SLAC-PUB-7431
SU-ITP-97-09
UTTG-10-97
March 1997
hep-th/yymmnnn

The Incredible Shrinking Torus[⋆]

W. Fischler[1 †], E. Halyo, A. Rajaraman[3 ‡] and L. Susskind[2 §]

[1] *Theory Group, Department of Physics, University of Texas, Austin, TX 78712*

[2] *Department of Physics, Stanford University Stanford, CA 94305-4060*

[3] *Stanford Linear Accelerator Center, Stanford University Stanford, CA 94305*

Using M(atrix) Theory, the dualities of toroidally compactified M-theory can be formulated as properties of super Yang Mills theories in various dimensions. We consider the cases of compactification on one, two, three, four and five dimensional tori. The dualities required by string theory lead to conjectures of remarkable symmetries and relations between field theories as well as extremely unusual dynamical properties. By studying the theories in the limit of vanishingly small tori, a wealth of information is obtained about strongly coupled fixed points of super Yang-Mills theory in various dimensions. Perhaps the most striking behavior, as noted by Rozali in this context, is the emergence of an additional dimension of space in the case of a four torus.

⋆ Submitted to Nuclear Physics B.
† Supported in part by NSF Grant PHY-9511632.
‡ Supported in part by the Department of Energy under contract no. DE-AC03-76SF00515.
§ Supported by NSF grant PHY-9219345.

1. Introduction

M(atrix) theory [1] is a nonperturbative formulation of string theory in terms of certain supersymmetric Yang-Mills theories. The known properties of the field theories can often illuminate the behaviour of string and M theory. For example the non-renormalization theorems of super Yang-Mills theories are essential to an understanding of causality in space time and the existence of asymptotic states. Another example is the use of electric-magnetic duality of four dimensional super Yang-Mills theory to establish T-duality non-perturbatively in the case of the three torus [2,3,4,5]. The other side of the coin is the use of known or expected properties of M- and string theory to formulate conjectures about field theory. In this paper we will be interested in the behavior of super Yang-Mills theories, with 16 real supersymmetries, that can be derived by considering M(atrix) theory compactifications on small tori. By small we will mean scales typically of order or smaller than the 11 dimensional Planck scale. In the following sections we will consider the cases of tori of dimension 1, 2, 3, 4, 5 and 6.

We will see that by studying the limit of vanishing size tori, we will be driven to the limit of strongly coupled super Yang-Mills theory. This limit is intrinsically interesting for the field theories, especially when the space-time dimensionality of the field theory is not 4. In these cases the theories are either super-renormalizable or non-renormalizable. In the former case the theory becomes strongly coupled in the infrared while in the latter the strong coupling occurs in the ultraviolet. In each case the difficult region is described by a strongly coupled fixed point which can have very nontrivial and often surprising features.

Let us consider a D dimensional super Yang-Mills theory compactified on a $D-1$ torus with dimensions all of order Σ. The Yang-Mills coupling constant, g^2 has dimension m^{4-D}. The coupling constant is unrenormalized due to the high degree of supersymmetry. However that does not mean that the strength of interaction does not run with mass scale. If we introduce a scale μ, a dimensionless running coupling constant $\tilde{g}^2$ may be defined.

$$\tilde{g}^2(\mu) = g^2 \mu^{D-4} \tag{1.1}$$

which measures the strength of interaction at length scale μ^{-1}. In particular, the coupling strength at scales of order the torus size Σ is defined to be $\tilde{g}^2$

$$\tilde{g}^2 = g^2 \Sigma^{4-D} \tag{1.2}$$

The behavior of the theory as the length scale varies can be studied by fixing g and letting the size of the torus vary. Thus we study the ultraviolet (infrared) properties of the theory by decreasing (increasing) Σ. Alternately we may keep Σ fixed and vary g. For $D < 4$, going to strong dimensionless coupling probes the infrared behavior of super Yang-Mills theory. On the other hand, for $D > 4$, strong coupling probes the ultraviolet behavior. We shall see that in both cases we obtain information by studying M(atrix) theory on vanishingly small tori.

Before stating our results we define some conventions. M(atrix) theory is defined by identifying the large N limit of $U(N)$ super Yang-Mills theory with the infinite momentum limit in which the 11th direction is chosen as the longitudinal direction. The theory is assumed to be compactified in this direction on a circle of size R. Physical quantities all become independent of R in the large N limit. We will consider compactifying some subset of the transverse dimensions on a d-torus. This d-torus never includes the 11th direction which may be assumed noncompact in the large N limit.

The various length scales that we encounter are the following. The 11 dimensional Planck scale of M-theory is defined to be l_{11}, the $d = D - 1$ dimensional torus that M-theory is compactified on has sides $L_1, L_2, ...L_d$. Each length is assumed to be of the same order of magnitude L. The super Yang-Mills theory is also compactified on a dual torus whose dimensions are defined to be $\Sigma_1, \Sigma_2, ...\Sigma_d$. Each of the Σ_i is of the same order of magnitude which is however, in general very different than the original length L_i. As mentioned above, the Yang-Mills coupling is g and the dimensionless coupling is $\tilde{g}^2$.

A few words about transversely compactified M(atrix) Theory will help eliminate later confusion. In the original discussion of [1] the starting point was the theory of $D0$-branes in type IIA theory when the compact direction is identified as the "longitudinal" direction. These objects became the carriers of longitudinal momentum (p_{11}) in an infinite momentum or light cone description. The number of these $D0$-branes is eventually allowed to go to infinity. In order to distinguish these carriers of p_{11} from other $D0$-branes which will emerge when we compactify transverse dimensions, we will introduce the notation $\tilde{D}0$-brane for the carriers of longitudinal momentum. Thus one would say that the infinite momentum limit is obtained by letting the number of $\tilde{D}0$-branes tend to infinity.

The results of our analysis are as follows.

