formal solution

$$U(x^-, x^\perp|A_-) = U(x^-{}_0, x^\perp|A_-)\,\tilde{\mathcal{P}}\, exp\left\{-ig\int_{x^-{}_0}^{x^-} dy^- A_-(y^-, x^\perp)\right\} \qquad (7.6)$$

where $\tilde{\mathcal{P}}$ indicates the anti-path-ordering along the longitudinal direction x^-. U has a series expansion in the powers of the coupling constant.

The Hamiltonian density in Feynman gauge is

$$\begin{aligned}\mathcal{H}^{LF} &= \mathcal{H}_0 + \mathcal{H}_{int} \\ &= -\frac{1}{2}g^{\mu\nu}A^a{}_\mu\,\partial^\perp\partial_\perp\,A^a{}_\nu - g\sqrt{2}\bar{\psi}^i_+\gamma^0 A_+{}^{ij}\psi_+{}^j \\ &- \bar{\psi}^i_+\left[\delta^{ij}(i\gamma^\perp\partial_\perp - m) + g\gamma^\perp A^{ij}{}_\perp\right]\psi_-{}^j + \frac{g}{2}f^{abc}(\partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu)A^{b\mu}A^{c\nu} \\ &+ \frac{g^2}{4}f^{abe}f^{cde}A^a{}_\mu A^b{}_\nu A^{c\mu}A^{d\nu} + \partial^\mu(\bar{\chi}^a)\partial_\mu\chi^a + gf^{abc}(\partial^\mu\bar{\chi}^a)\chi^b A^c{}_\mu \qquad (7.7)\end{aligned}$$

where $\psi_-{}^j$ is given above, we have set $\sqrt{2}\chi = (\chi_1 + i\chi_2)$, $\sqrt{2}\bar{\chi} = (\chi_1 - i\chi_2)$, and in $\mathcal{H}^{LF}$ the cubic and higher order terms belong to $\mathcal{H}_{int}$ which is also understood to be normal ordered. It is worth remarking that despite the presence of the longitudinal operators $a_\pm$ and $a^\dagger{}_\pm$ in the fields A_μ, there are no non-zero matrix elements involving these quanta as external lines in view of the commmutation relations of these operators as discussed in the previous section.

The perturbation theory expansion in the interaction representation where we time order with respect to the LF time τ is built following the Dyson-Wick procedure. We will illustrate it in our context through some explicit computations, for simplicity, in QED where $U(x|A_-) = exp\{-ie\int^{x^-} du^- A_-(\tau, u^-, x^\perp)\}$ and $D_\mu = (\partial_\mu - ieA_\mu)$. We observe

that a *seagull* term of the order e^2 is present in the interaction Hamiltonian at the tree level; like that found also in the scalar field QED.

Towards an illustration consider the computation of *Electron Self-Energy*. The contribution from the longitudinal components arises from

$$e^2\int d^4x_1d^4x_2 \qquad :\psi_+{}^\dagger(x_2)(m+i\not{\partial}_2{}^T)\int_{-\infty}^{\infty}\frac{1}{2}dy_2{}^-\epsilon(x_2{}^- - y_2{}^-)$$
$$\{\int_{y_2{}^-}^{x_2{}^-} du_2{}^-\dot{A}_-(u_2)\}(m-i\not{\partial}_2{}^T)\ddot{\psi}_+(y_2)\ddot{\psi}^\dagger_+(x_1)\psi_+(x_1)\dot{A}_+(x_1):\quad (7.8)$$

leading to

$$e^2\int d^4q\,\frac{\bar{u}^{(r)}(p)[\gamma^-(m+\not{q}^T)\gamma^+]u^{(s)}(p)}{[(p-q)^2+i\epsilon](q^2-m^2+i\epsilon)}\,(-g_{-+}) \qquad (7.9)$$

The graph with the A_+ and A_- interchanged gives rise to a similar expression with $g_{+-}\to g_{-+}$ while $\gamma^\pm\to\gamma^\mp$. The matrix elements following from the four graphs corresponding to the exchange of the (photon) fields A_1 and A_2 is also written down by simple inspection. As in the earlier case the expressions get simplified in virtue of (10) and acquire the covariant form encountered in the conventional covariant perturbation theory. The complete matrix element is found to be

$$e^2\int d^4q\,\frac{\bar{u}^{(r)}(p)[\gamma^\mu(m+\not{\tilde{q}})\gamma^\nu]u^{(s)}(p)}{[(p-q)^2+i\epsilon](q^2-m^2+i\epsilon)}\,(-g_{\mu\nu}) \qquad (7.10)$$

where $\tilde{q}^\mu\equiv((m^2+q^\perp q^\perp)/(2q^+),q^+,q^\perp)$ and the integration measure is $d^4q=d^2q^\perp dq^+dq^-$. Considering that the integrand has the pole at $q^2-m^2\approx 0$ we may regard the expression obtained [17] on the LF to be effectively identical to the one obtained in the conventional covariant theory framework. The discussion parallel to that given here may be followed also in the context of the light-cone gauge. The latter, however, demands the further introduction [59] of a light-like vector $n^\mu=(n^0,\vec{n})$ and its dual $\tilde{n}^\mu=(n^0,-\vec{n})$ in order to evaluate the corresponding Feynman integrals in a consistent manner.

8 Conclusions

Collected below are some of the interesting conclusions we seem to reach.

- The LF hyperplane is *equally valid and appropriate* as the conventional equal-time one for the field theory quantization.
- The appearence of the nonlocality along the longitudinal direction in the *front form* quantized theory is not unexpected; it does not conflict with the the microcausality (or cluster decomposition) principle.
- The covariant phase space and Fourier expansion considerations based on the description of the relativistic theory using light-cone coordinates lead to the LF commutator for the free scalar field, which is nonlocal in the longitudinal direction.
- The hyperplanes $x^\pm=0$ define the characteristic surfaces of a hyperbolic partial differential equation. From the mathematical theory of classical partial differential

equations [24] it is known that the Cauchy initial value problem would require us to specify the data on both the hyperplanes. From our studies we conclude [16] that it is sufficient in the *front form* theory to choose one of the two LF hyperplanes for canonically quantizing the theory.

