Multiple Scattering and Functional Integrals*

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Abstract

A functional integral method for performing the statistical averages implicit in problems involving multiple scattering is developed. The scattering fields in a non-crystalline medium are represented as a fixed set of fields for each incident particle, but are assumed to vary randomly from one particle in the beam to the next. The method is applied to eikonal treatments of both multiple scattering and photon radiation. This latter problem involves the Landau-Pomeranchuk-Migdal effect and correction terms to earlier treatments are evaluated.

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1 Introduction and Rationale

A general quantum treatment utilizing standard high energy scattering methods has been given for multiple scattering and for the Landau-Pomeranchuk[1]-Migdal [2] effect in a recent paper[3]. The LPM phenomenon describes the suppressive effect that multiple scattering has on the Bethe-Heitler[4] radiation process for the emission of soft photons. This discussion was valid for targets of finite thickness compared to the formation length of the photon and is of interest in a detailed comparison between theory and experiment [5]. The theoretical treatment was extended to targets with internal structure in Ref. [6].

In the treatment of Refs. [3] and [6], the description of electron propagation in a medium was based on a higher order eikonal treatment[7] developed for a different problem. This approach yielded simple expressions for the wave functions in the medium and for the transition probability. It is the latter that must be averaged over the fluctuating field of the medium corresponding to the different set of fields that each incident particle experiences as it traverses the medium from different entrance point. An alternative approach that treats more accurately the effect of the short range singularity of the coulomb force on multiple scattering has also been given for an infinite target [8].

In the above treatments of the LPM effect, certain approximations were made in performing the statistical averages over the fields. It is the purpose of this paper to give a general and exact method for carrying out these averages. Functional integrals have long been used to evaluate statistical averages of fluctuating quantities and will be used here for the same purpose. They were used in Ref. [9] to discuss both the LPM suppression effect and the amplification effect in crystalline materials discovered by Ter-Mikaelyan[10] and extended by Akhiezer and Shul'ga[11]. Finally, the main

correction term will be numerically evaluated for situations of physical interest.

2 Model of the Scattering Medium

The model of a medium used in reference[3] was based upon two physical features. First, when a beam particle enters a non-crystalline medium, it will be scattered from the (shielded) atoms. Since these scattering centers are encountered by chance, a random walk path is generated through the medium. Each beam particle will enter the medium at a different point and thereby encounter its own unique pattern of scatterers. Second, the eikonal approximation is an accurate treatment of the scattering of a high energy particle if the scattering potential is sufficiently smooth. The phase of the wave function is the primary characterization of the process, and this phase is given by longitudinal line integrals through the scatterers.

Consistent with these arguments, the potential in the medium is written as

$$V(z, b_{\perp}) = -b_{\perp} \cdot E_{\perp}(z) , \quad (1)$$

where $b_{\perp}$ is the impact parameter relative to a fixed chosen axis, and z is the longitudinal depth in the medium. The electric field $E_{\perp}$ in the medium consists of longitudinal slices. The electric field in each slice is purely transverse. For each beam particle, the field arrangement is fixed. The scattering and/or radiation must be computed for each incident particle for this fixed field pattern. The final computed probabilities of interest must then be averaged over the particles in the incident beam; recall that each beam particle sees a completely different field arrangement. In this non-crystalline medium, the fluctuating nature of the field from one beam particle to the next, is expressed by the ensemble average

$$\langle \vec{E}_{\perp}(z_2) \cdot \vec{E}_{\perp}(z_1) \rangle = \frac{\langle p_{\perp}^2 \rangle}{L(z_2)} \delta(z_2 - z_1) . \quad (2)$$

The average transverse momentum accumulated via multiple scattering while traversing one radiation length, denoted by $L(z_2)$, is given by $\langle p_{\perp}^2 \rangle$. The square of the accumulated transverse momentum is seen to grow linearly with the path length, the expected random walk behavior.

3 Averaging Random Scalar Fields

Consider the quadratic form for an arbitrary function $q(z_2, z_1)$

$$Q_2 = \int dz_1 dz_2 E(z_2) q(z_2, z_1) E(z_1) , \quad (3)$$

where the scalar fields $E(z)$ are Gaussian random variables with zero mean. The correlation function is given by (refer to eq. (3.3) in the LPM paper[3])

$$\langle E(z_2) E(z_1) \rangle = m^2 a_1(z_2) \delta(z_2 - z_1) , \quad (4)$$

where $a_1(z_2)$ is a measure of the field fluctuations in the medium at the point z_2 . The functional integrals over the fields $E(z)$ are conveniently performed by putting the fields on a lattice. Thus define $z_i = \Delta i$ and $E(i) = E(z_i)$, then

$$Q_2 = \Delta^2 \sum_{i,j} E(i) q(i, j) E(j) , \quad (5)$$

and $\langle E(i) E(j) \rangle$ is proportional to δ_{ij} .

