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A semi-classical treatment of channeling radiation reaction ⋆

Zhirong Huang, Pisin Chen and Ronald D. Ruth,

Stanford Linear Accelerator Center,
Stanford University, Stanford, CA 94309 USA

A semi-classical formalism is used to calculate the radiation reaction of a relativistic particle in a straight, continuous focusing system. Due to the absence of quantum excitation in such a focusing system, the radiation damping rate of the transverse action obtained using this formalism agrees exactly with the result from the classical Lorentz-Dirac radiation reaction equation. In the limit where the pitch angle of the particle is much smaller than the radiation opening angle, the transverse action damps exponentially with an energy-independent rate that is much faster than the energy decay rate. In the opposite limit, both the transverse action and the energy damp with power laws and their relative rates are comparable. The general time-dependence of the transverse action damping and the energy decay are obtained analytically from these rate equations.

1 Introduction

Radiation reaction including damping and quantum excitation has been studied extensively in synchrotrons and storage rings [1–3], where the effect is mainly due to the bending field. Recent development in advanced acceleration concepts [4] requires very strong transverse focusing to maintain beam stability, and novel ideas such as channeling acceleration [5,6] utilize the superstrong microscopic focusing field existing in a crystal channel. Motivated by these considerations, we have studied earlier the radiation reaction effect in a generic focusing channel and demonstrated its essential difference from that in a bending magnet [7]. We have shown that no quantum excitation is induced in such a focusing system, and that the transverse action of the channeled

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particle damps exponentially with an energy-independent damping rate if the pitch angle of the particle is much smaller than the radiation opening angle. More recently, we developed a semi-classical approach to calculate the radiation reaction of a relativistic particle and obtained the result in the opposite limit where the pitch angle of the particle is much larger than the radiation opening angle [8]. In this paper, we first review the semi-classical approach to the radiation reaction and then extend our calculation to obtain a unified expression of the radiation damping rate covering both limits and the intermediate regime. We also compare this result with the solution of the classical Lorentz-Dirac equation [9] applied to the case of channeling. Finally, we show how the transverse action and the energy damps after a particle with an arbitrary pitch angle is injected into a focusing channel.

2 Channeling kinematics

Let us consider a planar focusing system that provides a continuous parabolic potential $Kx^2/2$, where K is the focusing strength. A charged particle with energy $E = \gamma mc^2\,(\gamma \gg 1)$ oscillates in the transverse x direction while moving freely in the longitudinal z direction with a constant longitudinal momentum p_z in the absence of radiation. We define the pitch angle of the particle $\theta_p = p_{x,max}/p_z$ ($p_{x,max}$ being the maximum transverse momentum) and assume that θ_p is always much less than one. Then the motion of the particle can be decomposed into two parts: a free longitudinal motion and a transverse harmonic oscillation. This yields

$$\begin{aligned} E_{tot} &= E + Kx^2/2 = \sqrt{m^2c^4 + p_z^2c^2 + p_x^2c^2} + Kx^2/2 \\ &\simeq \sqrt{m^2c^4 + p_z^2c^2} + p_x^2c^2/2E_z + Kx^2/2 \\ &= E_z + E_x \quad , \end{aligned} \tag{1}$$

with $E_z = \sqrt{m^2c^4 + p_z^2c^2} \equiv \gamma_z mc^2$ and $E_x = p_x^2c^2/2E_z + Kx^2/2$.

A simple quantum mechanical analysis for such a system shows [7] that $E_x = (n + 1/2)\hbar\omega_z$, with $n = 0, 1, 2, ...$ being the transverse quantum number and $\omega_z = \sqrt{Kc^2/E_z}$ being the transverse oscillation frequency. The transverse action J_x is defined through the relation

$$J_x = \frac{E_x}{\omega_z} = (n + 1/2)\hbar \simeq n\hbar \quad \text{for large } n. \tag{2}$$

Note that this system has two invariants in the absence of radiation: a longitudinal one γ_z (or E_z, p_z) and a transverse one J_x (or E_x, n).

3 Radiation reaction

In Ref. [7], radiation reaction of the transverse energy level was calculated based on the the quantum mechanical transition rate between an initial and a final state. The calculation can be complex because the exact wavefunction contains special functions that depend on both n and p_z. However, when the typical photon energy emitted is much smaller than the energy of the particle, this transition rate can be approximately obtained from classical radiation theory [10]. It is well known that the energy radiated per unit solid angle per unit frequency is given by [11]

$$\left|\frac{d^2E}{d\Omega d\omega}\right| = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})e^{i\omega(t'-\boldsymbol{n}\cdot\boldsymbol{r}/c)}dt'\right|^2 \quad , \tag{3}$$

where $\boldsymbol{n}$ is the unit vector from the source to the observation point, $\boldsymbol{\beta}c$ and $\boldsymbol{r}$ are the velocity and position of the particle at the retarded time t'.