- In the quantized theory the equal-τ commutators of the field operators, at a fixed initial LF-time, form now a part of the initial data instead and we deal with operator differential equations.

- The information on the commutators on the other characteristic hyperplane seems already to be contained [15] in the quantized theory; it may not, in general, be required to specify it separately.

- The constrained phase space dynamics in the LF theory with one more kinematical generator and the inherent symmetry with regard to $x^{\pm}$ result in a reduced number of independent field operators. The discussion of the Hilbert space becomes more transparent compared to that in the conventional treatment. The lack of manifest covariance which appears problematic can be handled[15] by employing, for example, the *LF four-spinor* [15] and the Fourier transform of the spinor field as defined [17] in Sec. 4.

- On the LF the γ_5 symmetry of free massless Dirac equation can be generalized to a nonlocal (chiral) Γ_5 symmetry valid also in the massive case. The Weyl and Majorana spinors and the helicity operator may be defined on the LF in straightforward fashion.

- The zero-longitudinal-momentum modes of the fields are important for describing the nonperturbative effects on the LF. In the scalar and gauge theories they are dynamical variables in the frame work of the *standard* Dirac procedure. The *separation* $\phi(\tau, x^-, x^\perp) = \omega(\tau, x^\perp) + \varphi(\tau, x^-, x^\perp)$ introduced in Secs. 2.6, 5 correspond to the gauge-fixing conditions [25] required to be introduced in the theory for handling first class constraints.

  - In the case of the scalar theory we obtain constraint equations which enable us to describe SSB and the (tree level) Higgs mechanism. Associated to the local theory in the conventional coordinates we find a nonlocal LF Hamiltonian.

  - The gauge field zero modes play a crucial role in the description of the nonperturbative vacuum structures in the LF quantized SM and CSM. They also indicate that the (popular) light-cone gauge may not be accessible in the *front form* theory if we are concerned with the study of nonperturbative effects.

- The physical content following from the *front form* theory is the same, even though arrived at through different description on the LF, when compared with the one in the *instant form* case.

- Not all the constraints in the LF theory need to be solved first before considering its renormalization; it is sometimes convenient to obtain some of them as renormalized constraint equations [18] instead.

[15] See also, [54]

- In the conventional treatment we may be required to introduce external constraints in the quantized theory based on physical considerations, say, while describing the SSB. The analogous relevant constraints in the *front form* theory appear to be already contained in the quantized theory.

- On the LF the quantized theory of *chiral boson* appears straightforward (Sec. 3.4). The field commutator does not conflict with the microcausality principle.

- A theoretical demonstration of the well accepted notion that a classical model field theory must be upgraded first through its quantization before we confront it with the experimental data, seems to emerge.

- The LF quantized QCD employing covariant gauges [17] looks promising. All of the propagators become causal and the covariance of the theory is tractable. The semiclassical theory is found revealed at the tree level. The algebra of bilocals in the LF quantized theory may help reveal the string like structure as seems to be found [55], for example, in QCD_2.

- The recently proposed BRS-BFT [60] quantization procedure is extended straightforwardly on the LF (Appendix C).

- It is well known that topological considerations are often required in the field theory quantization employing the functional integral method, where the Euclidean theory action is employed. The corresponding ingredients seem to arise in the canonically quantized *front form* theory as well but with different interpretation. This is suggested, for example, from the studies of the SM, CSM, and the study of the *kink* solutions.

- In connection with the relativistic bound state problem, not touched upon in this article, the LF Tamm-Dancoff method [9, 56] and Bethe-Salpeter dynamics on the covariant null plane [19, 20, 21] seem to be promising alternatives to lattice gauge theory approach.

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Appendix A: Poincaré Generators on the LF

The Poincaré generators in coordinate system (x^0, x^1, x^2, x^3), satisfy $[M_{\mu\nu}, P_\sigma] = -i(P_\mu g_{\nu\sigma} - P_\nu g_{\mu\sigma})$ and $[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\rho}g_{\nu\sigma} + M_{\nu\sigma}g_{\mu\rho} - M_{\nu\rho}g_{\mu\sigma} - M_{\mu\sigma}g_{\nu\rho})$ where the metric is $g_{\mu\nu} = diag\,(1, -1, -1, -1)$, $\mu = (0, 1, 2, 3)$ and we take $\epsilon_{0123} = \epsilon_{-+12} = 1$. If we define $J_i = -(1/2)\epsilon_{ikl}M^{kl}$ and $K_i = M_{0i}$, where $i, j, k, l = 1, 2, 3$, we find $[J_i, F_j] = i\epsilon_{ijk}F_k$ for $F_l = J_l, P_l$ or K_l while $[K_i, K_j] = -i\epsilon_{ijk}J_k$, $[K_i, P_l] = -iP_0 g_{il}$, $[K_i, P_0] = iP_i$, and $[J_i, P_0] = 0$.