The normalized distribution function for the fields is then given by

$$[dE] = \prod_k \left[\sqrt{\frac{1}{2\pi A_1(k)}} dE(k) \exp\left(-\frac{E(k)^2}{2A_1(k)}\right) \right] \quad (6)$$

so that

$$\langle E(i) E(j) \rangle = \int [dE] E(i) E(j) = \delta_{ij} \int \sqrt{\frac{1}{2\pi A_1(i)}} dE(i) E(i)^2 \exp\left(-\frac{E(i)^2}{2A_1(i)}\right) \quad (7)$$

and hence one can identify on the lattice

$$\langle E(i) E(j) \rangle = A_1(i) \delta_{ij} \quad \text{with} \quad A_1(i) = \frac{m^2 a_1(i)}{\Delta}, \quad (8)$$

whereas in the continuum formulation $A_1(z) = a_1(z)$. Thus

$$\langle Q_2 \rangle = \Delta^2 \int [dE] \sum_{i,j} E(i) q(i,j) E(j) = \Delta^2 \sum_i A_1(i) q(i,i) \quad (9)$$

A continuum generating function for the statistical averages that are of interest here is defined as

$$J[F] = \int [dE] \exp \left[- \int dz_1 dz_2 E(z_2) F(z_2, z_1) E(z_1) \right]. \quad (10)$$

The discrete form of the generating function is then

$$J[F] = \int [dE] \exp \left[- \Delta^2 \sum_{i,j} E(i) F(i,j) E(j) \right] = \int [dE] \exp \left[- \Delta^2 E \cdot F \cdot E \right] \quad (11)$$

in an obvious matrix notation. The result of the integration is

$$J[F] = \frac{1}{\sqrt{\det[1 + 2\Delta^2 A_1 F]}} = \exp \left[- \frac{1}{2} \text{Tr} \log(1 + 2\Delta^2 A_1 F) \right]. \quad (12)$$

To lowest order in F this is

$$J[F] \sim \exp \left[- \Delta^2 \text{Tr} A_1 F \right] \sim 1 - \Delta^2 \text{Tr} A_1 F \quad (13)$$

which explicitly takes the discrete form

$$\langle [1 - \Delta^2 \sum_{i,j} E(i) F(i,j) E(j)] \rangle = 1 - \Delta^2 \sum_i A_1(i) F(i,i) \quad (14)$$

or restoring the continuum

$$\int dz_1 dz_2 \langle E(z_2) F(z_2, z_1) E(z_1) \rangle = \int dz_2 a_1(z_2) F(z_2, z_2) \quad (15)$$

which is in agreement with the initial formulation. From now on, all explicit factors of Δ will be omitted where there is no possible ambiguity; the reader can supply them wherever necessary.

4 Averaging Random Transverse Fields

Consider now the extended quadratic form

$$Q_{\perp} = \int dz_1 dz_2 E_{\perp}(z_2) \mathbf{q}(z_2, z_1) E_{\perp}(z_1) , \quad (16)$$

where $\mathbf{q}$ is a matrix in the transverse coordinates, $\mathbf{q} = q^{st}$, and the two dimensional fields $E_{\perp}(z) = (E^1(z), E^2(z))$ are Gaussian random variables with zero mean. The correlation function is now given by (see eq. (3.3) in reference [3])

$$\langle E^s(z_2) E^t(z_1) \rangle = \frac{1}{2} m^2 a(z_2) \delta(z_2 - z_1) \delta^{st} , \quad (17)$$

where $a(z) = \langle p_{\perp}^2 \rangle / m^2 L(z)$, and $L(z)$ is the radiation length in the medium[12] at the point z . The functional integrals over the fields are again performed by putting them on a lattice,

$$Q_{\perp} = \Delta^2 \sum_{i,j;s,t} E^s(i) q^{s,t}(i,j) E^t(j) , \quad (18)$$

and

$$\langle E^s(i) E^t(j) \rangle = \frac{1}{2} \left(\frac{m^2 a(i)}{\Delta} \right) \delta_{ij} \delta^{st} . \quad (19)$$