We can express Eq. (3) in the form of a double integral with respect to t_1 and t_2:

$$\left|\frac{d^2E}{d\Omega d\omega}\right| = \frac{e^2\omega^2}{4\pi^2 c}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2(\boldsymbol{\beta_1}\cdot\boldsymbol{\beta_2}-1)e^{i(\Phi_1-\Phi_2)} \quad , \tag{4}$$

where we have introduced the notation $\boldsymbol{\beta}_{1,2} = \boldsymbol{\beta}(t_{1,2})$, $\boldsymbol{r}_{1,2} = \boldsymbol{r}(t_{1,2})$ and $\Phi_{1,2} = \omega(t_{1,2} - \boldsymbol{n}\cdot\boldsymbol{r}_{1,2}/c)$. Going over to the new variables of integration t and τ via the transformation $t_1 = t+\tau$ and $t_2 = t$, and treating the integrand in the integral with respect to t as the angular spectral distribution of the radiated power at time t, we have

$$\begin{aligned}\left|\frac{d^3E}{dt d\Omega d\omega}\right| = \frac{e^2\omega^2}{4\pi^2 c}&\int_{-\infty}^{\infty} d\tau(\boldsymbol{\beta}(t+\tau)\cdot\boldsymbol{\beta}(t)-1) \\ &\times\exp\left\{i\omega\left[\tau - \frac{1}{c}\boldsymbol{n}\cdot(\boldsymbol{r}(t+\tau)-\boldsymbol{r}(t))\right]\right\} \quad .\end{aligned} \tag{5}$$

Let us define the number of photons emitted per unit time with energies between $\hbar\omega$ and $\hbar(\omega+d\omega)$ in the directions between Ω and $\Omega+d\Omega$ to be

$$R(\omega,\Omega) = \frac{1}{\hbar\omega}\left|\frac{d^3E}{dt d\Omega d\omega}\right| \quad . \tag{6}$$

$R(\omega, \Omega)$ corresponds to the transition rate between states that allow for such photon emissions. The rate of change of a physical quantity F due to the radiation, averaged over one oscillation period $T_z = 2\pi/\omega_z$ (indicated by $\langle\rangle$), is then given by [8]

$$\begin{aligned}\left\langle \frac{dF}{dt} \right\rangle &= \int_0^{T_z} \frac{dt}{T_z} \int d\Omega \int d\omega \Delta F(\omega, \Omega) R(\omega, \Omega) \\ &= \int_0^{T_z} \frac{dt}{T_z} \int d\Omega \int d\omega \frac{\Delta F(\omega, \Omega)}{\hbar\omega} \left| \frac{d^3 E}{dt d\Omega d\omega} \right| \quad ,\end{aligned} \tag{7}$$

where $\Delta F(\omega, \Omega)$ is the change of F after a photon with energy $\hbar\omega$ is emitted in the direction Ω. We use the convention that $\Delta F(\omega, \Omega)$ is negative if F decreases after the emission (contrary to the convention used in Ref. [8]). For example, the rate of energy decay $\langle dE/dt \rangle$ is obtained by replacing ΔF with $\Delta E = -\hbar\omega$ and is just the average energy decay rate.

In general, this semi-classical approach can give rise to radiation damping as well as quantum excitation because it incorporates both the quantum picture of a single photon emission and the continuous, classical spectrum of radiation. Let us consider here the radiation effect to the transverse action of the channeled particle: From Eq. (2), any small change of J_x is given by

$$\Delta J_x \simeq \frac{\Delta E_x}{\omega_z} - \frac{\Delta \omega_z E_x}{\omega_z^2} \quad , \tag{8}$$

Suppose the change is caused by emission of a photon with energy $\hbar\omega$ and with an angle θ from the longitudinal z direction, then the energy and the longitudinal momentum conservation require $\Delta E_x \simeq -\hbar\omega(1 - \beta_{zz} \cos\theta)$[7], where

$$\beta_{zz} \equiv \frac{p_z c}{E_z} = \sqrt{1 - \frac{1}{\gamma_z^2}} \simeq 1 - \frac{1}{2\gamma_z^2} \quad . \tag{9}$$

The frequency is also changed by $\Delta\omega_z \simeq \hbar\omega\omega_z/2E_z \simeq \hbar\omega\omega_z\theta_p^2/4E_x$, where we have used the approximation

$$\theta_p^2 = \frac{p_{x,max}^2}{p_z^2} \simeq \frac{2E_x}{E_z} = \frac{2J_x\omega_z}{E_z} \quad . \tag{10}$$