The LF generators are P_+, P_-, P_1, P_2, $M_{12}=-J_3$, $M_{+-}=-K_3$, $M_{1-}=-(K_1+J_2)/\sqrt{2}\equiv -B_1$, $M_{2-}=-(K_2-J_1)/\sqrt{2}\equiv -B_2$, $M_{1+}=-(K_1-J_2)/\sqrt{2}\equiv -S_1$ and $M_{2+}=-(K_2+J_1)/\sqrt{2}\equiv -S_2$. We find $[B_1,B_2]=0$, $[B_a,J_3]=-i\epsilon_{ab}B_b$, $[B_a,K_3]=iB_a$, $[J_3,K_3]=0$, $[S_1,S_2]=0$, $[S_a,J_3]=-i\epsilon_{ab}S_b$, $[S_a,K_3]=-iS_a$ where $a,b=1,2$ and $\epsilon_{12}=-\epsilon_{21}=1$. Also $[B_1,P_1]=[B_2,P_2]=iP^+$, $[B_1,P_2]=[B_2,P_1]=0$, $[B_a,P^-]=iP_a$, $[B_a,P^+]=0$, $[S_1,P_1]=[S_2,P_2]=iP^-$, $[S_1,P_2]=[S_2,P_1]=0$, $[S_a,P^+]=iP_a$, $[S_a,P^-]=0$, $[B_1,S_2]=-[B_2,S_2]=-iJ_3$, $[B_1,S_1]=[B_2,S_2]=-iK_3$. For $P_\mu=i\partial_\mu$, and $M_{\mu\nu}\to L_{\mu\nu}=i(x_\mu\partial_\nu-x_\nu\partial_\mu)$ we find $B_a=(x^+P^a-x^aP^+)$, $S_a=(x^-P^a-x^aP^-)$, $K_3=(x^-P^+-x^+P^-)$ and $J_3=(x^1P^2-x^2P^1)$. Under the conventional *parity* operation $\mathcal{P}$: ($x^\pm\leftrightarrow x^\mp, x^{1,2}\to -x^{1,2}$) and ($p^\pm\leftrightarrow p^\mp, p^{1,2}\to -p^{1,2}$), we find $\vec{J}\to\vec{J}$, $\vec{K}\to-\vec{K}$, $B_a\to -S_a$ etc.. The six generators P_l, M_{kl} leave $x^0=0$ hyperplane invariant and are called *kinematical* while the remaining P_0, M_{0k} the *dynamical* ones. On the LF there are *seven* kinematical generators : P^+,P^1,P^2,B_1,B_2,J_3 and K_3 which leave the LF hyperplane, $x^0+x^3=0$, invariant and the three *dynamical* ones S_1,S_2 and P^- form a mutually commuting set. The K_3 which was dynamical becomes now a kinematical; it generates scale transformations of the LF components of x^μ, P^μ and $M^{\mu\nu}$. We note that each of the set $\{B_1,B_2,J_3\}$ and $\{S_1,S_2,J_3\}$ generates an $E_2\simeq SO(2)\otimes T_2$ algebra; this will be shown below to be relevant for defining the *spin* for massless particle. Including K_3 in each set we find two subalgebras each with four elements. Some useful identities are $e^{i\omega K_3}\,P^\pm\,e^{-i\omega K_3}=e^{\pm\omega}\,P^\pm$, $e^{i\omega K_3}\,P^\perp\,e^{-i\omega K_3}=P^\perp, e^{i\bar v.\bar B}\,P^-\,e^{-i\bar v.\bar B}=P^-+\bar v.\bar P+\frac{1}{2}\bar v^2P^+, e^{i\bar v.\bar B}\,P^+\,e^{-i\bar v.\bar B}=P^+, e^{i\bar v.\bar B}\,P^\perp\,e^{-i\bar v.\bar B}=P^\perp+v^\perp P^+, e^{i\bar u.\bar S}\,P^+\,e^{-i\bar u.\bar S}=P^++\bar u.\bar P+\frac{1}{2}\bar u^2P^-, e^{i\bar u.\bar S}\,P^-\,e^{-i\bar u.\bar S}=P^-, e^{i\bar u.\bar S}\,P^\perp\,e^{-i\bar u.\bar S}=P^\perp+u^\perp P^-$ where $P^\perp\equiv\bar P=(P^1,P^2)$, $v^\perp\equiv\bar v=(v_1,v_2)$ and $(v^\perp.P^\perp)\equiv(\bar v.\bar P)=v_1P^1+v_2P^2$ etc. Analogous expressions with P^μ replaced by X^μ can be obtained if we use $[P^\mu,X_\nu]\equiv[i\partial^\mu,x_\nu]=i\delta^\mu_\nu$.

Appendix B[16]: LF Spin Operator. Hadrons in LF Fock basis

The Casimir generators of the Poincaré group are : $P^2\equiv P^\mu P_\mu$ and W^2, where $W_\mu=(-1/2)\epsilon_{\lambda\rho\nu\mu}M^{\lambda\rho}P^\nu$ defines the Pauli-Lubanski pseudovector. It follows from $[W_\mu,W_\nu]=i\epsilon_{\mu\nu\lambda\rho}W^\lambda P^\rho$, $[W_\mu,P_\rho]=0$ and $W.P=0$ that in a representation charactarized by particular eigenvalues of the two Casimir operators we may simultaneously diagonalize P^μ along with just one component of W^μ. We have $W^+=-[J_3P^+ +B_1P^2-B_2P^1], W^-=J_3P^- +S_1P^2-S_2P^1, W^1=K_3P^2+B_2P^--S_2P^+$, and $W^2=-[K_3P^1+B_1P^--S_1P^+]$ and it shows that W^+ *has a special place* since it contains only the kinematical generators [15]. On the LF we define $\mathcal{J}_3=-W^+/P^+$ as the *spin operator*. It may be shown to commute with P_μ,B_1,B_2,J_3, and K_3. For $m\neq 0$ we may use the parametrizations $p^\mu:(p^-=(m^2+p^{\perp^2})/(2p^+),p^+=(m/\sqrt{2})e^\omega,p^1=-v_1p^+,p^2=-v_2p^+)$ and $\tilde p^\mu:(1,1,0,0)(m/\sqrt{2})$ in the rest frame. We have $P^2(p)=m^2I$ and $W(p)^2=W(\tilde p)^2=-m^2[J_1^2+J_2^2+J_3^2]=-m^2s(s+1)I$ where s assumes half-integer values. Starting from the rest state $|\tilde p;m,s,\lambda,..\rangle$ with $J_3\,|\tilde p;m,s,\lambda,..\rangle=\lambda\,|\tilde p;m,s,\lambda,..\rangle$ we may build an arbitrary eigenstate of $P^+,P^\perp,\mathcal{J}_3$ (and P^-) on the LF by

$$|p^+,p^\perp;m,s,\lambda,..\rangle=e^{i(\bar v.\bar B)}e^{-i\omega K_3}|\tilde p;m,s,\lambda,..\rangle$$

[16]See, P.P. Srivastava, *Lightfront quantization of field theory* in *Topics in Theoretical Physics, Festschrift for Paulo Leal Ferreira*, eds., V.C. Aguilera-Navarro et al., pgs. 206-217, IFT-São Paulo, SP, Brasil (1995); hep-th/9610044; 9610149.