The normalized distribution function for the fields is then given by

$$[dE_{\perp}] = \prod_k \left[\frac{d^2 E(k)}{\pi A(k)} \exp\left(-\frac{E_{\perp}(k)^2}{A(k)}\right) \right] \quad (20)$$

so that

$$\langle E^s(i) E^t(j) \rangle = \int [dE_{\perp}] E^s(i) E^t(j) \quad (21)$$

$$= \delta_{ij} \delta^{st} \int \frac{d^2 E(i)}{\pi A(i)} E^s(i)^2 \exp\left(-\frac{E_{\perp}(i)^2}{A(i)}\right) \quad (22)$$

and hence one can identify on the lattice

$$\langle E_i^s E_j^t \rangle = \frac{1}{2} A(i) \delta_{ij} \delta^{st} \quad \text{with} \quad A(i) = \frac{m^2 a(i)}{\Delta} , \quad (23)$$

whereas in the continuum formulation $A(z) = m^2 a(z)$. Thus

$$\langle Q_\perp \rangle = \Delta^2 \int [dE] \sum_{i,j} E_\perp(i) \mathbf{q}(i,j) E_\perp(j) = \frac{1}{2} \Delta^2 \sum_i A(i) \text{tr} \mathbf{q}(i,i) \quad (24)$$

As in the earlier argument, a continuum generating function is introduced as

$$J[\mathbf{F}] = \int [dE_\perp] \exp \left[- \int dz_1 dz_2 E_\perp(z_2) \mathbf{F}(z_2, z_1) E_\perp(z_1) \right], \quad (25)$$

where $\mathbf{F}(z_2, z_1)$ is a symmetric transverse matrix, and the sum over transverse components is implied. The discrete form of the generating function is then

$$J[\mathbf{F}] = \int [dE_\perp] \exp \left[- \sum_{i,j} E_\perp(i) \mathbf{F}(i,j) E_\perp(j) \right] = \int [dE_\perp] \exp [-E_\perp \cdot \mathbf{F} \cdot E_\perp]. \quad (26)$$

The result of this integration is

$$J[\mathbf{F}] = \frac{1}{\sqrt{\det[1 + A \mathbf{F}]}} = \exp \left[-\frac{1}{2} \text{Tr} \log(1 + A \mathbf{F}) \right], \quad (27)$$

where the trace now includes a trace over the transverse elements of the matrix $\mathbf{F}$.

Recall that A is also symmetric. To lowest order

$$J[\mathbf{F}] \sim \exp \left[-\frac{1}{2} \text{Tr} A \mathbf{F} \right] \sim 1 - \frac{1}{2} \text{Tr} A(i) \mathbf{F} \quad (28)$$

which explicitly takes the discrete form

$$\langle [1 - \sum_{i,j} E_\perp(i) \mathbf{F}(i,j) E_\perp(j)] \rangle = 1 - \frac{1}{2} \sum_i A(i) \text{tr} \mathbf{F}(i,i), \quad (29)$$

where tr is the trace over transverse matrix coordinates. Restoring the continuum, we find

$$\int dz_1 dz_2 \langle E_\perp(z_2) \mathbf{F}(z_2, z_1) E_\perp(z_1) \rangle = \frac{1}{2} m^2 \int dz_2 a(z_2) \text{tr} \mathbf{F}(z_2, z_2) \quad (30)$$

which confirms the definition of a .

A more general generating function will be required, defined as

$$J[G, \mathbf{F}] = \int [dE_\perp] \exp [-E_\perp \cdot G - G \cdot E_\perp - E_\perp \cdot \mathbf{F} \cdot E_\perp]. \quad (31)$$

The quadratic form is diagonalized by the transformation

$$E_{\perp} = E'_{\perp} - (1 - A\mathbf{F})^{-1}AG \quad (32)$$

which then yields

$$J[G, \mathbf{F}] = \exp \left[+G(1 - A\mathbf{F})^{-1}AG - \frac{1}{2} \text{Tr} \log(1 + A\mathbf{F}) \right] . \quad (33)$$

5 Phase Averages

In discussing problems such as the propagation of a wave packet or the LPM effect, the model of a medium describing multiple scattering requires that one perform statistical averages over functions of the form

$$\langle LPM \rangle = \int [dE_{\perp}] Q[E_{\perp}] \sin \Phi[E_{\perp}] , \quad (34)$$

where

$$\Phi[E_{\perp}] = \phi + \int dz_1 dz_2 E_{\perp}(z_2) \mathbf{f}(z_2, z_1) E_{\perp}(z_1) , \quad (35)$$

and

$$Q[E_{\perp}] = 1 + \int dz_1 dz_2 E_{\perp}(z_2) \mathbf{q}(z_2, z_1) E_{\perp}(z_1) . \quad (36)$$

