SLAC–PUB–6959
January 1996

Saturation of the Ion Induced Transverse Blow-up Instability*

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Presented at the International Workshop on Collective Effecs and Impedance for B Factories
Tsukuba, Japan, June 12–17, 1995

* Work supported by Department of Energy contract DE–AC03–76SF00515.

Introduction

In a recent paper[1], T. Raubenheimer and F. Zimmermann described a new, fast transverse instability caused by the interaction of a train of bunches with the residual gas. Ions produced by transversely offset bunches in the head of a train induce oscillations of the tail of the train. The ions may be cleared out by a gap after one revolution, but the memory remains in the train. Amplitude of oscillations keeps growing exponentially as $exp \sqrt{s/s_c}$ until the amplitude of a bunch centroid is on the order of the transverse rms σ of a bunch. The rise time s_c of the oscillations of a bunch centroid for the PEP-II HER was found to be a fraction of a millisecond, even taking into account the spread of ion frequencies[2].

Computer simulations[1] [2] confirm the exponential growth. However, the results of the simulations show that the exponential regime holds only for a short period of time and then changes to a much slower growth. Initial growth is rapid; it would be difficult to observe it directly in experiments. From a practical point of view, the important questions are, what is the amplitude at which a transition to slow growth takes place and, secondly, what is the growth rate after that transition compared to the rate, which could be handled with a reasonable feedback system.

The exponential regime is limited by nonlinearity of the beam-ion interaction. As a result, exponential growth at large amplitudes is replaced by a linear dependence of the amplitude on time. The transition from exponential growth to a linear regime depends on the initial conditions: exponential growth is noticeable only for very small initial amplitudes. An estimate of the growth rate at large amplitude is obtained and compared with computer simulations.

For completeness, the basic formulas of the original paper are re-derived.

Basic equations

First, we reproduce the basic equations of the paper [1]. Consider a train of bunches $n = 1, .., n_b$. The head of the n-th bunch is located at the position

⋆ Work supported by Department of Energy contract DE–AC03–76SF00515.

The vertical motion of electrons of the n-th bunch is described by the equation

$$\frac{\partial^2 y(s,z_n)}{\partial s^2} + \omega_b^2 y(s,z_n) = \frac{r_e}{\gamma}\frac{\partial U}{\partial y}\,, \tag{1}$$

where ω_b is a betatron frequency (in cm^{-1}), and the potential U of the beam-ion interaction is

$$U(\vec{r},s) = \quad \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|}\, n_i(\vec{r}',x). \tag{2}$$

The ion density $n_i(\vec{r},s)$ is given by the initial density n_i^0;

$$n_i(\vec{r},s) = \quad d\vec{r}'\delta[x - X_k(\vec{r}',s)]\delta[y - Y_k(\vec{r}',s)]\delta[z - z']n_i^0(\vec{r}')\,. \tag{3}$$

k

Here, $X_k(\vec{r},s)$ and $Y_k(y',z,s)$ are trajectories of ions generated by the k-th bunch at the location z of the ring. The initial value y' for Y_k, $Y_k(y',z,s_k) = y'$, is equal to the offset $y' = y_b(s_k,z_k)$ of the k-th bunch at the location z of the ring at the moment $s_k < s$, $s_k = z + z_k$.

Ions are generated uniformly along the ring with the rate

$$\frac{dN_i}{dz} = 0.06(\frac{p}{\mathrm{torr}})(\frac{300K^o}{T})N_b \quad \mathrm{cm}^{-1}\,, \tag{4}$$

where p is the residual gas pressure and the ionization cross-section is assumed to be 2 Mbarn. Initial distribution of ions is defined by the density of the parent bunch

$$n_i^0(\vec{r}) = \frac{dN_i}{dz}\rho(x)\rho(y - y_b(z+z_k,z_k))\,, \tag{5}$$

where $\rho(x)$, $\rho(y)$ are Gaussian distributions with rms σ_x, σ_y, respectively. We assume a flat beam $\sigma_x >> \sigma_y$, and take $\sigma_{x,y}$ constant around the ring, and for all bunches. Therefore, the ion frequencies for ions with atomic number A,

$$\omega_{x,y}^2 = \frac{2N_b r_p}{A s_b \sigma_y \sigma_x}\,, \tag{6}$$

are also constant. The frequency spread in a nonlinear regime is produced automatically by nonlinearity of the motion.