If we make use of the following *identity* [10]

$$\mathcal{J}_3(p) = \; J_3 + v_1 B_2 - v_2 B_1 = \quad e^{i(\bar{v}.\bar{B})}\; J_3\; e^{-i(\bar{v}.\bar{B})}$$

we find $\mathcal{J}_3\,|p^+, p^\perp; m, s, \lambda, ..\rangle = \lambda\,|p^+, p^\perp; m, s, \lambda, ..\rangle$. Introducing also the operators $\mathcal{J}_a = -(\mathcal{J}_3 P^a + W^a)/\sqrt{P^\mu P_\mu}$, $a = 1, 2$, which do, however, contain dynamical generators, we verify that $[\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk}\mathcal{J}_k$.

For $m = 0$ case when $p^+ \neq 0$ a convenient parametrization is $p^\mu : (p^- = p^+ {v^\perp}^2/2,\, p^+, p^1 = -v_1 p^+, p^2 = -v_2 p^+)$ and $\tilde{p} : (0, p^+, 0^\perp)$. We have $W^2(\tilde{p}) = -(S_1^2 + S_2^2){p^+}^2$ and $[W_1, W_2](\tilde{p}) = 0$, $[W^+, W_1](\tilde{p}) = -ip^+ W_2(\tilde{p})$, $[W^+, W_2](\tilde{p}) = ip^+ W_1(\tilde{p})$ showing that W_1, W_2 and W^+ generate the algebra $SO(2) \otimes T_2$. The eigenvalues of W^2 are hence not quantized and they vary continuously. This is contrary to the experience so we impose that the physical states satisfy in addition $W_{1,2}|\,\tilde{p};\, m = 0, ..\rangle = 0$. Hence $W_\mu = -\lambda P_\mu$ and the invariant parameter λ is taken to define as the *spin* of the massless particle. From $-W^+(\tilde{p})/\tilde{p}^+ = J_3$ we conclude that λ assumes half-integer values as well. We note that $W^\mu W_\mu = \lambda^2 P^\mu P_\mu = 0$ and that on the LF the definition of the spin operator appears unified for massless and massive particles. A parallel discussion based on $p^- \neq 0$ may also be given.

As an illustration consider the three particle state on the LF with the total eigenvalues p^+, λ and $p^\perp$. In the *standard frame* with $p^\perp = 0$ it may be written as $(|x_1 p^+, k_1^\perp; \lambda_1\rangle|x_2 p^+, k_2^\perp; \lambda_2\rangle|x_3 p^+, k_3^\perp; \lambda_3\rangle\;)$ with $\sum_{i=1}^3 x_i = 1$, $\sum_{i=1}^3 k_i^\perp = 0$, and $\lambda = \sum_{i=1}^3 \lambda_i$. Aplying $e^{-i(\bar{p}.\bar{B})/p^+}$ on it we obtain $(|x_1 p^+, k_1^\perp + x_1 p^\perp; \lambda_1\rangle|x_2 p^+, k_2^\perp + x_2 p^\perp; \lambda_2\rangle|x_3 p^+, k_3^\perp + x_3 p^\perp; \lambda_3\rangle\;)$ now with $p^\perp \neq 0$. The x_i and $k_i^\perp$ indicate relative (invariant) parameters[17] and do not depend upon the reference frame. The x_i is the fraction of the total longitudinal momentum carried by the i^{th} particle while $k_i^\perp$ its transverse momentum. The state of a pion with momentum $(p^+, p^\perp)$, for example, may be expressed as an expansion over the LF Fock states constituted by the different number of partons

$$|\pi : p^+, p^\perp\rangle = \sum_{n,\lambda} \int \bar{\Pi}_i \frac{dx_i d^2 {k^\perp}_i}{\sqrt{x_i}\, 16\pi^3} |n : x_i p^+, x_i p^\perp + {k^\perp}_i, \lambda_i\rangle\; \psi_{n/\pi}(x_1, {k^\perp}_1, \lambda_1; x_2, ...)$$

where [8] the summation is over all the Fock states n and spin projections λ_i, with $\bar{\Pi}_i dx_i = \Pi_i dx_i\; \delta(\sum x_i - 1)$, and $\bar{\Pi}_i d^2 k_i^\perp = \Pi_i dk_i^\perp\; \delta^2(\sum k_i^\perp)$. The wave function of the parton $\psi_{n/\pi}(x, k^\perp)$ indicates the probability amplitude for finding inside the pion the partons in the Fock state n carrying the 3-momenta $(x_i p^+, x_i p^\perp + k_i^\perp)$.

The *discrete symmetry* transformations may also be defined on the LF Fock states [8, 15] For example, under the conventional parity $\mathcal{P}$ the spin operator $\mathcal{J}_3$ is not invariant. We may rectify this by defining *LF Parity operation* by $\mathcal{P}^{lf} = e^{-i\pi J_1}\mathcal{P}$. We find then $B_1 \to -B_1, B_2 \to B_2, P^\pm \to P^\pm, P^1 \to -P^1, P^2 \to P^2$ etc. such that $\mathcal{P}^{lf}\,|p^+, p^\perp; m, s, \lambda, ..\rangle \simeq |p^+, -p^1, p^2; m, s, -\lambda, ..\rangle$. Similar considerations apply for charge conjugation and time inversion. For example, it is straightforward to construct [15] the free *LF Dirac spinor* $\chi(p) = [\sqrt{2}p^+\Lambda^+ + (m - \gamma^a p^a)\,\Lambda^-]\tilde{\chi}/\sqrt{\sqrt{2}p^+ m}$ which is also an eigenstate of $\mathcal{J}_3$ with eigenvalues $\pm 1/2$. Here $\Lambda^\pm = \gamma^0\gamma^\pm/\sqrt{2} = \gamma^\mp\gamma^\pm/2 = (\Lambda^\pm)^\dagger$, $(\Lambda^\pm)^2 = \Lambda^\pm$,