These will be rewritten in an obvious matrix notation which is valid for both the continuum and the discrete formulation.

$$\Phi[E_{\perp}] = \phi + E_{\perp} \mathbf{f} E_{\perp} , \quad (37)$$

$$Q[E_{\perp}] = 1 + E_{\perp} \mathbf{q} E_{\perp} , \quad (38)$$

where $\mathbf{f}$ and $\mathbf{q}$ are symmetric matrices in the coordinates z_2 and z_1 (in the discrete case, $\mathbf{f}$ and $\mathbf{q}$ also have explicit factors of Δ^2).

Rewrite $\langle LPM \rangle$ as the imaginary part of the integral $\langle lpm \rangle$:

$$\langle lpm \rangle = \int [dE_{\perp}] Q[E_{\perp}] \exp i\Phi[E_{\perp}] . \quad (39)$$

Consider first the $E_\perp$ independent term in Q .

$$\langle lpm \rangle_1 = \int [dE_\perp] \exp i\Phi[E_\perp] . \quad (40)$$

This functional integral was evaluated before, eq. (12). Identifying $F = -i\mathbf{f}$, the result is of the form

$$\langle lpm \rangle_1 = \exp \left[i\phi - \frac{1}{2} \text{Tr} \log(1 - iA\mathbf{f}) \right] . \quad (41)$$

This can be written as

$$\langle lpm \rangle_1 \equiv R \exp i\phi_{tot} , \quad (42)$$

where the magnitude is given by

$$R = \exp \left[-\frac{1}{4} \text{Tr} [\log(1 - iA\mathbf{f}) + \log(1 + iA\mathbf{f})] \right] \quad (43)$$

$$= \exp \left[-\frac{1}{4} \text{Tr} \log(1 + A\mathbf{f}A\mathbf{f}) \right] \quad (44)$$

and the phase is

$$\phi_{tot} = \phi - \frac{1}{4i} \text{Tr} [\log(1 - iA\mathbf{f}) - \log(1 + iA\mathbf{f})] . \quad (45)$$

To first order in $\mathbf{f}$, the magnitude is unity and the phase becomes

$$\phi_{tot} = \phi + \frac{1}{2} \text{Tr} [A\mathbf{f}] \dots \quad (46)$$

$$= \phi + \frac{1}{2} m^2 \int dz_2 a(z_2) \text{tr} \mathbf{f}(z_2, z_2) \dots \quad (47)$$

When $\mathbf{f}$ is proportional to the unit matrix, this is in agreement with the ‘LPM’ approximation. The correction terms to both the magnitude and the phase are down by extra factors of $(A\mathbf{f})^2$.

The $\mathbf{q}$ term in $Q_\perp$ is most easily evaluated by writing

$$Q[E_\perp] = 1 + E_\perp \mathbf{q} E_\perp \sim \exp[E_\perp \mathbf{q} E_\perp] , \quad (48)$$

and expanding the final result to zeroth and first order in $\mathbf{q}$. With the identification $\mathbf{F} = -(\mathbf{if} + \mathbf{q})$, the result is

$$\langle lpm \rangle = \exp \left[i\phi - \frac{1}{2} \text{Tr} \log(1 - A(\mathbf{if} + \mathbf{q})) \right] . \quad (49)$$

Expanding in $\mathbf{q}$ this becomes

$$\langle lpm \rangle = \left[1 + \frac{1}{2} \text{Tr} [(1 - iA\mathbf{f})^{-1} A\mathbf{q}] \right] \exp \left[i\phi - \frac{1}{2} \text{Tr} \log(1 - iA\mathbf{f}) \right] . \quad (50)$$

The full result can be rewritten in a more transparent form as

$$\langle lpm \rangle = \left[1 + \frac{1}{2} \text{Tr} [(1 - iA\mathbf{f})^{-1} A\mathbf{q}] \right] R \exp i\phi_{tot} , \quad (51)$$

where R and ϕ_{tot} were defined earlier. Thus

$$\begin{aligned} \langle LPM \rangle = & \left[1 + \frac{1}{2} \text{Tr} [(1 + A\mathbf{f}A\mathbf{f})^{-1} A\mathbf{q}] \right] R \sin \phi_{tot} \\ & + \frac{1}{2} \text{Tr} [(1 + A\mathbf{f}A\mathbf{f})^{-1} A\mathbf{f}A\mathbf{q}] R \cos \phi_{tot} . \end{aligned} \quad (52)$$