1. For the case of the 1-torus, shrinking yields a theory which is equivalent, through the M(atrix) connection, to uncompactified perturbative type IIA string theory.

   a) The spectrum of the theory is in one to one correspondence with the spectrum of perturbative string states. The limit $L \to 0$ is described by a stongly coupled super Yang-Mills theory [2] with dimensionless coupling $\tilde{g}^2 = (2\pi)^4 \frac{l_{11}^3}{L^3}$. On the string side it describes IIA string theory with string coupling $g_s^2 = \frac{2\pi}{\tilde{g}^2}$. Thus the limit $L \to 0$ is free string theory. The super Yang-Mills theory has a strong coupling expansion which is just the usual string loop expansion.

   b) The $U(N)$ super Yang-Mills theory has a normalizable ground state with unbroken $U(N)$ symmetry and an excited state spectrum with a mass gap equal to $\frac{2\pi}{N\Sigma}$. The theory also has a variety of Higgsed vacua with symmetry broken to products of smaller $U(N_i)$ factors. The normalizability of the ground state has implications for the behavior of D1-branes in IIB theory. It requires the existence of normalizable bound states for a collection of N D1-branes.

   c) We identify the states of the super Yang-Mills theory which describe the $D0$-branes of the transverse compactification.

2. For compactification on a 2-torus, the vanishing torus limit is 10 dimensional type IIB theory [11]. More generally, compactification on a 2-torus leads to type IIB string theory with 1 compact space dimension. The infinite momentum description has a manifest $O(7)$ rotation invariance. The coupling constant is of order 1 unless the sides of the torus are very different in which case it becomes weakly coupled. Shrinking the torus decompactifies the compact dimension and leads to an enhanced rotational symmetry $O(8)$. This implies that the 2+1 dimensional super Yang-Mills theory also develops the enhanced $O(8)$ symmetry at strong coupling. Since in this case, it is the infrared behavior which is governed by strong coupling, the implication is that the I.R. behavior is $O(8)$ invariant [3,9].

3. The shrunken 3-torus limit leads back to decompactified M-theory. This can be seen from T-duality. On the super Yang-Mills theory side, the symmetry under T-duality becomes symmetry under inverting the coupling constant $g \to \frac{2\pi}{g}$. This is just the Montonen Olive electric-magnetic duality of four dimensional super Yang-Mills theory. Had it not already been known, this would have been the prediction following from

compactification on a 3-torus. In this case, the theory is scale invariant.

4. The four torus is perhaps the most interesting example of all. It is described by 4+1 dimensional super Yang-Mills theory [6].

   a) Shrinking the 4-torus to zero leads to weakly coupled string theory in 7 noncompact space-time dimensions. The coupling vanishes in the limit and the compactification radii are all of string scale.

   b) The U-duality group of the theory is $SL(5;Z)$. This is the symmetry group of the 5-torus. Following Rozali [13], we find that the super Yang-Mills theory develops a new 5th spatial direction whose radius grows as the M-theory torus shrinks. The duality group is a reflection of the 5-dimensional behavior.

   b) The origin of the new direction is light states which come down as the Yang-Mills coupling increases. These light states carry instanton charge which may be identified as the momentum in the 5th spatial direction.

   c) In this case strong coupling behavior governs the ultraviolet behavior of the theory. This means that the short distance behavior of 4+1 dimensional super Yang-Mills theory is controlled by a fixed point describing a five dimensional field theory with one very large dimension!

5. Compactification on the 5-torus [7] exhibits a phenomenon which in a sense is the reverse of what happens in the 4-torus case. The strongly coupled limit of this theory is described by a 4+1 dimensional system, namely super Yang-Mills theory with a coupling of order unity. On the M-theory side the limiting theory can be dualized to M-theory on a 4-torus whose size is of order the 11 dimensional Planck scale. It can also be described as IIA string theory at intermediate coupling, compactified on a torus of string scale size.

6. For d equal to or greater than 6 we find that we are unable to dualize the theory into any form in which it can be reliably analyzed. Any attempt to dualize it to weak or intermediate coupling produces ultra-small compactification radii. Applying T-duality to increase the compactification radii inevitably leads to ultra-large coupling. From the field-theoretic side this may be connected with the lack of superconformal fixed points in space-time dimension greater than 6. What it means on the string or M-theoretic side is unclear. Perhaps it signals some sort of non-perturbative breakdown

of toroidal compactification. Resolving this point could have important implications for 4-dimensions.

2. The 1-torus

Compactification of M(atrix) theory on a circle of circumference L leads to a $1+1$ dimensional $U(N)$ super Yang-Mills theory containing a vector field, eight scalars and their fermionic partners [2]. We will take the compact transverse direction of the M(atrix) theory to be the 9th direction. The Lagrangian is

$$L = \frac{1}{g^2} \int^{\Sigma} d\sigma Tr \left(-\frac{1}{4} F_{\mu\nu}^2 + D_\mu \phi_i^2 + [\phi_i, \phi_j]^2 + fermions \right) \tag{2.1}$$

where ϕ_i represents the 8 adjoint scalars, D_μ is the covariant derivative, Σ is the circumference of the circle parametrized by σ.

To determine the relation between L and the field theoretic parameters g^2, Σ we use the method of [4] which entails comparing certain energy scales. Suppose that the $U(N)$ symmetry is broken to $U(N-1) \times U(1)$ with the $U(1)$ factor describing the motion of a single $\tilde{D}0$-brane. In the temporal gauge $A_0 = 0$ the Lagrangian for the homogeneous mode of the $U(1)$ gauge field is

$$\frac{1}{2g^2} \Sigma \dot{A}_1^2 \tag{2.2}$$

Furthermore $A\Sigma$ is an angle whose momentum is quantized in integers. Thus the energy stored in the electic field has the form $\frac{g^2}{2} \Sigma n^2$ with n being an integer. This energy is to be identified with the kinetic energy of a $\tilde{D}0$-brane which has transverse momentum n in the 9th direction. This energy is of the form $\frac{{p_9}^2}{2p_{11}}$ which is equal to $\frac{(2\pi)^2 n^2 R}{2L^2}$. Thus we find

$$g^2 \Sigma = \frac{(2\pi)^2 R}{L^2} \tag{2.3}$$

Next we equate the energy of a membrane wrapped around R and L to the energy of a single quantum in the field theory. The membrane energy is $\frac{RL}{2\pi l_{11}^3}$ and the energy of the quantum is $\frac{2\pi}{\Sigma}$. This gives

$$\frac{2\pi}{\Sigma} = \frac{RL}{2\pi l_{11}^3} \tag{2.4}$$

The parameter Σ is the size of the circle upon which the super Yang-Mills field theory is compactified. We will refer to it as the field theory compactification radius. From (2.4) we see that Σ becomes large when L becomes small.