Thus, Eq. (8) can be simplified to be

$$\Delta J_x \simeq -\frac{\hbar\omega}{\omega_z}\left(1-\beta_{zz}\cos\theta+\frac{\theta_p^2}{4}\right) \quad , \tag{11}$$

which is always negative definite, and hence the transverse action always decreases after a emission process for all possible values of the emission angle θ. As pointed out in Ref. [7,8], this result reveals a unique feature of channeling radiation reaction: while the particle recoils against the emitted photon in the longitudinal direction to conserve the longitudinal momentum of the system, the existence of a transverse focusing environment absorbs the recoil in the transverse direction. As a result, the net recoil of the particle is not opposite to the direction of the photon emission, but always has a component pointing towards the focusing axis. Thus, quantum excitation of the transverse action is absent in such a focusing system.

The damping rate of the transverse action $\langle dJ_x/dt\rangle$ is obtainable by replacing ΔF in Eq. (7) with ΔJ_x in Eq. (11). Since $\Delta J_x/\hbar\omega \simeq -(1-\beta_{zz}\cos\theta+\theta_p^2/4)/\omega_z$ is independent of the frequency, we can integrate Eq. (5) over $d\omega$ and change Eq. (7) to

$$\left\langle\frac{dJ_x}{dt}\right\rangle = -\int_0^{T_z}\frac{dt}{2\pi}\int d\Omega(1-\beta_{zz}\cos\theta+\theta_p^2/4)\left|\frac{d^2E}{dtd\Omega}\right| \quad , \tag{12}$$

where the angular power spectrum $d^2E/(dtd\Omega)$ can be expressed in terms of the instantaneous momentum $\boldsymbol{p}$, the energy E and their time derivatives ($\dot{\boldsymbol{p}}$ and $\dot{E}$) in the absence of radiation [12]:

$$\left|\frac{d^2E}{dtd\Omega}\right| = \frac{e^2}{4\pi\gamma^2m^2c^3}\left[\frac{\dot{\boldsymbol{p}}^2-\dot{E}^2/c^2}{(1-\boldsymbol{n}\cdot\boldsymbol{p}c/E)^3}-\frac{1}{\gamma^2}\frac{(\boldsymbol{n}\cdot\dot{\boldsymbol{p}}-\dot{E}/c)^2}{(1-\boldsymbol{n}\cdot\boldsymbol{p}c/E)^5}\right] \quad . \tag{13}$$

Without losing generality, we choose the direction of the instantenous momentum $\boldsymbol{p}$ as the z' axis, $\boldsymbol{p}$ and $\dot{\boldsymbol{p}}$ plane as the $x'z'$ plane (Fig. 1). We also assume that $\boldsymbol{p} \propto \cos\omega_z t$ and $\dot{\boldsymbol{p}} = -K\boldsymbol{x} \propto \sin\omega_z t$, so that the longitudinal direction z oscillates with an angle $\theta_p\cos\omega_z t$ from the z' axis. Then in the $x'y'z'$ coordinate system, we have

$$\begin{aligned}
\boldsymbol{n} &= (\sin\psi\cos\phi, \sin\psi\sin\phi, \cos\psi) \quad ,\\
\boldsymbol{\beta} &= \boldsymbol{p}c/E = (0,0,\beta) \quad ,\\
\boldsymbol{z} &= (\sin(\theta_p\cos\omega_z t), 0, \cos(\theta_p\sin\omega_z t)) \quad ,\\
\dot{\boldsymbol{p}} &= (\dot{p}_\perp, 0, \dot{p}_{||}) \quad .
\end{aligned}$$

Thus, $\dot{E}/c = \boldsymbol{\beta}\cdot\dot{\boldsymbol{p}} = \beta\dot{p}_{||}$ and

$$\cos\theta = \boldsymbol{n}\cdot\boldsymbol{z} = \sin\psi\sin(\theta_p\cos\omega_z t)\cos\phi + \cos\theta\cos(\theta_p\sin\omega_z t) \quad . \tag{14}$$

Equation (13) then takes the form

$$\begin{aligned}\left|\frac{d^2E}{dtd\Omega}\right| = \frac{e^2}{4\pi\gamma^2 m^2c^3}\Bigg\{ & \dot{p}_{\|}^2\Big[\frac{(1-\beta^2)^2}{(1-\beta\cos\psi)^3} - \frac{1}{\gamma^2}\frac{(\cos\psi-\beta)^2}{(1-\beta\cos\psi)^5}\Big] \\ & + \dot{p}_{\perp}^2\Big[\frac{1}{(1-\beta\cos\psi)^3} - \frac{1}{\gamma^2}\frac{\sin^2\psi\cos^2\phi}{(1-\beta\cos\psi)^5}\Big] \\ & - \dot{p}_{\|}\dot{p}_{\perp}\frac{2}{\gamma^2}\frac{(\cos\psi-\beta)\sin\psi\cos\phi}{(1-\beta\cos\psi)^5}\Bigg\} \quad .\end{aligned} \tag{15}$$