Neglecting correction terms down by order $\mathbf{f}^2$, the result is

$$\langle LPM \rangle = \left[1 + \frac{1}{2} \text{Tr} [A\mathbf{q}] \right] \sin \phi_{tot} + \frac{1}{2} \text{Tr} [A\mathbf{f}A\mathbf{q}] \cos \phi_{tot} , \quad (53)$$

where ϕ_{tot} to first order is given by eq. (47). The term involving the trace of the product, $[\mathbf{f}\mathbf{q}]$ is *not* present in the the ‘LPM’ approximation which neglects correlations between the fluctuating amplitude, $[E\mathbf{q}E]$, and the fluctuating part of the phase, $[E\mathbf{f}E]$.

6 Wave Packet Propagation

In Ref. [3], the behavior of a wave packet traversing a homogeneous medium while undergoing multiple scattering was discussed. After passing through the medium, the packet takes the form

$$|\phi(r, t)|^2 = \exp \left[-\Phi/(2w^2) \right] , \quad (54)$$

where

$$\Phi(r, t) = (\vec{R}(z, b_\perp) - \vec{v}^i t)^2, \quad (55)$$

$$R_z(z) = z + \frac{1}{2(p^i)^2} \left[(z-l)A_\perp^2(l) + \int_0^l dz' A_\perp(z') \cdot A_\perp(z') \right] \quad (56)$$

$$R_\perp(z, b_\perp) = b_\perp - \frac{1}{p^i} \left[(z-l)A_\perp(l) + \int_0^l dz' A_\perp(z') \right]. \quad (57)$$

The phase function Φ expanding to second order in the transverse fields $E_\perp$ is

$$\Phi(r, t) = \Phi_0(r, t) - g \cdot E_\perp - E_\perp \cdot g + E_\perp \cdot \mathbf{f} \cdot E_\perp, \quad (58)$$

where

$$\Phi_0(r, t) = (z - v^i t)^2 + b_\perp^2, \quad (59)$$

$$g \cdot E_\perp = \frac{1}{p} \int_0^l dz_1 (z - z_1) b_\perp \cdot E_\perp(z_1), \quad (60)$$

$$E_\perp \cdot \mathbf{f} \cdot E_\perp = \frac{1}{p^2} \int_0^l dz_2 \int_0^l dz_1 E_\perp(z_2) \cdot E_\perp(z_1) \times \quad (61)$$

$$\left[(z - v^i t)(z - z_>) + (z - z_2)(z - z_1) \right], \quad (62)$$

where $z_>$ is the larger of z_2 or z_1 , and $z > l$. The discussion will now be restricted to a homogeneous medium in which the radiation length does not depend upon position.

The statistical average that must be performed over the random fields is

$$\langle |\phi(r, t)|^2 \rangle = \exp \left[-\Phi_{eff}/(2w^2) \right] = \int [dE_\perp] \exp \left[-\Phi/(2w^2) \right], \quad (63)$$

with Φ given above. The functional integral that must be performed is of the form of eq. (33) that was evaluated earlier. The result is

$$\Phi_{eff} = \Phi_0(r, t) - \frac{A}{2w^2} g \cdot (1 + \frac{A}{2w^2} \mathbf{f})^{-1} \cdot g + w^2 \text{Tr} \log(1 + \frac{A}{2w^2} \mathbf{f}). \quad (64)$$

The $g \dots$ term is proportional to $b_\perp^2$ and marks the spreading of the packet and an increase in its packet width due to multiple scattering. To lowest order in $\mathbf{f}$, this result becomes

$$\Phi_{eff} = \Phi_0(r, t) - \frac{A}{2w^2} g \cdot g + \frac{1}{2} A \text{Tr} \mathbf{f}. \quad (65)$$

Carrying out the traces yields

$$\text{Tr } \mathbf{f} = 2 \frac{l}{p^2} \left[(z - v^i t) \left(z - \frac{1}{2} l \right) + \left(z^2 - z l + \frac{1}{3} l^2 \right) \right] \quad (66)$$

$$g \cdot g = b_{\perp}^2 \frac{l}{p^2} \left(z^2 - z l + \frac{1}{3} l^2 \right), \quad (67)$$

so that

$$\Phi_{eff} = (z - v^i t)^2 + b_{\perp}^2 \left[1 - \frac{A l}{2 w^2 p^2} \left(z^2 - z l + \frac{1}{3} l^2 \right) \right] \quad (68)$$

$$+ \frac{A l}{p^2} \left[(z - v^i t) \left(z - \frac{1}{2} l \right) + \left(z^2 - z l + \frac{1}{3} l^2 \right) \right]. \quad (69)$$