For a flat beam we take $X_k(\vec{r},s) = x\cos\psi_x$, $\psi_x = \omega_x(s-s_k)$, and $s_k = z + z_k$. Then

$$U(\vec{r},s) = \frac{}{4\pi}\frac{dN_i}{dz}\hat{\rho}(x) \quad {}_k \quad {}_0^{\infty} \frac{d\lambda}{\lambda^2} \quad dy' e^{-\lambda^2[y-Y_k(y',z,s)]^2}\rho(y' - y_b(z+z_k, z_k)). \quad (7)$$

Here $\hat{\rho}(x)$ has rms $\sigma_x^2\cos^2\psi_x \simeq \sigma_x^2/2$. The equation for a bunch centroid is obtained by averaging Eq. (1) with the density $\rho_n(\vec{r},s) = N_b\rho(x)\rho(y-y_b(s,z_n))\rho(z-(s-z_n))$ of the n-th bunch. The following relation simplifies the result

$$\quad {}_0^{\infty} \frac{d\lambda}{[1+2\lambda^2\sigma_y^2]^{3/2}} e^{-\frac{\lambda^2 y^2}{1+2\lambda^2\sigma_y^2}} = \frac{1}{\overline{2}\sigma_y} \quad {}_0^{1} d\eta e^{-\frac{\eta^2 y^2}{2\sigma_y^2}}. \quad (8)$$

Averaging Eq. (7) gives

$$< \frac{\partial U(\vec{r},s)}{\partial y} >= -\frac{2}{\sigma_x\sigma_y} \quad \frac{\overline{2}}{3}\frac{dN_i}{dz} \quad {}_{k,z_k<z_n} \quad dy'[y_b(s,z_n) - Y_k(y', s-z_n, s)]$$

$$\rho[y' - y_b(s - z_n + z_k, z_k)] \quad {}_0^{1} d\eta e^{-\frac{\eta^2}{2\sigma_y^2}[y_b(s,z_n)-Y_k(y',s-z_n,s)]^2}. \quad (9)$$

Here $y_b(s - z_n + z_k, z_k)$ is the offset of the k-th bunch at the moment $s_k = z + z_k$ when it generates ions at the same location $z = s - z_n$, where the n-th bunch is situated at the moment s.

Define the position $\hat{Y}_m(s, z_0)$ of the group of ions generated by the m-th bunch at the location z_0

$$\hat{Y}_m(s, z_0) \equiv Y_m(y_b(z_0 + z_m, z_m), z_0, s) \quad (10)$$

with the initial condition $\hat{Y}_m(z_0+z_m, z_0) = y_b(z_0+z_m, z_m)$ at $s = z_0+z_m$. Equation

(1) gives after averaging

$$\frac{\partial^2 y_b(s,z_n)}{\partial s^2}+\omega_b^2 y(s,z_n) = -\kappa s_b \quad [y_b(s,z_n)-\hat{Y}_k(s,s-z_n)] \quad d\eta e^{-\frac{\eta^2}{2\sigma_y^2}[y_b(s,z_n)-\hat{Y}_k(s,s-z_n)]^2}. \tag{11}$$

$$k, z_k<z_n \qquad 0$$

Here

$$\kappa = \frac{4r_e}{3\gamma s_b}\frac{dN_i}{ds}\frac{1}{\sigma_x\sigma_y}. \tag{12}$$

Note that parameter K of the paper[1] is $K = \kappa z_0$ where z_0 is the total length of the bunch train.