Before we proceed to calculate the damping rate of the transverse action, it is helpful to first obtain the total rate of energy decay by integrating Eq. (15) over $\int d\Omega = \int d\phi \int d\psi \sin\psi$. Immediately we see that the $\dot{p}_{\|}\dot{p}_{\perp}$ term drops out while the $\dot{p}_{\|}^2$ term and the $\dot{p}_{\perp}^2$ term recombine to give the total radiated power

$$\left|\frac{dE}{dt}\right| = \frac{2e^2}{3m^2c^3}\gamma^2(\dot{\boldsymbol{p}}^2 - \frac{1}{c^2}\dot{E}^2) \tag{16}$$

as expected. In the case of channeling, we have $\gamma \simeq \gamma_z[1+(1/2)\theta_p^2\cos^2\omega_z t]$, $(\dot{\boldsymbol{p}})^2 = (Kx)^2 = 2KE_x\sin^2\omega_z t$ and $\dot{E}^2/c^2 = \beta^2\dot{p}_{\|}^2 \sim \theta_p^2(\dot{\boldsymbol{p}})^2$. Averaging over one oscillation period and keeping only the leading order term in θ_p^2, we find

$$\left\langle\frac{dE}{dt}\right\rangle \simeq \left\langle\frac{dE_z}{dt}\right\rangle \simeq -\frac{4e^2}{3m^2c^3}\gamma_z^2 KE_x\langle\sin^2\omega_z t\rangle \simeq -\frac{2}{3}\frac{r_eK}{mc}\gamma_z^2 J_x\omega_z \quad , \tag{17}$$

where $r_e = e^2/mc^2$ is the classical electron radius.

By putting Eq. (9), Eq. (14) and Eq. (15) into Eq. (12) and carrying out the angular integration and the time averaging as above, it is now straightforward to show that the damping rate of transverse action is

$$\begin{aligned}\left\langle\frac{dJ_x}{dt}\right\rangle &= \frac{1}{\omega_z}\Big[\Big(\frac{1}{\gamma_z^2}+\frac{\theta_p^2}{4}\Big)\left\langle\frac{dE}{dt}\right\rangle + \frac{\theta_p^2}{2}\left\langle\frac{dE}{dt}\cos^2\omega_z t\right\rangle\Big] \\ &= \frac{1}{\omega_z}\Big[\Big(\frac{1}{\gamma_z^2}+\frac{3\theta_p^2}{8}\Big)\Big]\left\langle\frac{dE}{dt}\right\rangle \qquad (18) \\ &= -\frac{2}{3}\frac{r_eK}{mc}\Big(J_x + \frac{3}{4}\sqrt{\frac{K}{m^3c^4}}\gamma_z^{1/2}J_x^2\Big) \quad , \qquad (19)\end{aligned}$$

where we have used the relation

$$\left\langle \frac{dE}{dt} \cos^2 \omega_z t \right\rangle = 2\left\langle \frac{dE}{dt} \right\rangle \langle \cos^2 \omega_z t \sin^2 \omega_z t \rangle = \frac{1}{4}\left\langle \frac{dE}{dt} \right\rangle \quad . \tag{20}$$

It is clear from Eq. (18) that the damping of the transverse action consists of two competing terms that differ by a factor of $\gamma_z^2\theta_p^2$. In the "undulator regime" where the pitch angle θ_p is much smaller than the radiation opening angle $1/\gamma_z$, the first term in Eq. (18) dominates and gives

$$\left\langle \frac{dJ_x}{dt} \right\rangle \simeq \frac{1}{\omega_z}\frac{1}{\gamma_z^2}\left\langle \frac{dE}{dt} \right\rangle = -\frac{2}{3}\frac{r_e K}{mc} J_x \equiv -\Gamma_c J_x \quad , \tag{21}$$

which shows that the transverse action damps exponentially with an energy-independent damping constant $\Gamma_c = (2r_e K/3mc)$ [7,8]. We also notice that the relative damping rate of the transverse action is much faster than that of the energy in this regime since

$$\Gamma_c = \left| \frac{1}{J_x}\left\langle \frac{dJ_x}{dt} \right\rangle \right| \gg \left| \frac{1}{E}\left\langle \frac{dE}{dt} \right\rangle \right| \simeq \frac{\Gamma_c}{2}\gamma_z^2\theta_p^2 \quad . \tag{22}$$