Recalling that this expression is valid only to order $\langle E_{\perp}^2 \rangle$, it can be manipulated into a more interpretable physical form. The transit time T through the medium is clearly

$$v^i T = l + \frac{A l^2}{4 p^2}. \quad (70)$$

Therefore we can write

$$\Phi_{eff} = \left[\left(1 + \frac{A l}{2 p^2} \right) (z - l) - v^i (t - T) \right]^2 \quad (71)$$

$$+ b_{\perp}^2 \left[1 + \frac{A l}{2 w^2 p^2} \left(z^2 - z l + \frac{1}{3} l^2 \right) \right]^{-1} + \dots \quad (72)$$

The packet width is increased and that the velocity in the z direction is decreased and is given by

$$v^i \left(l + \frac{A l}{2 p^2} \right)^{-1} \sim v^i \left(l - \frac{A l}{2 p^2} \right) = v^i \langle \cos \theta(l) \rangle, \quad (73)$$

$$\text{where} \quad \langle \theta^2(l) \rangle = \frac{A l}{p^2} = \frac{\langle p_{\perp}^2 \rangle l}{p^2 L}. \quad (74)$$

To this order, the effective width in $b_{\perp}$ at large times varies as

$$\sqrt{2 w^2 + \sin^2 \langle \theta \rangle} \left(z^2 - z l + \frac{1}{3} l^2 \right) \sim \sin \langle \theta \rangle v^i t. \quad (75)$$

Note that the transit time given above is consistent, since its average value should be given by

$$v^i T = \int_0^l \langle \frac{dz}{\cos(\theta)} \rangle \sim \int_0^l dz [1 + \frac{1}{2} \langle \theta(z)^2 \rangle] \quad (76)$$

$$= \int_0^l dz [1 + \frac{A z}{2 p^2}] = l + \frac{A l^2}{4 p^2} \quad (77)$$

which is in agreement with eq. (70).

7 LPM Radiation

In Refs.[3] and [6] on the LPM effect, the probability for the emission of a photon for a fixed distribution of the target scatterers was given as

$$\pi(1-x) \frac{dP(x)}{dx} = I(\text{tot}) = \int_{-\infty}^{\infty} db_2 \int_{-\infty}^{b_2} db_1 I(b_2, b_1, b_l), \quad (78)$$

where $I(\text{tot})$ includes the total effect of multiple scattering

$$I(b_2, b_1, b_l) = 2 \frac{C(b_2)C(b_l)}{b} \left\{ \left[1 + \frac{1}{2} r(x) \lambda(b_2, b_1, b_l) \right] \sin(c) - \sin(b) \right\},$$

$$c = b [1 + \eta(b_2, b_1, b_l)], \quad (79)$$

and dimensionless variables are introduced by $b_i = z_i/l_f$, $b = (z_2 - z_1)/l_f$, and the formation length has been defined as $l_f = 2xp/(m^2(1-x))$. The cutoff functions $C(b)$ can be set equal to one at a later stage of the calculation. The field dependent quantities $\lambda(z_2, z_1, l)$ and $\eta(z_2, z_1, l)$ are given by ($z = z_2 - z_1 \geq 0$)

$$m^2 \lambda(z_2, z_1, l) = [\vec{A}_\perp(z_2, z_1)]^2, \quad (80)$$

$$z^2 m^2 \eta(z_2, z_1, l) = z \int_{z_1}^{z_2} dz [\vec{A}_\perp(z, z_1)]^2 - [\int_{z_1}^{z_2} dz \vec{A}_\perp(z, z_1)]^2, \quad (81)$$

$$\text{where } \vec{A}_\perp(z, z_1) = \int_{z_1}^z dz' \vec{E}_\perp(z'). \quad (82)$$

The matrix functions $\mathbf{q}$ and $\mathbf{f}$ of eq.34 can be identified from:

$$Q = 1 + \frac{r(x)}{2m^2} A_{\perp}^2(z_2, z_1) \quad (83)$$

$$\Phi = b + \frac{b}{m^2 z_2^2} \left[z \int_{z_1}^{z_2} dz' A_{\perp}^2(z', z_1) - \left[\int_{z_1}^{z_2} dz' A_{\perp}(z', z_1) \right]^2 \right] . \quad (84)$$

Both $\mathbf{q}$ and $\mathbf{f}$ are proportional to the unit transverse matrix and are given by

$$\mathbf{f}(z'_2, z'_1) = \frac{b}{m^2 z_2^2} [z(z_2 - z'_>) - (z_2 - z'_2)(z_2 - z'_1)] \theta(z_2 > z'_2, z'_1 > z_1) , \quad (85)$$

where $z'_>$ is the greater of z'_2 or z'_1 , and

$$\mathbf{q}(z'_2, z'_1) = \frac{r(x)}{2m^2} \theta(z_2 > z'_2, z'_1 > z_1) . \quad (86)$$

The function $\theta(z_2 > z'_2, z'_1 > z_1)$ is zero unless both z'_2 and z'_1 are between z_2 and z_1 .