Equations (2.3) and (2.4) determine the parameters Σ and g^2. From these a single dimensionless parameter $\tilde{g}^2$ can be obtained

$$\tilde{g}^2 = g^2\Sigma^2 = (2\pi)^4 \frac{l_{11}^3}{L^3} \tag{2.5}$$

The parameter $\tilde{g}^2$ is a measure of the strength of interaction at length scales of order Σ, that is the largest scale in the theory. In what follows we will be especially interested in the limit in which L is much smaller than l_{11}. This is the limit in which the dimensionless coupling $\tilde{g}^2$ is very large. We know very little about this theory but if we believe the M(atrix) theory connection then we can determine a great deal by connecting it with weakly coupled type IIA string theory. Before we do so it is worth pointing out that this same field theory governs the behavior of a collection of N D1-branes wound around a large cycle in type IIB string theory.

The connection with type IIA theory comes from the fact that M-Theory compactified on a circle of circumference L becomes weakly coupled IIA string theory as L tends to zero. The precise connection is that the string length and string coupling g_s are given by

$$\frac{l_s^2}{2} = \alpha' = \frac{2\pi l_{11}^3}{L} \tag{2.6}$$

and

$$g_s^2 = \frac{1}{(2\pi)^3} \frac{L^3}{l_{11}^3} \tag{2.7}$$

Evidently the dimensionless Yang-Mills coupling $\tilde{g}^2$ and the IIA string coupling are inversely related to one another.

$$g_s^2 = \frac{2\pi}{\tilde{g}^2} \tag{2.8}$$

Thus the limit of strongly coupled super Yang-Mills theory is equivalent to free string theory. As an example, consider the properties of the ground state. It is not known whether the field theory has a normalizable ground state or if the ground state wave function spreads out along the flat directions of the moduli space. To answer this we note that the ground state of string theory in the infinite momentum frame with longitudinal momentum R/N is the graviton whose energy is $\frac{Rp^2}{2N}$ where p is the transverse momentum in the uncompactified directions. For $p=0$ the energy is minimized. This state, being an eigenstate of transverse momentum is of course not normalizable but this center of mass factor is associated with the $U(1)$ factor of the gauge group. This can be trivially factored off so that we are really considering the normalizability of the $SU(N)$ theory. Now apart from its center of mass motion a single graviton should be a normalizable state. Thus it follows that the $SU(N)$ 1+1 dimensional super Yang-Mills theory on a circle has a normalizable ground state. This fact has implications for the theory of D1-branes in type IIB theory. It says that a collection of N D1-branes wound around a compact cycle has a single normalizable bound state. Note that this result is in direct conflict with claims in the literature .

Free string theory has superselection sectors corresponding to states with any number of disconnected strings. The corresponding phenomenon in super Yang-Mills theory is the Higgs phenomenon in which $U(N)$ breaks down into $U(N_1)\times U(N_2)\times U(N_3)$... [8,9,10] The individual factors describe the separated strings. Let us concentrate on the unbroken sector of the super Yang-Mills theory and consider the excited state spectrum. This corresponds to the spectrum of excited free strings. According to free string theory in the infinite momentum frame the energy of the first excitation is

$$E=\frac{M^2}{2P_{11}}=M^2\frac{R}{2N} \tag{2.9}$$

where $M^2=\frac{2}{\alpha'}$. This translates to an energy gap in the super Yang-Mills theory.

$$E=\frac{2\pi}{N\Sigma} \tag{2.10}$$

This gap is surprisingly small. The natural gap for a massless field theory defined on a circle of size Σ would be $\frac{2\pi}{\Sigma}$. The gap in (2.10) is N times smaller.This fact has also been noted by Motl and by Banks and Seiberg and is believed to be related to the phenomenon reported in [14].

The string theory connection indicates that there should exist a strong coupling expansion for the field theory. Since the super Yang-Mills coupling is inverse to the string coupling, the field theoretic strong coupling expansion is just the familiar genus expansion of 10 dimensional IIA theory.

Consider next the states which are excited by having a single unit of momentum along the compactified 9th direction. The momentum in the compactified direction is proportional to the $U(1)$ electric field (Note that in this case we are not discussing the $U(1)$ factor which occurs when $U(N) \to U(N-1) \times U(1)$ but rather the $U(1)$ which commutes with $SU(N)$). A single quantum of electric field carries energy

$$E_E = \frac{g^2 \Sigma}{2N} = \frac{R}{2N} \left(\frac{2\pi}{L} \right)^2 \tag{2.11}$$

This field theoretic energy is interpreted as an energy in the infinite momentum frame of M-theory. It corresponds to a mass

$$M_E = \frac{2\pi}{L} = \frac{1}{g_s \sqrt{\alpha'}} \tag{2.12}$$

This is exactly the mass of a $D0$-brane of the weakly coupled type IIA string theory. We emphasize that this object is not the original $\tilde{D}0$-brane that carries longitudinal momentum and whose number goes to infinity in the infinite momentum limit. These new objects are the massive $D0$-branes of the transversely compactified theory whose existence is required by weakly coupled string theory. Their presence in the spectrum of the super Yang-Mills theory constitutes a consistency check for M(atrix) theory. It would be very interesting to study the sector of the theory with n units of electric energy corresponding to $nD0$-branes. Using purely field theoretic methods it should be possible to discover that this sector is described by $U(n)$ super Yang-Mills theory in 0+1 dimensions.