In the opposite regime where $\theta_p \gg 1/\gamma_z$, the second term in Eq. (18) dominates and gives

$$\left\langle \frac{dJ_x}{dt} \right\rangle \simeq \frac{1}{\omega_z}\frac{3\theta_p^2}{8}\left\langle \frac{dE}{dt} \right\rangle \quad . \tag{23}$$

By applying Eq. (10), we obtain

$$\frac{1}{J_x}\left\langle \frac{dJ_x}{dt} \right\rangle \simeq \frac{3}{4}\frac{1}{E}\left\langle \frac{dE}{dt} \right\rangle \quad , \tag{24}$$

which shows that the relative damping rate of the transverse action is comparable to the relative energy decay rate, with the numerical difference due to the chromatic effect and the sinusoidal variation of the focusing field [8].

Therefore, by using the above semi-classical approach, we have obtained general results for the absence of quantum excitation and for the asymmetric radiation damping of a channeled particle with arbitrary $\gamma_z\theta_p$. For very large $\gamma_z\theta_p$ ($\gg 1$), the photon emission takes place in a small fraction of one oscillation period [8], and the focusing environment is ineffective in suppressing the transverse action, so that both the transverse and the longitudinal dimensions of the particle damp with comparable rates. As $\gamma_z\theta_p$ becomes smaller (~ 1), the radiation is formed in a larger fraction of the oscillation period, and the focusing environment takes away a bigger portion of the transverse momentum of the particle resulting in faster damping of the transverse dimension. In the limit of very small $\gamma_z\theta_p$ ($\ll 1$), the radiation is formed over many oscillation periods, and the radiation reaction is predominantly transverse. Thus the transverse action damps exponentially, much faster than the energy decay.

4 Classical Lorentz-Dirac equation

Since quantum excitation is absent in the system studied here, we expect the radiation damping rate derived from the classical Lorentz-Dirac equation to coincide with the above result. Following Ref. [3], the fourth component of the Lorentz-Dirac equation simply reproduces Eq. (16), the equation for the radiated power. The first three components give rise to the Lorentz force (the focusing force) plus the radiation reaction force:

$$\frac{d\boldsymbol{p}}{dt} = -K\boldsymbol{x} + \left[\tau_0 \frac{d}{dt}\left(\gamma\frac{d\boldsymbol{p}}{dt}\right) + \frac{\boldsymbol{v}}{c^2}\frac{dE}{dt}\right] \quad , \tag{25}$$

where $\tau_0 = 2e^2/3mc^3 = 2r_e/3c = 6.26\times10^{-24}$sec must be small compared with $T_z = 2\pi/\omega_z$. We can then treat the radiation reaction force as perturbation and substitute $d\boldsymbol{p}/dt$ on the right hand side of equation with $-K\boldsymbol{x}$. Thus, the x component of this equation is

$$\begin{aligned}\frac{dp_x}{dt} &= -Kx + \left[\tau_0\frac{d}{dt}\Big(-\gamma Kx\Big) + \frac{v_x}{c^2}\frac{dE}{dt}\right] \\ &\simeq -Kx + \left[-\tau_0\gamma K v_x + \frac{v_x}{c^2}\frac{dE}{dt}\right]\end{aligned} \tag{26}$$

to the first order in τ_0.

Instead of solving for the instantaneous coordinates of the particle, we look for the evolution of the transverse action, which is an invariant in the absence of radiation. Since $J_x = E_x/\omega_z \propto E_x E_z^{1/2}$, we have

$$\frac{dJ_x}{dt} = \frac{J_x}{E_x}\frac{dE_x}{dt} + \frac{J_x}{2E_z}\frac{dE_z}{dt} \simeq \frac{1}{\omega_z}\frac{dE_x}{dt} + \frac{\theta_p^2}{4\omega_z}\frac{dE_z}{dt} \quad . \tag{27}$$

Taking the time derivative of $E_x = Kx^2/2 + p_x^2c^2/2E_z$ and applying the Lorentz-Dirac Eq. (26), we find

$$\frac{dE_x}{dt} = kxv_x + \frac{p_xc^2}{E_z}\frac{dp_x}{dt} - \frac{p_x^2c^2}{2E_z^2}\frac{dE_z}{dt} \simeq kxv_x + v_x\frac{dp_x}{dt} - \frac{v_x^2}{2c^2}\frac{dE}{dt}$$

$$=-\tau_0\gamma K v_x^2+\frac{v_x^2}{2c^2}\frac{dE}{dt}=-\Gamma_c\frac{p_x^2c^2}{E_z}+\frac{\theta_p^2}{2}\frac{dE}{dt}\cos^2\omega_z t \quad , \tag{28}$$

where we have used the relation $\Gamma_c=\tau_0 K/m$.