Using $A(z) = 2\pi m^2 / (\alpha L(z))$, it now follows that the magnitude factor is

$$1 + \frac{1}{2} \text{Tr} [A \mathbf{q}] = 1 + r(x) \frac{\pi}{\alpha} \int_{z_1}^{z_2} \frac{dz'}{L(z')} \quad (87)$$

$$= 1 + 3 r(x) T \bar{\lambda}(z_2, z_1, l) , \quad (88)$$

the phase function is

$$\phi_{tot} = b + \frac{1}{2} \text{Tr} [A \mathbf{f}] \quad (89)$$

$$= b + \frac{2\pi b}{\alpha z_2^2} \int_{z_1}^{z_2} \frac{dz'}{L(z')} [z(z_2 - z') - (z_2 - z')^2] \quad (90)$$

$$= b + 6 b T \bar{\eta}(z_2, z_1, l) , \quad (91)$$

and the leading ‘cross term’ correction is

$$\frac{1}{2} \text{Tr} [A \mathbf{f} A \mathbf{q}] = r(x) m^2 \left(\frac{\pi}{\alpha} \right)^2 \int_{z_1}^{z_2} \frac{dz'_2}{L(z'_2)} \frac{dz'_1}{L(z'_1)} f(z'_2, z'_1) \quad (92)$$

$$= 9 r(x) b T^2 \bar{\delta}(z_2, z_1, l) . \quad (93)$$

The next order term that has been neglected arises from the normalization factor R , and involves $\text{Tr} [\mathbf{A} \mathbf{f} \mathbf{A} \mathbf{f}]$. The normalized and dimensionless functions that measure the effects of multiple scattering are given by

$$\bar{\lambda}(z_2, z_1, l) = \frac{1}{J} \int_{z_1}^{z_2} \frac{dz'}{L(z')} \quad (94)$$

$$\bar{\eta}(z_2, z_1, l) = \frac{1}{J^2 z^2} \int_{z_1}^{z_2} \frac{dz'}{L(z')} (z_2 - z')(z' - z_1) \quad (95)$$

$$\bar{\delta}(z_2, z_1, l) = \frac{1}{J^2 z^2} \int_{z_1}^{z_2} \frac{dz'}{L(z')} \frac{dz''}{L(z'')} [z(z_2 - z_>) - (z_2 - z')(z_2 - z'')] , \quad (96)$$

where $z_>$ is the greater of z' or z'' , and J is a measure of the total radiation length of the target,

$$J = \int_0^l \frac{dz'}{L(z')} . \quad (97)$$

The explicit statistical averages for a homogeneous target plate of thickness l are readily evaluated. The results for b_2 and b_1 in the regions before, inside, and after the target, denoted by $(-, 0, +)$ respectively, are

Region ($b_2 b_1$)	$\lambda(b_2, b_1, b_l)$	$6 b^2 \bar{\eta}(b_2, b_1, b_l)$	$12 b^2 \bar{\delta}(b_2, b_1, b_l)$
(--)	0	0	0
(++)	0	0	0
(0 -)	b_2/b_l	$b_2^2 [3b - 2b_2]/b_l$	$b_2^3 [4b - 3b_2]/b_l^2$
(+ 0)	$(b_l - b_1)/b_l$	$(b_l - b_1)^2 [3b - 2(b_l - b_1)]/b_l$	$(b_l - b_1)^3 [4b - 3(b_l - b_1)]/b_l^2$
(0 0)	b/b_l	b^3/b_l	b^4/b_l^2
(+ -)	1	$[3(b_2 + b_1)b_l - 2b_l^2 - 6b_2b_1]$	$[4b_2b_l + 2b_1b_l - 3b_l^2 - 12b_2b_1]$

These formulas join smoothly at the intersections of the regions.