3. The 2-torus

Next consider M(atrix) Theory compactified on a small 2-torus. The length of the sides of the torus are L_1, L_2. The appropriate field theory is 2+1 dimensional super Yang-Mills theory on a 2-torus with sides Σ_1, Σ_2 and coupling constant g. Following the procedure in the previous section [4] we find that these parameters are connected by the following equations.

$$g^2 = (2\pi)^2 \frac{R}{L_1 L_2} \tag{3.1}$$

$$\Sigma_i = (2\pi)^2 \frac{l_{11}^3}{L_i R} \tag{3.2}$$

where $i = 1, 2$. A dimensionless coupling constant $\tilde{g}^2$ can also be defined by multiplying g by the appropriate power of the area $\Sigma_1 \Sigma_2$.

$$\tilde{g}^2 = g^2 (\Sigma_1 \Sigma_2)^{\frac{1}{2}} = (2\pi)^4 \frac{l_{11}^3}{(L_1 L_2)^{\frac{3}{2}}} \tag{3.3}$$

Again, when the volume $L_1 L_2$ becomes small, the dimensionless Yang-Mills coupling becomes large. The new feature in this case is the existence of magnetic flux through the $\Sigma_1 \Sigma_2$ torus. The abelian magnetic flux is quantized in integer multiples of 2π. One easily finds that the energy associated with the flux is given in terms of Yang-Mills quantities by

$$E_M = \frac{(2\pi)^2}{2N g^2 \Sigma_1 \Sigma_2} \tag{3.4}$$

It can be reexpressed in terms of the L_i, R and l_{11}.

$$E_M = \frac{R}{2N} \frac{L_1^2 L_2^2}{(2\pi)^4 l_{11}^6} = \frac{1}{2p_{11}} \frac{L_1^2 L_2^2}{(2\pi)^4 l_{11}^6} \tag{3.5}$$

The mass associated with this energy is

$$M_M = \frac{L_1 L_2}{(2\pi)^2 l_{11}^3} \tag{3.6}$$

The expression in eq. (3.6) is precisely the mass of a two brane in M-theory [5], wrapped on the 2-torus. When the 2-torus shrinks to zero size, the magnetic energy gives rise to a dense spectrum of low energy states. In all the higher dimensional examples we will consider, a similar infinity of light states come down as the torus shrinks. The interpretation of these states is one of the main themes of this paper. Surprisingly, the interpretation is quite different in each dimension.

In the present case the interpretation is well known [11,3]. A new dimension, Y, opens up and becomes decompactified as $L_i \to 0$. The quantized flux becomes the Kaluza-Klein momentum in the new direction. To compute the circumference L_y of the new direction we equate the energy in eq. (3.6) to the energy of the first Kaluza-Klein excited state. This gives

$$L_y = (2\pi)^3 \frac{l_{11}^3}{L_1 L_2}$$

or more symmetrically

$$L_y L_1 L_2 = (2\pi)^3 l_{11}^3 \tag{3.7}$$

Let us next consider the energy stored in the $U(1)$ electric field. Following the same logic as in eqs. (2.11) and (2.12) we find that the energy in a single quantum of electric field in the 1 direction is

$$E_E = \frac{g^2}{2N}\frac{\Sigma_1}{\Sigma_2} = \frac{R}{2N}\left(\frac{L_y L_2}{(2\pi)^2 l_{11}^3}\right)^2 \tag{3.8}$$

The mass of the state is

$$M_E = \frac{L_y L_2}{(2\pi)^2 l_{11}^3} \tag{3.9}$$

This mass corresponds to the energy of a membrane wrapped around the directions $(2, Y)$. As L_2 tends to zero, the configuration becomes a string, wound around the Y direction. In fact it is a type IIB string. The coupling constant of the type IIB string theory is given by

$$g_s = \frac{L_2}{L_1} \tag{3.10}$$

Thus we see that the electric field in the 1 direction corresponds to the winding number of type IIB strings. If the ratio $\frac{L_2}{L_1}$ is small these strings are weakly coupled. We can also consider the electric field in the 2 direction which by an identical argument must also correspond to some kind of string. These strings are easily identified as the D-strings of type

IIB theory. The $SL(2;Z)$ symmetry of the IIB theory is just the geometric symmetry of the shrinking two torus. All of these facts lead to the conclusion that in the large N limit, 2+1 dimensional super Yang-Mills theory is equivalent to type IIB string theory.

The consequences of this conclusion for the super Yang-Mills theory include many predictions about the spectrum of the theory. For example the energy gap in the un-Higgsed sector must be of the form

$$E = \frac{1}{2N} \frac{1}{(\Sigma_1 \Sigma_2)^{\frac{1}{2}}} F\left(\frac{\Sigma_1}{\Sigma_2}\right) \tag{3.11}$$

where $F\left(\frac{\Sigma_1}{\Sigma_2}\right)$ is a function symmetric under interchange of $\Sigma_{1,2}$. In the limit in which $\Sigma_2 >> \Sigma_1$ the function F behaves like $\left(\frac{\Sigma_1}{\Sigma_2}\right)^{\frac{1}{2}}$. As in the 1+1 dimensional case, the energy gap is required to be surprisingly small as N increases.

The most interesting prediction about 2+1 super Yang-Mills theory follows from the spatial rotational invariance of string theory. The 2+1 dimensional super Yang-Mills theory has explicit $O(7)$ invariance associated with the 7 scalar fields. This invariance is just the explicit rotational invariance of the uncompactified transverse directions of the original M-Theory. The emergence of the new noncompact direction Y, in the limit of vanishing torus size, requires the symmetry to be enhanced to the group of 8 dimensional rotations. This of course only happens in the strong coupling limit. For the abelian $U(1)$ part of the theory the invariance can be made manifest. In addition to the 7 scalar free fields, the bosonic content of the abelian theory includes the gauge field which in 2+1 dimensions can be dualized to give an 8th free scalar field, completing a vector multiplet of $O(8)$. The fermions also group into a spinor. From the point of view of this paper, the $O(8)$ symmetry of the strongly coupled nonabelian theory will be regarded as a prediction. Since in this case, as in the previous example, strong coupling governs the infrared properties of the super Yang-Mills theory, the correct statement is that the fixed point describing the infrared has $O(8)$ symmetry. Evidence for the correctness of the prediction has been given in [3,9].