Inserting Eq. (28) into Eq. (27) and averaging over one oscillation period with the help of Eq. (20), we obtain

$$\begin{aligned}\left\langle\frac{dJ_x}{dt}\right\rangle &\simeq\frac{1}{\omega_z}\left[-\Gamma_c\left\langle\frac{p_x^2c^2}{E_z}\right\rangle+\frac{\theta_p^2}{2}\left\langle\frac{dE}{dt}\cos^2\omega_z t\right\rangle+\frac{\theta_p^2}{4}\left\langle\frac{dE}{dt}\right\rangle\right]\\ &=\frac{1}{\omega_z}\left[-\Gamma_c E_x+\frac{\theta_p^2}{8}\left\langle\frac{dE}{dt}\right\rangle+\frac{\theta_p^2}{4}\left\langle\frac{dE}{dt}\right\rangle\right]\\ &=\frac{1}{\omega_z}\left[\left(\frac{1}{\gamma_z^2}+\frac{3\theta_p^2}{8}\right)\right]\left\langle\frac{dE}{dt}\right\rangle \quad ,\end{aligned} \tag{29}$$

which is exactly Eq. (18) obtained by the semi-classical formalism. We note that, although the Lorentz-Dirac radiation reaction force can be applied successfully to calculate the average damping effect in cases such as ours, it in general suffers from conceptual difficulties related to its purely classical origins (e.g., "runaway" solutions and "pre-acceleration") [11]. On the other hand, our semi-classical treatment of radiation reaction by-passes such difficulties because it is based on the overall kinematics of the quantum emission process instead of the detailed interaction between the charged particle and its own field. Moreover, since this formalism is the natural approximation of the quantum mechanical treatment of radiation reaction when the emitted photon energy is much smaller than the energy of the particle, it not only has a classical limit that agrees with the Lorentz-Dirac equation, but also allows for quantum excitation.

5 Time evolution

The simple analytical results (i.e. Eq. (17) and Eq. (19)) of channeling radiation reaction, derivable by both the classical Lorentz-Dirac equation and the semi-classical approach, can be regarded as two coupled differential equations between J_x and γ_z. Defining a constant $C=\sqrt{K/m^3c^4}$ and omitting the bracket on the left hand sides of Eq. (17) and Eq. (19), we obtain

$$\begin{aligned}\frac{d\gamma_z}{dt}&=-\Gamma_c C\gamma_z^{3/2}J_x \quad ,\\ \frac{dJ_x}{dt}&=-\Gamma_c J_x-\frac{3}{4}\Gamma_c C\gamma_z^{1/2}J_x^2 \quad .\end{aligned} \tag{30}$$

Suppose that the particle enters the channel with transverse action J_{x0}, energy $\gamma_{z0}mc^2$ and thus a pitch angle $\theta_{p0} = \sqrt{2J_{x0}C/\gamma_{z0}^{3/2}}$, then the general solution to these coupled equations is

$$J_x(t) = J_{x0}\Big\{1 + \frac{5}{8}\gamma_{z0}^2\theta_{p0}^2\big[1 - \exp{(-\Gamma_c t)}\big]\Big\}^{-3/5} \exp{(-\Gamma_c t)} \quad ,$$
$$\gamma_z(t) = \gamma_0\Big\{1 + \frac{5}{8}\gamma_{z0}^2\theta_{p0}^2\big[1 - \exp{(-\Gamma_c t)}\big]\Big\}^{-4/5} \quad . \tag{31}$$

To illustrate the time evolution of the transverse action and energy, we plot the normalized transverse action J_x/J_{x0} and the normalized energy γ_z/γ_{z0} versus the scaled time $\Gamma_c t$ in Fig. 2 for three different initial conditions of $\gamma_{z0}\theta_{p0}$. For large $\gamma_{z0}\theta_{p0}$, both the transverse action and the energy initially damp with power laws, i.e.,

$$\text{when } \Gamma_c t \ll 1, \; J_x(t) \simeq J_{x0}\Big(1 + \frac{5}{8}\gamma_{z0}^2\theta_{p0}^2\Gamma_c t\Big)^{-3/5} \quad ,$$
$$\gamma_z(t) \simeq \gamma_{z0}\Big(1 + \frac{5}{8}\gamma_{z0}^2\theta_{p0}^2\Gamma_c t\Big)^{-4/5} \quad . \tag{32}$$