Referring to Eq.53, to this order the resultant integrand in eq.78 has become

$$I(b_2, b_1, b_l) = 2 \frac{C(b_2)C(b_1)}{b} \left\{ [1 + 3 r(x) T \bar{\lambda}(b_2, b_1, b_l)] \sin(c) - \sin(b) + 9 r(x) T^2 \bar{\delta}(b_2, b_1, b_l) b \cos(c) \right\} , \quad (98)$$

where

$$c = \phi_{tot} = b [1 + 6 T \bar{\eta}(b_2, b_1, b_l)] . \quad (99)$$

To this order, the only difference from the result of Ref. [3] is the $\cos(c)$ term that arose from the cross correlation. Now the numerical effect of this term will be discussed.

8 Numerical Results

In Figure 1 the photon spectra from a 25GeV electron beam incident upon a set of Au targets with $T = 0.1, 1.0$, and 10.0 (0.07, 0.7, and 7% of a radiation length respectively) are shown. A form factor that extracts the Bethe-Heitler behavior of the emission probability is conveniently introduced as

$$I(\text{tot}) = I(\text{BH}) F(k, T, x) , \quad (100)$$

where $I(\text{BH}) = 4T$ is computed to first order in $\bar{\lambda}$ and $\bar{\eta}$ but dropping $\bar{\delta}$. The solid curves are with the leading cross term correlation while the dashed curves are without. The formation length scales as the square of the incident electron energy.

For small k , or large formation length l_f , the internal structure of the target becomes unimportant. The problem becomes one of radiation from a effectively thin target. In this limit one finds

$$\bar{\lambda}(b_2, b_1, b_l) \sim \theta(b_2 - B)\theta(B + b - b_2) \quad (101)$$

$$\bar{\eta}(b_2, b_1, b_l) \sim \bar{\lambda}(b_2, b_1, b_l) (b_2 - B)(B + b - b_2)/b^2 \quad (102)$$

$$\bar{\delta}(b_2, b_1, b_l) \sim \bar{\lambda}(b_2, b_1, b_l) [b(b_2 - B) - (b_2 - B)^2]/b^2 , \quad (103)$$

where B is the position of the center of the target.

Writing $b_2 = B + bw$ with $db_2 = b dw$, allows the b integral to be performed and the form factor becomes

$$F(l_f \gg l, T, r) \sim \frac{1}{2T} \int_0^1 dw \left\{ \left(1 + \frac{3}{2}rT\right)/D(w) + \frac{3}{2}rT/D^2(w) - 1 \right\} , \quad (104)$$

where $D(w) = 1 + 6Tw(1 - w)$. For $T \rightarrow 0$, the form factor approaches one, the BH result. Again, this result does not depend upon the detailed structure or geometric

arrangement of the target, only the integral variable T . A plot of this limit is given in Figure 2 and compared with the the LPM approximation.

Finally, in Figure 3 the effect of the cross correlation term on the interference between two plates are shown. The cross correlation term seems to have its maximum effect for $T \sim 1$. Consider a target with $T = 1$ that is divided into two identical plates which are then separated by a gap. The effect on the photon spectra is illustrated in the last figure. The solid curve includes the correlation effect, the dashed curve is without. The shape of the spectra are changed, but the presence of interference between the two plates is still evident. The dimensionless gap G is measured and defined just like the target thickness T . That is, a gap of $G = T$ means that the total gap between all the plates is equal to their original total thickness.

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- [12] Note that a can be thought of as being twice as big as the a_1 of the previous section, since $\langle p_\perp^2 \rangle = \langle p_1^2 \rangle + \langle p_2^2 \rangle$. This explains the apparent mismatch of factors of 2 between this and the previous section.

FIGURE CAPTIONS

FIGURE 1. A plot of the form factor $F(k,T,x)$ vs k at three different values of T . The solid curves are the result with the cross correlation term; the dashed curves are the LPM approximation.

FIGURE 2. A plot of the form factor $F(k,T,x)$ vs T for small k , that is, for $l_f \gg l$ and $x \sim 1$.

FIGURE 3. A plot of the form factor $F(k,T,x)$ vs k for two plates at different values of G . The solid curves are the result with the cross correlation term; the dashed curves are the LPM approximation.

Figure 1

A plot of the form factor $F(k, T, x)$ vs k at selected T values. The solid curves include the cross correlation term; the dashed curves are the LPM approximation.

Figure 2

A plot of the form factor $F(k, T, x)$ vs T for small k , that is, for $l_f \gg l$ and $x \sim 1$.

Figure 3

A plot of the form factor $F(k, T, x)$ vs k for two plates at different values of G . The solid curves are the result with the cross correlation term; the dashed curves are the LPM approximation.