4. The 3-torus

Matrix theory compactified on a 3-torus leads to 3+1 dimensional $U(N)$ super Yang-Mills theory with 4 supersymmetries. The compactification lengths are L_1, L_2, L_3 on the M-theory side and $\Sigma_1, \Sigma_2, \Sigma_3$ on the super Yang-Mills theory side. The Yang-Mills coupling constant g^2 in this case is dimensionless. We find,

$$g^2 = \tilde{g}^2 = (2\pi)^4 \frac{l_{11}^3}{L_1 L_2 L_3} \tag{4.1}$$

$$\Sigma_i = (2\pi)^2 \frac{l_{11}^3}{R L_i} \tag{4.2}$$

where $i = 1, 2, 3$. Now let us consider the states which become light as the L_i shrink. The integer quantized magnetic fluxes can be labelled n_{ij}. We find the magnetic energy is given by

$$E_M = (2\pi)^2 \left(\frac{\Sigma_3}{2g^2 \Sigma_1 \Sigma_2} n_{12}^2 + \frac{\Sigma_1}{2g^2 \Sigma_2 \Sigma_3} n_{23}^2 + \frac{\Sigma_2}{2g^2 \Sigma_3 \Sigma_1} n_{31}^2 \right) \tag{4.3}$$

For a single unit of flux in the 1,2 plane the energy is

$$E_M = (2\pi)^2 \frac{\Sigma_3}{2g^2 \Sigma_1 \Sigma_2} = \frac{R}{2N} \frac{L_1^2 L_2^2}{(2\pi)^4 l_{11}^6} \tag{4.4}$$

corresponding to a mass

$$M_M = \frac{L_1 L_2}{(2\pi)^2 l_{11}^3} \tag{4.5}$$

and similarly for the other two directions. We interpret these light states as the Kaluza-Klein excitations of 3 decompactifying directions whose lengths are

$$\bar{L}_3 = (2\pi)^3 \frac{l_{11}^3}{L_1 L_2} \tag{4.6}$$

and so forth.

Evidently as the 3-torus shrinks, 3 new directions open up and restore the theory to an 11 dimensional theory. The only reasonable candidate is M-Theory itself. We will shortly see what the interpretation of this is in the super Yang-Mills theory.

To determine the parameters of the new 11 dimensional theory we consider the electric energy for a field in the 1 direction.

$$E_E = \frac{g^2 \Sigma_1}{2N\Sigma_2\Sigma_3} = \frac{R}{2N}\frac{(2\pi)^2}{L_1^2} \tag{4.7}$$

leading to a mass

$$M_E = \frac{2\pi}{L_1} \tag{4.8}$$

The natural interpretation of this energy in the new M-Theory is a membrane wrapped around the 2,3 face of a growing 3 torus. The area of the membrane is $\bar{L}_2\bar{L}_3$ and its energy is expected to have the form $\frac{\bar{L}_2\bar{L}_3}{(2\pi)^2\bar{l}_{11}^3}$. Using eq. (4.8) we can identify the new 11 dimensional planck length.

$$\bar{l}_{11}^3 = (2\pi)^3 \frac{l_{11}^6}{L_1 L_2 L_3} \tag{4.9}$$

The new M-Theory must also be equivalent to a super Yang-Mills theory whose parameters are given in terms of the barred versions of eqs. (4.1),(4.2). One finds

$$\bar{g} = \frac{2\pi}{g} \tag{4.10}$$

This relation between coupling constants is just the S-duality of four dimensional super Yang-Mills theory which interchanges electric and magnetic fields. In order that the new 11 dimensional theory be exactly M-Theory, the full $U(N)$ theory must have electric-magnetic duality. In this way, if we did not already know of the duality of super Yang-Mills theory, we might be led to predict it.

The full duality group of 3-dimensionally compactified M-theory is now easily understood as the product of the $SL(2;Z)$ electric-magnetic duality and the $SL(3;Z)$ geometric symmetry group of the 3-torus.

Thus far we have made use of the properties of 2- branes but we have not had need for the 5-branes of M-Theory. Later we will find them indispensible. However this is a good place to discuss them. The 5-branes have proved much more elusive than the 2-branes which are found as classical configurations of M(atrix) theory [12]. Following [5] we can gain some understanding of 5-brane states by considering a 5-brane wrapped around the 3-torus. The other 2 dimensions of the brane form an unwrapped infinitely extended 2-brane. This object together with an ordinary unwrapped 2-brane form a doublet under the $SL(2;Z)$ duality. This can be seen by observing that under T-duality the two configurations are interchanged. Since the T-duality operation is represented by the electric-magnetic S-duality of super Yang-Mills theory we find that to construct the 5-brane it is necessary to understand the action of S-duality on the fields of the super Yang-Mills theory. Since electric-magnetic duality is not a symmetry of classical field theory, we do not expect the 5-brane to be represented by a classical configuration. In fact it probably can not be represented locally in terms ot the fields of the original theory. We will however assume that it exists as a quantum state of super Yang-Mills theory. A deeper understanding of the 5-brane will probably await a more concrete operator construction of electric-magnetic duality.

5. The 4 Torus

Up to this point we have been mainly reviewing some well known connections. In this section we will see some surprising departures from the pattern of the previous sections. What we have seen up to now is that light states associated with magnetic flux are interpreted in terms of new dimensions of space. Thus if we compactify K dimensions and let the torus shrink, the pattern has been that we regain $K(K-1)/2$ large dimensions and the theory becomes $11 + \frac{K(K-3)}{2}$ dimensional. If this pattern were to persist, compactification on a shrinking 4-torus would lead to a 13 dimensional theory! This would be quite surprising since there are no known formulations of supergravity above 11 dimensions. We shall see in what follows that the interpretation of the light states is entirely different for tori of dimension 4 or higher. The analysis that we will present of this case is based on an important observation of Rozali [13]. Rozali observed that the group of dualities for 4 compact toroidal dimensions is $SL(5;Z)$, which also happens to be the group of symmetries of a 5-torus. Rozali suggested that the 4+1 dimensional super Yang-Mills theory is actually equivalent to a 5+1 dimensional theory of some kind which lives on a 5-torus. We will see substantial evidence for this in what follows.