[17] We note ${p_i}^+ = x_i p^+$, ${p_i}^\perp = x_i p^\perp + {k_i}^\perp$, and $(p \cdot p) = (2p^+p^- - p^\perp p^\perp) = \sum_i \left[({m_i}^2 + {k_i}^\perp {k_i}^\perp)/x_i\right]$ where $(p_i \cdot p_i) = {m_i}^2$ and $\sum {p_i}^\mu = p^\mu$.

and $\chi(\tilde{p}) \equiv \tilde{\chi}$ with $\gamma^0\tilde{\chi} = \tilde{\chi}$. The conventional (equal-time) spinor can also be constructed by the procedure analogous to that followed for the LF spinor and it has the well known form $\chi_{con}(p) = (m + \gamma.p)\tilde{\chi}/\sqrt{2m(p^0+m)}$. Under the conventional parity operation $\mathcal{P} : \chi'(p') = c\gamma^0\chi(p)$ (since we must require $\gamma^\mu = L^\mu{}_\nu\, S(L)\gamma^\nu S^{-1}(L)$, etc.). We find $\chi'(p) = c[\sqrt{2}p^-\Lambda^- + (m - \gamma^a p^a)\,\Lambda^+]\,\tilde{\chi}/\sqrt{\sqrt{2}p^- m}$. For $p \neq \tilde{p}$ it is not proportional to $\chi(p)$ in contrast to the result in the case of the usual spinor where $\gamma^0\chi_{con}(p^0, -\vec{p}) = \chi_{con}(p)$ for $E > 0$ (and $\gamma^0\eta_{con}(p^0, -\vec{p}) = -\eta_{con}(p)$ for $E < 0$). However, applying parity operator twice we do show $\chi''(p) = c^2\chi(p)$ hence leading to the usual result $c^2 = \pm 1$. The LF parity operator over spin 1/2 Dirac spinor is $\mathcal{P}^{lf} = c\,(2J_1)\,\gamma^0$ and the corresponding transform of χ is shown to be an eigenstate of $\mathcal{J}_3$.

Appendix C: BRS-BFT Quantization on the LF of the CSM

We apply here the recently proposed BFT procedure [60] which is elegant and avoids the computation of Dirac brackets. It would thus get tested [61] on the LF as well and it also allows us to construct (new) effective Lagrangian theories.

We convert the two second class constraints of the bosonized CSM with $a > 1$ into first class constraints according to the BFT formalism. We obtain then the first class Hamiltonian from the canonical Hamiltonian and recover the DB using Poisson brackets in the extended phase space. The corresponding first class Lagrangian is then found by performing the momentum integrations in the generating functional.

(a) Conversion to First Class Constrained Dynamical System

The bosonized CSM model (for $a > 1$) is described by the action

$$S_{CSM} = \int d^2x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + eA_\nu(\eta^{\mu\nu} - \epsilon^{\mu\nu})\partial_\mu\phi + \frac{1}{2}ae^2A_\mu A^\mu\right], \quad \text{(C. 1)}$$

where a is a regularization ambiguity which enters when we calculate the fermionic determinant in the fermionic CSM. The action in the LF coordinates takes the form

$$S_{CSM} = \int d^2x^- \left[\frac{1}{2}(\partial_+A_- - \partial_-A_+)^2 + \partial_-\phi\,\partial_+\phi + 2eA_-\partial_+\phi + ae^2A_+A_-\right], \quad \text{(C. 2)}$$

We now make the *separation*, in the scalar field (a generalized function) : $\phi(\tau, x^-) = \omega(\tau) + \varphi(\tau, x^-)$. The Lagrangian density then becomes

$$\mathcal{L} = \frac{1}{2}(\partial_+A_- - \partial_-A_+)^2 + \partial_-\varphi\,\partial_+\varphi + ae^2[A_+ + \frac{2}{ae}(\partial_+\varphi + \partial_+\omega)]A_-, \quad \text{(C. 3)}$$

We note that the dynamical fields are A_- and φ while A_+ has no kinetic term. On making a redefinition of the (auxiliary) field A_+ we can recast the action on the LF in the following form

$$S_{CSM} = \int dx^- \left[\dot{\varphi}\varphi' + \frac{1}{2}(\dot{A}_- - A_+{}')^2 - 2e\dot{A}_-\varphi + ae^2A_+A_-\right], \quad \text{(C. 4)}$$

The canonical momenta are given by

$$\begin{aligned} \pi^+ &= 0, \\ \pi^- &= \dot{A}_- - A'_+ - 2e\varphi, \\ \pi_\varphi &= \varphi'. \end{aligned} \tag{C. 5}$$

We follow now the Dirac's standard procedure in order to build an Hamiltonian framework on the LF. The definition of the canonical momenta leads to two primary constraints

$$\pi^+ \approx 0, \tag{C. 6}$$

$$\Omega_1 \equiv (\pi_\varphi - \varphi') \approx 0 \tag{C. 7}$$

and we derive one secondary constraint

$$\Omega_2 \equiv \partial_- \pi^- + +2e\varphi' + ae^2 A_- \approx 0. \tag{C. 8}$$

This one follows when we require the τ independence (e.g., the persistency) of the primary constraint π^+ with respect to the preliminary Hamiltonian

$$H' = H_c{}^{l.f.} + \int dx\ u_+ \pi^+ + \int dx\ u_1 \Omega_1, \tag{C. 9}$$

where H_c is the canonical Hamiltonian

$$H_c{}^{l.f.} = \int dx\ \left[\frac{1}{2}(\pi^- + 2e\varphi)^2 + (\pi^- + 2e\varphi)A'_+ - ae^2 A_+ A_- \right], \tag{C. 10}$$

and we employ the standard equal-τ Poisson brackets. The u_+ and u_1 denote the Lagrange multiplier fields. The persistency requirement for Ω_1 give conditions to determine u_1. The Hamiltonian is next extended to include also the secondary constraint

$$H_e{}^{l.f.} = H_c{}^{l.f.} + \int dx\ u_+ \pi^+ + \int dx\ u_1 \Omega_1 + \int dx\ u_2 \Omega_2 \tag{C. 11}$$