However, the exponential damping factor in the transverse action becomes more important for longer time and takes over when the particle damps to the "undulator regime", i.e.,

$$\text{when } \Gamma_c t \gg 1, \; J_x(t) \simeq J_{x0}\Big(1 + \frac{5}{8}\gamma_{z0}^2\theta_{p0}^2\Big)^{-3/5} \exp{(-\Gamma_c t)} \quad ,$$
$$\gamma_z(t) \simeq \gamma_{z0}\Big(1 + \frac{5}{8}\gamma_{z0}^2\theta_{p0}^2\Big)^{-4/5} \quad . \tag{33}$$

For small $\gamma_{z0}\theta_{p0}$, the exponential damping begins immediately, i.e.,

$$J_x(t) \simeq J_{x0} \exp{(-\Gamma_c t)} \quad ,$$
$$\gamma_z(t) \simeq \gamma_{z0}\big[1 - \frac{1}{2}\gamma_{z0}^2\theta_{p0}^2(1 - \exp{(-\Gamma_c t)})\big] \quad . \tag{34}$$

Therefore, the transverse action always has an exponential damping factor that dominates in the asymptote, while the energy eventually reaches a constant value: a factor of $(1 + 5\gamma_{z0}^2\theta_{p0}^2/8)^{4/5}$ reduction from the initial energy.

Finally, we note that in the definition of the transverse action J_x (i.e., Eq.(2)), we have thus far ignored $\hbar/2$ (due to the zero point energy of a harmonic oscillator) for large quantum number n. In fact, because of the excitation-free radiation damping in the transverse dimension, the particle eventually damps to its transverse ground state ($n = 0$) that is stable against further radiation. In this ground state the transverse action reaches the minimum value $J_{min} = \hbar/2$, corresponding to a minimum normalized emittance: one half of the Compton wavelength [7]. This minimum is the fundamental emittance limited by the uncertainly principle.

6 Conclusions

In this paper, we have given a complete account of channeling radiation reaction that is fundamentally different from synchrotron radiation reaction occurred in a bending magnet. Several assumptions have been made about the system we study, that is, a straight, planar, continuous focusing channel. We think that the only crucial requirement is the focusing. First, it is possible to extend the basic results obtained here to bent systems provided that the focusing field dominates over the bending field [7,8]. Secondly, when focusing exists in both transverse dimensions, the total transverse energy (i.e., $E_\perp = E_x + E_y$) can be shown to damp without quantum excitation [7]. Moreover, the parabolic potential model is used to simplify the analysis and can be generalized to different focusing potentials.

It appears that the excitation-free, asymmetric radiation reaction found in focusing systems can have interesting applications in beam handling, cooling, and acceleration. For example, in a sufficiently low-energy, focusing-dominated electron ring [7,8], the absolute transverse damping could perhaps be utilized to obtain ultra-cool beams in transverse phase space with fractional energy loss. The existence of a transverse ground state and hence a minimum normalized emittance for the accelerated particles might also be quite relevant and important. However, when applying these results to realistic systems, effects other than radiation reaction must be included and some of the results may be modified. For instance, we have investigated other excitation mechanisms such as bremsstrahlung and multiple Coulomb scattering as a charged particle transports through a crystal channel [6]. We have shown that multiple Coulomb scattering is a main factor restricting the radiation damping effect, and that the actual equilibrium beam emittance will primarily depend upon the energy of the particle and the electron density near the channeling axis.

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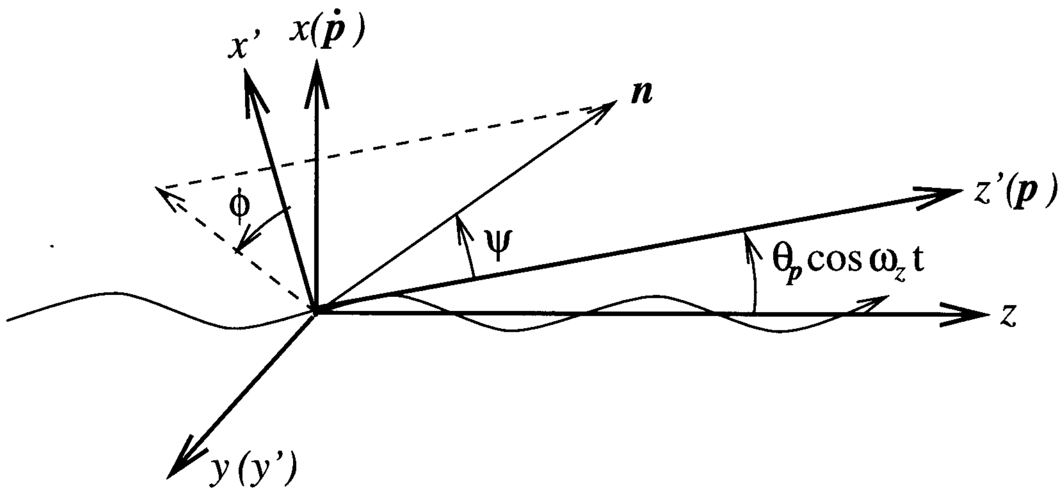

Fig. 1. The primed and the unprimed coordinate systems.