The connection between M-theoretic and field theoretic quantities is

$$\Sigma_i = (2\pi)^2 \frac{l_{11}^3}{RL_i} \tag{5.1}$$

$$\tilde{g}^2 = \frac{g^2}{(\Sigma_1\Sigma_2\Sigma_3\Sigma_4)^{\frac{1}{4}}} = (2\pi)^4 \frac{l_{11}^3}{(L_1L_2L_3L_4)^{\frac{3}{4}}} \tag{5.2}$$

The magnetic and electric energies are

$$E_M = \frac{R}{2N}\frac{L_i^2L_j^2}{(2\pi)^4 l_{11}^6} \tag{5.3}$$

$$E_E = \frac{R}{2N}\frac{(2\pi)^2}{L_i^2} \tag{5.4}$$

and the corresponding masses are

$$M_M = \frac{L_iL_j}{(2\pi)^2 l_{11}^3} \tag{5.5}$$

$$M_E = \frac{2\pi}{L_i} \tag{5.6}$$

In addition to the electric and magnetic fluxes, a new quantum number appears in the 4+1 super Yang-Mills theory that did not occur in the lower dimensional examples. Ordinary Yang-Mills instanton solutions occur as static solitons. The new quantum number is just the integer valued instanton charge Q,

$$Q = \frac{1}{32\pi^2}\int d^4\sigma F \wedge F \tag{5.7}$$

The energy associated with instanton number Q is

$$E_I = \frac{(2\pi)^2 Q}{g^2} \tag{5.8}$$

The energies of the electric and magnetic fluxes were generally quadratic in the integer quantum numbers. This was essential to their interpretation as infinite momentum frame energies. By contrast, the energy stored in instanton charge is linear in Q. To interpret this we note that the energy of a "photon" in the super Yang-Mills theory is also linear in its integer valued wave number. This suggests that Q should be identified with wave number in a new 5th direction in the quantum field theory. In other words, in the limit that g^2 becomes large, the 4-dimensional super Yang-Mills theory somehow develops a new spatial direction. The size of this dimension can be read off from eq. (5.8) and is given by

$$\Sigma_5 = \frac{g^2}{2\pi} = (2\pi)^5 \frac{l_{11}^6}{RL_1L_2L_3L_4} \tag{5.9}$$

This does not mean that a new dimension opens up in the spacetime M-theory. What it does mean is that the original field theory on the 4-torus becomes some kind of quantum field theory with the symmetry and low lying level density of a theory on a five torus. In fact as the L_i shrink this fifth dimension becomes the largest of the Σ_i.

The geometric symmetry of the 5-torus is $SL(5;Z)$ which includes the discrete five dimesnional rotations. Thus the states and operators of the theory will be classified in $SL(5;Z)$ multiplets. The first such multiplet is the 4 field-theoretic momenta and the instanton charge $\frac{Q}{g^2}$. The 6 magnetic fluxes also must belong to a multiplet. They can be combined with the 4 components of electric field to form a 10 component antisymmetric tensor. Let us denote the fluxes in the following way.

$$\begin{aligned} \Phi_{ij} =& \epsilon_{ijkl} n_{kl} \\ \Phi_{i5} =& m_i \end{aligned} \tag{5.10}$$

where n_{ij} and m_i are the integer valued magnetic and electric fluxes. We can now write the energies corresponding to a unit flux in the symmetrical form

$$E_{ab} = \frac{\pi}{N}\left(\frac{\Sigma_a^2 \Sigma_b^2}{\Sigma_1\Sigma_2\Sigma_3\Sigma_4\Sigma_5}\right) \tag{5.11}$$

where $a, b = 1, 2, 3, 4, 5$.

Let us now turn to the string theoretic interpretation of the theory. Consider the spectrum of string-like excitations that can be created by wrapping M-Theory 2-branes around the cycles of the 4-torus. There are four obvious independent string states that correspond to a membrane wrapped around L_1, L_2, L_3, L_4. If the theory is to have the symmetry of the 5-torus then there must be an additional 5th state. In fact there is such a string configuration arising from the wrapping of an M-theory 5-brane around the 4-torus. In fact the formula for the tensions of the various types of strings exhibits a high degree of symmetry. Consider first, the string tension of a 2-brane wrapped around L_i.

$$T_i = \frac{L_i}{(2\pi)^2 l_{11}^3} \tag{5.12}$$

For the 5-brane wrapped on the 4-torus the tension is

$$T_5 = \frac{L_1 L_2 L_3 L_4}{(2\pi)^5 l_{11}^6} \tag{5.13}$$

These can be combined into a single formula

$$T_a = \frac{1}{R\Sigma_a} \tag{5.14}$$

which reveals the 5 dimensional symmetry. From eq. (5.13)we find that the string length scale for the wrapped 5-brane is given by

$$\frac{l_s^2}{2} = \alpha' = (2\pi)^4 \left(\frac{l_{11}^6}{L_1 L_2 L_3 L_4} \right) \tag{5.15}$$

It is now possible to understand the meaning of the light magnetic flux states which occur as the M-theory torus shrinks. Assuming the four L_i shrink in fixed proportion, the mass scale of these fluxes tend to zero as

$$M^2 \sim \frac{L^4}{l_{11}^6} \tag{5.16}$$

as seen from eq. (5.5). Let us compare this with the tension of the lightest of the various strings. In the limit $L \to 0$, the lightest string is the wrapped 5 brane whose tension is given in eq. (5.13). The two scales clearly agree with one another. Evidently, the light magnetic flux states are connected with the physics of the lightest strings.