and the procedure is now repeated with respect to the extended Hamiltonian. For the case $a > 1$, no more secondary constraints are seen to arise and we are left only with the persistency conditions which determine the multipliers u_1 and u_2 while u_+ is left undetermined. We also find[18] $\{\Omega_i, \Omega_j\} = D_{ij}\ (-2\partial_x \delta(x-y))$ where $i, j = 1, 2$ and $D_{11} = 1,\ D_{22} = ae^2,\ D_{12} = D_{21} = -e$ and π^+ is shown to have vanishing brackets with $\Omega_{1,2}$. The $\pi^+ \approx 0$ constitutes a first class constraint on the phase space; it generates local transformations of A_+ which leave the H_e invariant, $\{\pi^+, H_e\} = \Omega_2 \approx 0$. The Ω_1, Ω_2 constitute a set of second class constraints and do not involve A_+ or π^+. It is very convenient, though not necessary, to add to the set of constraints on the phase space the (accessible) gauge fixing constraint $A_+ \approx 0$. It is evident from that such a gauge freedom is *not* available at the Lagrangian level. We will also implement (e.g., turn into strong equalities) the (trivial) pair of weak constraints $A_+ \approx 0,\ \pi^+ \approx 0$ by defining the Dirac brackets with respect to them. It is easy to see that for the other remaining dynamical variables the corresponding Dirac brackets coincide with the standard Poisson brackets. The variables A_+, π^+ are thus removed from the discussion, leaving behind

[18] We make the convention that the first variable in an equal- τ bracket refers to the longitudinal coordinate $x^- \equiv x$ while the second one to $y^- \equiv y$

a constrained dynamical system with the two second class constraints $\Omega_1,\ \Omega_2$ and the light-front Hamiltonian

$$H^{l.f.} = \frac{1}{2}\int dx\ (\pi^- + 2e\varphi)^2 + \int dx\ u_1\Omega_1 + \int dx\ u_2\Omega_2 \tag{C. 12}$$

which will be now handled by the BFT procedure.

We introduce the following linear combinations $\top_i$, $i = 1, 2$, of the above constraints

$$\begin{aligned} \top_1 &= c_1(\Omega_1 + \frac{1}{M}\Omega_2) \\ \top_2 &= c_2(\Omega_1 - \frac{1}{M}\Omega_2) \end{aligned} \tag{C. 13}$$

where $c_1 = 1/\sqrt{2(1-e/M)}$, $c_2 = 1/\sqrt{2(1+e/M)}$, $M^2 = ae^2$, and $a > 1$. They satisfy

$$\{\top_i, \top_j\} = \delta_{ij}(-2\partial_x\delta(x-y)) \tag{C. 14}$$

and thus diagonalize the constraint algebra.

We now introduce new auxiliary fields Φ^i in order to convert the second class constraint $\top_i$ into first class ones in the extended phase space. Following BFT [60] we require these fields to satisfy

$$\begin{aligned} \{A^\mu(\text{or } \pi_\mu), \Phi^i\} &= 0, \quad \{\varphi(\text{or } \pi_\varphi), \Phi^i\} = 0, \\ \{\Phi^i(x), \Phi^j(y)\} &= \omega^{ij}(x,y) = -\omega^{ji}(y,x), \end{aligned} \tag{C. 15}$$

where ω^{ij} is a constant and antisymmetric matrix. The strongly involutive modified constraints $\widetilde{\top}_i$ satisfying the abelian algebra

$$\{\widetilde{\top}_i, \widetilde{\top}_j\} = 0 \tag{C. 16}$$

as well as the boundary conditions, $\widetilde{\top}_i\ |_{\Phi^i=0} = \top_i$ are then postulated to take the form of the following expansion

$$\widetilde{\top}_i(A^\mu, \pi_\mu, \varphi, \pi_\varphi; \Phi^j) = \top_i + \sum_{n=1}^{\infty} \widetilde{\top}_i^{(n)}, \qquad \top_i^{(n)} \sim (\Phi^j)^n. \tag{C. 17}$$

The first order correction terms in this infinite series are written as

$$\widetilde{\top}_i^{(1)}(x) = \int dy X_{ij}(x,y)\Phi^j(y). \tag{C. 18}$$

The first class constraint algebra of $\widetilde{\top}_i$ then leads to the following condition:

$$\{\top_i, \top_j\} + \{\widetilde{\top}_i^{(1)}, \widetilde{\top}_i^{(1)}\} = 0 \tag{C. 19}$$

or

$$(-2\partial_x\delta(x-y))\delta_{ij} + \int dw\ dz\ X_{ik}(x,w)\omega^{kl}(w,z)X_{jl}(y,z) = 0. \tag{C. 20}$$

There is clearly some arbritrariness in the appropriate choice of ω^{ij} and X_{ij} which corresponds to the canonical transformation in the extended phase space. We can take without any loss of generality the simple solutions,

$$\begin{aligned} \omega^{ij}(x,y) &= -\delta^{ij}\epsilon(x-y) \\ X_{ij}(x,y) &= \delta_{ij}\partial_x\delta(x-y), \end{aligned} \tag{C. 21}$$

Their inverses are easily shown to be

$$\begin{aligned} {\omega^{-1}}_{ij}(x,y) &= -\frac{1}{2}\delta_{ij}\partial_x\delta(x-y) \\ (X^{-1})^{ij}(x,y) &= \frac{1}{2}\delta^{ij}\epsilon(x-y), \end{aligned} \tag{C. 22}$$

With the above choice, we find up to the first order

$$\begin{aligned} \widetilde{\top}_i &= \top_i + \widetilde{\top}_i^{(1)} \\ &= \top_i + \partial\Phi^i, \end{aligned} \tag{C. 23}$$

and a strongly first class constraint algebra

$$\{\top_i + \widetilde{\top}_i^{(1)}, \top_j + \widetilde{\top}_j^{(1)}\} = 0. \tag{C. 24}$$

The higher order correction terms (suppressing the integration operation)

$$\widetilde{\top}_i^{(n+1)} = -\frac{1}{n+2}\Phi^l{\omega^{-1}}_{lk}(X^{-1})^{kj}B_{ji}^{(n)} \qquad (n \geq 1) \tag{C. 25}$$

with

$$B_{ji}^{(n)} \equiv \sum_{m=0}^{n}\{\widetilde{\top}_j^{(n-m)}, \widetilde{\top}_i^{(m)}\}_{(A,\pi,\varphi,\pi_\varphi)} + \sum_{m=0}^{n-2}\{\widetilde{\top}_j^{(n-m)}, \widetilde{\top}_i^{(m+2)}\}_{(\Phi)} \tag{C. 26}$$

automatically vanish as a consequence of the proper choice of ω^{ij} made above. The Poisson brackets are to be computed here using the standard canonical definition for A_μ and φ as postulated above. We have now only the first class constraints in the extended phase space and in view of the proper choice only $\widetilde{\top}_i^{(1)}$ contributes in the infinite series above.