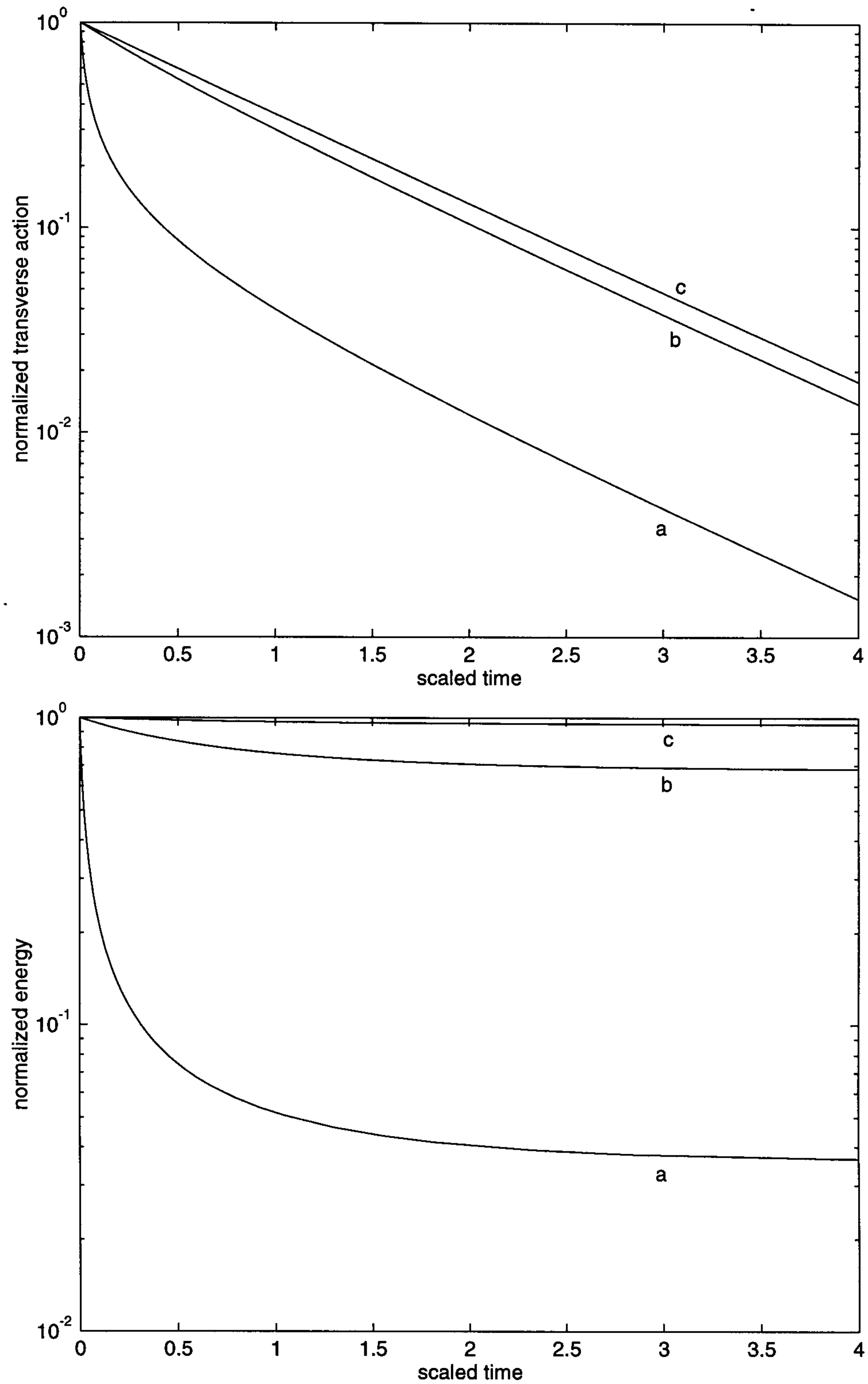

Fig. 2. (a) $\gamma_{z0}^2\theta_{p0}^2 = 100$, (b)$\gamma_{z0}^2\theta_{p0}^2 = 1$ and (c) $\gamma_{z0}^2\theta_{p0}^2 = 0.1$.