To further understand the system as a string theory it is useful to choose one of the four directions of the torus, say L_1, and treat it as the direction identified with the dilaton. In other words we view the system as type IIA string theory on a 3-torus with sides L_2, L_3, L_4. We first permute the directions L_2, L_3. Now we perform T-duality, inverting the lengths

L_2, L_3. The identical T-duality was studied in [4] where it was shown that the sides L_1, L_2, L_3 transform into dual dimensions $\bar{L}_1, \bar{L}_2, \bar{L}_3$, given by

$$\begin{aligned}\bar{L}_1 =&(2\pi)^3\frac{l_{11}^3}{L_2L_3}\\ \bar{L}_2 =&(2\pi)^3\frac{l_{11}^3}{L_1L_3}\\ \bar{L}_3 =&(2\pi)^3\frac{l_{11}^3}{L_1L_2}\end{aligned} \tag{5.17}$$

All the remaining dimensions are unaffected. Thus

$$\bar{L}_4 = L_4 \tag{5.18}$$

In addition the T-dual theory has a transformed 11 dimensional Planck scale $\bar{l}_{11}$.

$$\bar{l}_{11}^3 = (2\pi)^3\frac{l_{11}^6}{L_1L_2L_3} \tag{5.19}$$

From eqs. (5.15) and (5.17) it is seen that the new compactification scales are of the same order of magnitude as the scale governing the lightest strings, the wrapped 5-branes. It is now clear what the fate of the light magnetic fluxes is. We have already seen that their energy scale is the same as the wrapped 5-brane string scale. The natural interpretation of the 6 magnetic fluxes is as winding number and momentum modes of the light strings wound on the three cycles of the $\bar{L}$ torus. To make this correspondence explicit, first uplift the type IIA theory to 11-dimensional M-theory on four radii R_1, R_2, R_3, R_4 such that R_4 corresponds to the short direction $\bar{L}_4$. Now reduce M-theory again to type IIA theory by compactifying on R_4. This type IIA theory is weakly coupled, since R_4 is small.

The string coupling constant for the light strings is easily found to be

$$\bar{g}_s^2 = \frac{L_1L_2L_3L_4^3}{(2\pi)^3l_{11}^6} \tag{5.20}$$

which indicates that the string is weakly coupled.

In this picture, the light fluxes in the original picture become momentum and winding modes. In particular, the correspondences are

$$\begin{aligned}
\phi_{23} =& \textit{momentum along } 1 \\
\phi_{13} =& \textit{momentum along } 2 \\
\phi_{12} =& \textit{momentum along } 3 \\
\phi_{14} =& \textit{winding along } 1 \\
\phi_{24} =& \textit{winding along } 2 \\
\phi_{34} =& \textit{winding along } 3
\end{aligned} \tag{5.21}$$

The result of the T-duality transformation can also be thought of as a new M(atrix)-theory description, equivalent to, but different from the original. In fact it is related to the original description by a rotation of the 5-torus $\Sigma_1\Sigma_2\Sigma_3\Sigma_4\Sigma_5$ which rotates by $\pi/2$ in both the $4,5$ and $2,3$ planes.

In addition to the multiplet of 5 string states, the theory contains other BPS multiplets of 0, 2 and 3 branes. We have found that these branes form 10, 5 and 10 dimensional multiplets of the 5 dimensional rotations.

To understand the meaning of these results let us consider fixing g and varying the dimensionless coupling $\tilde{g}^2$ by varying the overall scale of the $\Sigma_{1,2,3,4}$. According to eq. (5.9) the size of the fifth dimension Σ_5 is the fixed constant g. In other words the theory behaves like a five dimensional system with a fixed size for the 5th dimension and a varying size for the 4-torus. When the 4-torus, $\Sigma_{1,2,3,4}$, is very large, the behavior of the system at that scale is 4 dimensional. On the other hand, when the four torus, $\Sigma_{1,2,3,4}$ is much smaller than Σ_5 the system behaves in a 1+1 dimensional fashion. Between these limits all 5 directions are equally important and the dynamics is 5 dimensional.

6. The 5 Torus

We next consider compactification on a 5-torus. The analysis will follow the same lines as the previous case. We begin by noting that the U-duality group in this case is $SO(5,5;Z)$. This implies that super Yang-Mills theory on a 5-torus also has this symmetry and that the states should transform covariantly under it. Thus we are led to conclude that super Yang-Mills theory in 5+1 dimensions also has such a duality structure. As an example consider states carrying a unit of electric or magnetic flux. There are 10 magnetic and 5 electric fluxes. Since there is no 15-dimensional representation of $SO(5,5;Z)$, we must combine these states with one or more additional states. If we return to the 4+1 dimensional case we recall that the strings originating from 2-branes formed a multiplet with the string formed from a wrapped 5-brane. If we compactify an additional dimension it is obvious that we should combine the fully wrapped 5-brane with the wrapped 2-branes in a single multiplet. Since the 2-brane states are identified with magnetic flux we must group these quantities with the wrapped 5-brane charge. As we have remarked previously it is not known how to represent this quantity in terms of the super Yang-Mills theory quantum fields. Assuming it exists it combines with the 15 fluxes to form a 16 dimensional spinor of $SO(5,5;Z)$. For simplicity, in what follows we will write the formulas for the special case in which all the L_i are equal.

T-duality can be used to transform the theory. Choosing the 1 direction to be identified with the string coupling, we apply T-duality to the other 4 cycles.

$$\begin{aligned} \frac{\bar{L}_1}{\bar{l}_{11}} &= (2\pi)^4 \frac{l_{11}^3}{L^3} \\ \frac{\bar{L}_i}{\bar{l}_{11}} &= 2\pi \quad (i = 2,3,4,5) \\ \bar{l}_{11} &= (2\pi)^2 \frac{l_{11}^3}{L^2} \end{aligned} \tag{6.1}$$

Thus in the dual theory, one of the compactification scales, $\bar{L}_1$ decompactifies while the other 4 are of order the 11 dimensional Planck scale. This means that an element of the duality group has led us back to M-theory, compactified on a 4-torus with radius of order unity in 11 dimensional Planck units. Another equivalent description is IIA string theory on a 3-torus with string coupling of order 1 and compactification radius of order $\sqrt{\alpha'}$.

Surprisingly, the corresponding gauge theory is 4+1 dimensional super Yang-Mills theory at a value of $\tilde{g}^2$ of order 1. Evidently, this is the theory which governs the short distance behavior of 5+1 dimensional super Yang-Mills theory.