(b)- First Class Hamiltonian and Dirac Brackets

We next introduce modified ("gauge invariant") dynamical variables $\widetilde{F} \equiv (\widetilde{A}_\mu, \widetilde{\pi}^\mu, \widetilde{\varphi}, \widetilde{\pi}_\varphi)$ corresponding to $F \equiv (A_\mu, \pi^\mu, \varphi, \pi_\varphi)$ over the phase space by requiring the the following strong involution condition for $\widetilde{F}$ with the first class constraints in our extended phase space, viz,

$$\{\widetilde{\top}_i, \widetilde{F}\} = 0 \tag{C. 27}$$

with

$$\widetilde{F}(A_\mu, \pi^\mu, \varphi, \pi_\varphi; \Phi^j) = F + \sum_{n=1}^{\infty}\widetilde{F}^{(n)}, \qquad \widetilde{F}^{(n)} \sim (\Phi^j)^n \tag{C. 28}$$

and which satisfy the boundary conditions, $\widetilde{F}\mid_{\Phi^i=0}= F$.

The first order correction terms are easily shown to be given by

$$\widetilde{F}^{(1)}(x) = -\int\, du\, dv\, dz\ \Phi^j(u)\omega^{-1}{}_{jk}(u,v)X^{-1}{}^{kl}(v,z)\ \{\top_l(z), F(x)\}_{(A,\pi,\varphi,\pi_\varphi)}. \qquad \text{(C. 29)}$$

We find

$$\begin{aligned}
\widetilde{A}_-^{(1)} &= \frac{1}{2M}\partial(c_1\Phi^1 - c_2\Phi^2) \\
\widetilde{\pi}^{-(1)} &= \frac{M}{2}(c_1\Phi^1 - c_2\Phi^2) \\
\widetilde{\varphi}^{(1)} &= -\frac{1}{2}(c_1\Phi^1 + c_2\Phi^2), \\
\widetilde{\pi}_\varphi^{(1)} &= \frac{1}{2}\partial\left[c_1(1-\frac{2e}{M})\Phi^1 + c_2(1+\frac{2e}{M})\Phi^2\right]
\end{aligned} \qquad \text{(C. 30)}$$

where only the combinations $(c_1\Phi^1 \pm c_2\Phi^2)$ of the auxiliary fields are seen to occur. Furthermore, since the modified variables $\widetilde{F} = F + \widetilde{F}^{(1)} + ...$, up to the first order corrections, are found to be strongly involutive as a consequence of the proper choice made above, the higher order correction terms

$$\widetilde{F}^{(n+1)} = -\frac{1}{n+1}\Phi^j\omega_{jk}X^{kl}G_l^{(n)}, \qquad \text{(C. 31)}$$

with

$$G_l^{(n)} = \sum_{m=0}^{n}\{\top_i^{(n-m)}, \widetilde{F}^{(m)}\}_{(A,\pi,\phi,\pi_\phi)} + \sum_{m=0}^{n-2}\{\top_i^{(n-m)}, \widetilde{F}^{(m+2)}\}_{(\Phi)} + \{\top_i^{(n+1)}, \widetilde{F}^{(1)}\}_{(\Phi)} \qquad \text{(C. 32)}$$

again vanish. In principle we may follow similar procedure for any functional of the phase space variables; it may get, however, involved.

We make a side remark on the Dirac formulation for dealing with the systems with second class constraints by using the Dirac bracket (DB), rather than extending the phase space. In fact, the Poisson brackets of the modified (gauge invariant) variables $\widetilde{F}$ in the BFT formalism are related [60] to the DB, which implement the constraints $\top_i \approx 0$ in the problem under discussion, by the relation $\{f,g\}_D = \{\tilde{f},\tilde{g}\}\mid_{\Phi^i=0}$. In view of only the linear first order correction in CSM the computation of the right hand side is quite simple. We list some of the Dirac brackets

$$\begin{aligned}
\{\pi^-,\pi^-\}_D &= \{\widetilde{\pi^-},\widetilde{\pi^-}\}|_{\Phi=0} \\
&= \{\widetilde{\pi^-}^{(1)},\widetilde{\pi^-}^{(1)}\} = \frac{a^2e^2}{(a-1)}(-\frac{1}{4}\epsilon(x-y)), \\
\{\varphi,\varphi\}_D &= \{\widetilde{\varphi},\widetilde{\varphi}\}|_{\Phi=0} \\
&= \{\widetilde{\varphi}^{(1)},\widetilde{\varphi}^{(1)}\} = \frac{a}{(a-1)}(-\frac{1}{4}\epsilon(x-y)) \\
\{\varphi,\pi^-\}_D &= \{\widetilde{\varphi}^{(1)},\widetilde{\pi^-}^{(1)}\} = \frac{ae}{(a-1)}(-\frac{1}{4}\epsilon(x-y)).
\end{aligned} \qquad \text{(C. 33)}$$