In each of the above cases, the shrunken torus limit led to a theory which was to some extent analyzable. In the 1-torus and 4-torus cases we found weakly coupled string theories with compactification radii (in the 4-torus case) of order the string scale. In the 3-torus case we were led back to 11 dimensional M-theory. For the 2 and 5 torus we found string theories at intermediate coupling and no compactification radii smaller than $\sqrt{\alpha'}$. In the case of the 6-torus things become worse. We are unable to eliminate either infinite coupling or vanishing string compactification radii. Either of these renders the theory unanalyzable by current methods. We don't know what this means but it could potentially signal a non-perturbative anomaly for toroidal compactification to space-time dimension lower than 6.

7. Conclusions

By combining the assumptions of M(atrix) Theory with known properties of toroidally compactified M-theory we can derive properties of super Yang-Mills theory , some of which have not been previously known. By considering the limit in which the M-theory torus shrinks to zero size we obtain information about the strongly coupled fixed points which govern either the infrared or ultraviolet behavior of the super Yang-Mills theory. These properties can either be thought of as predictions about super Yang-Mills theory or as necessary conditions for M(atrix) theory to correctly describe M-theory. The detailed predictions are listed in the introduction, and will not be repeated here.

Many of the conclusions involve symmetries between different kinds of fluxes, electric, magnetic and the mysterious 5-brane wrapping number. These fluxes often play the role of momenta in various compact dimensions. Symmetries between them frequently relate to space-time symmetries such as rotational invariance [3] as in the 2-torus case. There is one more quantity which is not traditionally thought of as a flux but which also represents a discrete momentum and that is N itself. Indeed, the identification of N/R as the longitudinal momentum of the infinite momentum frame was the original basis for M(atrix) theory. This raises an interesting speculation. Perhaps the entire collection of $U(N)$ theories are combined together into a single "master gauge theory" in which N appears as a flux. The dualities of this master theory should contain operations which interchange the usual fluxes with N. These dualities will be the basis for the full spacetime symmetry group including the notoriously difficult symmetries which rotate the longitudinal direction into transverse directions.

We already have some evidence that something like this is the case. Recall that in section 2 we identified $D0$-branes as the quanized units of energy stored in the electric field. These $D0$-branes are not the original $\tilde{D}0$-brane which carry longitudinal momentum but are related by the interchange of a compact transverse direction with the longitudinal direction. The fact that the parameters of the $D0$-branes come out correctly indicates that this interchange is working as it should.

All of this raises another issue which up to now we have swept under the rug. Which of the results of the kind we have derived apply only in the large N limit and which are valid for arbitrary N? From the viewpoint of [1], super Yang-Mills theory only becomes M-theory in the $N \to \infty$ limit. We would like to suggest that the validity of the M-theory, super Yang-Mills theory connection is not really limited in this way. In the literature on light cone quantization there is a formulation called "Discrete Light Cone Quantization" or DLCQ for short[15]. In DLCQ the light like direction X^- is assumed periodic with radius R and the spectrum of the light-like momentum p^+ is discrete. The starting point is slightly different than that in [1] where a spacelike direction X^{11} was taken to be compact. In the limit of large momentum $N \to \infty$ the difference in not important but for finite momentum it is. For example in [1] it was argued that systems with negative longitudinal momentum such as anti-$\tilde{D}0$-branes decouple in the infinite momentum or large N limit. In DLCQ the physical interpretation also requires a large N limit in which $N = p_{11}R$ tends to infinity. However in the DLCQ formulation the negative p_{11} systems decouple for all N, not just large N.

String theory can also be formulated in DLCQ. Basically it looks like ordinary light cone quantization except for two things. The first is that all longitudinal string momenta p_{11}^+ are discrete multiples of $1/R$. The second difference is that strings which wind around the longitudinal direction must be introduced. The suggestion we wish to make is that the matrix model at finite N is the DLCQ formulation of M-theory.

Now one of the features of the DLCQ of string theory is that the T-dualities associated with transverse directions is preserved even for finite N. The only symmetries which are not operative at finite N are those which mix longitudinal and transverse directions. If this view is correct then the various dualities studied in this paper should be valid at finite N. It is interesting from this point of view that the electric magnetic duality of the 3-torus is expected to be correct at finite N. Furthermore, following [3] we can use this to prove that the rotational invariance is restored as the 2-torus shrinks to zero with N fixed. It then seems reasonable to conjecture the validity of the $SL(5;Z)$ and $SO(5,5;Z)$ symmetries of the 4 and 5 torus.

8. Acknowledgements

We would like to thank Tom Banks and Barak Kol for discussions.

W. F. thanks the ITP at Stanford for its hospitality and was supported in part by the Robert Welch Foundation and NSF Grant PHY-9511632. A.R. was supported in part by the Department of Energy under contract no. DE-AC03-76SF00515. L.S. is supported by NSF grant PHY-9219345.

REFERENCES

1. T. Banks,W. Fischler,S. Shenker and L. Susskind,hep-th/9610043.
2. W.Taylor,hep-th/9611042.
3. S. Sethi and L. Susskind,hep-th/9702101.
4. L. Susskind,hep-th/9611164.
5. O.J. Ganor, S. Ramgoolam and W.Taylor,hep-th/9611202.
6. N. Seiberg, hep-th/9608111.
7. N. Seiberg, hep-th/9609161.
8. L. Motl,hep-th/9701025.
9. T. Banks and N. Seiberg,hep-th/9702187.
10. R. Dijkgraaf,E. Verlinde and H. Verlinde, hep-th/9703030.
11. J. Schwarz, Phys. Lett. **B360** (1995) 13, hep-th/9508143; P. Aspinwall, hep-th/9508154.
12. T.Banks, N.Seiberg and S.Shenker,hep-th/9612157.
13. M. Rozali,hep-th/9702136.
14. J. Maldacena and L. Susskind, Nucl. Phys. **B475** (1996) 670, hep-th/9604042.
15. H.C. Pauli and S.J. Brodsky,Phys.Rev.**D32** (1985) 1993.