The other ones follow on using the now strong relations $\Omega_1 = \Omega_2 = 0$ with respect to $\{,\}_D$ and from $H^{l.f}$ it follows that the LF Hamiltonian reduces effectively to

$$H_D^{l.f.} = \frac{1}{2}\int dx\,(\pi^- + 2e\varphi)^2. \tag{C. 34}$$

The first class LF Hamiltonian $\widetilde{H}$ which satisfies the boundary condition $\widetilde{H}\mid_{\Phi^i=0} = H_D^{l.f.}$ and is in strong involution with the constraints $\widetilde{\top}_i$, e.g., $\{\widetilde{\top}_i, \widetilde{H}\} = 0$, may be constructed following the BT procedure or simply guessed for the CSM. It is given by

$$\widetilde{H} = \frac{1}{2}\int dx(\widetilde{\pi}^- + 2e\widetilde{\varphi})^2 \tag{C. 35}$$

which is just the expression in of $H_D^{l.f.}$ with field variables F replaced by the $\widetilde{F}$ variables, which already commute with the constraints $\widetilde{T}_i$. We do also check that $\{\widetilde{H}, \widetilde{H}\} = 0$ and we may identify $\widetilde{H}$ with the BRS Hamiltonian. This completes the operatorial conversion of the original second class system with the Hamiltonian H_c and constraints Ω_i into the first class one with the Hamiltonian $\widetilde{H}$ and (abelian) constraints $\widetilde{T}_i$.

(c)- First Class Lagrangian

We consider now the partition function of the model in order to construct the Lagrangian corresonding to $\widetilde{H}$ in the canonical Hamiltonian formulation discussed above.

We start by representing each of the auxiliary field Φ^i by a pair of fields $\pi^i, \theta^i,\ i = 1, 2$ defined by

$$\Phi^i = \frac{1}{2}\pi^i - \int du\ \ \epsilon(x-u)\,\theta^i(u) \tag{C. 36}$$

such that π^i, θ^i satisfy

$$\{\pi^i, \theta^j\} = -\delta^{ij}\delta(x-y) \quad etc., \tag{C. 37}$$

e.g., the (standard Heisenberg type) canonical Poisson brackets.

Then, The Phase Space Partition Function Is Given By the Faddeev formulae

$$Z = \int \mathcal{D}A_-\mathcal{D}\pi^-\mathcal{D}\varphi\mathcal{D}\pi_\varphi\mathcal{D}\theta^1\mathcal{D}\pi^1\mathcal{D}\theta^2\mathcal{D}\pi^2\prod_{i,j=1}^{2}\delta(\widetilde{\top}_i)\delta(\Gamma_j)\mathrm{det}\mid\{\widetilde{\top}_i, \Gamma_j\}\mid e^{iS}, \tag{C. 38}$$

where

$$S = \int d^2x\left(\pi^-\dot{A}_- + \pi_\varphi\dot{\varphi} + \pi^1\dot{\theta}^1 + \pi^2\dot{\theta}^2 - \widetilde{\mathcal{H}}\right) \equiv \int d^2x\,\mathcal{L}, \tag{C. 39}$$

with the Hamiltonian density $\widetilde{\mathcal{H}}$ corresponding to the Hamiltonian $\widetilde{H}$ which is now expressed in terms of (θ^i, π_i) rather than in terms of Φ^i. The gauge-fixing conditions Γ_i are chosen such that the determinants occurring in the functional measure are nonvanishing. Moreover, Γ_i may be taken to be independent of the momenta so that they correspond to the Faddeev-Popov type gauge conditions.

We will now verify in the *unitary gauge*, defined by the original second class constraints: $\Gamma_i \equiv \Omega_i = 0$, i=1,2 being employed in the partition function, do in fact lead to the original Lagrangian. We check that the determinants in the functional measure are non-vanishing and field independent while the product of delta functionals reduces to

$$\delta(\pi_\varphi - \varphi')\delta(\pi^{-\prime} + 2e\varphi' + M^2A_-)\delta(\pi^{1\prime} - 4\theta^1)\delta(\pi^{2\prime} - 4\theta^2) \tag{C. 40}$$

Since π_φ is absent from $\widetilde{H}$ we can perform functional integration over it using the first delta functional. The second delta functional is exponentiated as usual and we name the integration variable as A_+ for convenience. The functional integral over θ^1 and θ^2 are easily performed due to the presence of the delta functionals and it also reduces $\widetilde{\mathcal{H}}$ to $(\pi^- + 2e\varphi)^2/2$. The functional integrations over the then decoupled variables π^1 and π^2 give rise to constant factors which are absorbed in the normalization. The partition function in the unitary gauge thus becomes

$$Z = \int \mathcal{D}A_- \mathcal{D}\pi^- \mathcal{D}\varphi \mathcal{D}A_+ e^{iS}, \tag{C. 41}$$

with

$$S = \int d^2x \left[\pi^- \dot{A}_- + \varphi' \dot{\varphi} + (\pi^{-\prime} + 2e\varphi' + M^2 A_-)A_+ - \frac{1}{2}(\pi^- + 2e\varphi)^2\right], \tag{C. 42}$$

Performing the shift $\pi^- \to \pi^- - 2e\varphi$ and doing subsequently a Gaussian integral over π^- we obtain the original bosonized Lagrangian with ω eliminated by the field redefinition of A_+. It is interesting to recall that while constructing the LF Hamiltonian framework we eliminated the variable A_+ making use of the gauge freedom on the LF phase space and it gave rise to appreciable simplification. However, on going over to the first class Lagrangian formalism using the partition functional this variable reappears as it should, since the initial bosonized action is not gauge invariant due to the presence of the mass term for the gauge field. Making other acceptable choices for gauge-functions we can arrive at different effective Lagrangians for the system under consideration. It is interesting to recall that in the fermionic Lagrangian the right-handed component of the fermionic field describes a free field and only the left-handed one is gauged. It is also clear from our discussion that $\widetilde{H}$ proposed above is not unique and we could modify it so that it still leads to the original Lagrangian in the unitary gauge. The corresponding first class Lagrangian would produce still other gauge-fixed effective Lagrangians.

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[51] Here it is understood that an unsymmetrical expression like $\bar\psi\gamma^\mu\partial_\mu\psi$ is to be replaced by its symmetrized form $\left[\bar\psi\gamma^\mu\partial_\mu\psi - \partial_\mu\bar\psi\gamma^\mu\psi\right]/2$. It is convenient to work on the LF in terms of the projected four spinors $\psi_\pm$.

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