In the linear approximation $|y_b - Y_k| << \sigma_y$, we can take the dependence of the trajectory $Y_k(y,z,s)$ on the initial condition y in the form $Y_k(y,z,s) = y\cos\psi_y + Y_0(s)$ and obtain

$$< \frac{\partial U}{\partial y} >= -\frac{4}{3}\frac{1}{\sigma_x\sigma_y}\frac{dN_i}{dz} \quad [y_b(s,z_n) - \hat{Y}_k(s,s-z_n)]. \tag{13}$$

$$k$$

In another extreme case $\sigma_x >> |y_b - Y_k| >> \sigma_y$, Eqs. (8), and (9) give

$$< \frac{\partial U}{\partial y} >= -\frac{2}{\sigma_x} \quad \frac{\overline{\pi}}{3}\frac{dN_i}{dz} \quad \mathrm{sign}[y_b(s,z_n) - Y_k(y_{nk}, s-z_n, s)]. \tag{14}$$

$$k$$

The force Eq. (11) depends on a sign of $(y_b - Y_k)$.

The motion of ions at the location z_0 can be considered similarly. The ion centroid is described by the equation

$$\frac{\partial^2 \hat{Y}_m(s,z_0)}{\partial s^2} = -\omega_y^2 e^{-\frac{x_m^2(s,z_0)}{2\sigma_x^2}}[\hat{Y}_m(s,z_0) - y_b(s,s-z_0)] \quad d\eta e^{-\frac{\eta^2[\hat{Y}_m(s,z_0)-y_b(s,s-z_0)]^2}{2\sigma_y^2}}. \tag{15}$$

$$0$$

Equations (10), (11), and (15) can be generalized for the case when ions are not cleared out after one turn. A centroid of ions, generated by the m-th bunch on

the $l \geq 0$-th turn, is described by $\hat{Y}_m^l(s, z_0) \equiv Y_m(y_b(z_0 + z_m + lC, z_m), z_0, s)$ with the initial condition $\hat{Y}_m^l(z_0 + z_m + lC, z_0) = y_b(z_0 + z_m + lC, z_m)$ at $s = z_0 + z_m + lC$. Equations (11), (15) take the form

$$\frac{\partial^2 y_b(s, z_n)}{\partial s^2} + \omega_b^2 y(s, z_n) = -\kappa s_b \sum_{k,l} [y_b(s, z_n) - \hat{Y}_k^l(s, s_C - z_n)] \int_0^1 d\eta e^{-\frac{\eta^2}{2\sigma_y^2}[y_b(s,z_n) - \hat{Y}_k^l(s, s_C - z_n)]^2},$$

$$\frac{\partial^2 \hat{Y}_m^l(s, z_0)}{\partial s^2} = -\omega_y^2 e^{-\frac{x_m^2(s,z_0)}{2\sigma_x^2}} [\hat{Y}_m^l(s, z_0) - y_b(s, (s - z_0)_C)] \int_0^1 d\eta e^{-\frac{\eta^2[\hat{Y}_m^l(s,z_0) - y_b(s,(s-z_0)_C)]^2}{2\sigma_y^2}}.$$

Here C is the circumference of the ring, $s_C = s \bmod (C)$, and the sum over k is taken over bunches with $z_k < z_n$.

Linear approximation

Equations (11) and (15) give in the linear approximation $|y_b(s, z) - Y| << \sigma_y$ the system of equations

$$\frac{\partial^2 y_b(s, z_n)}{\partial s^2} + \omega_b^2 y(s, z_n) = -\kappa s_b \sum_{k, z_k < z_n} [y_b(s, z_n) - \hat{Y}_k(s, s - z_n)],$$

$$\frac{\partial^2 \hat{Y}_m(s, z)}{\partial s^2} = -\omega_y^2 [\hat{Y}_m(s, z) - y_b(s, s - z)]. \tag{17}$$

The second of Eqs. (17) has a solution

$$\hat{Y}_m(s, z) = y_b(z + z_m, z_m) \cos[\omega_y(s - z - z_m)] + Y_m^0(s, z), \tag{18}$$

$$Y_m^0(s, z) = \omega_y \int_{z+z_m}^{s} ds' y_b(s', s' - z) \sin[\omega_y(s - s')].$$

Equations (17) and (18) show that effect of the ions is equivalent to a transverse wakefield.

The solution was found in the original paper[1] in power series

$$y_b(s,z) = A_0 \sum_m y^{(m)}(s,z), \quad y^{(m)} = \frac{1}{(m!)^2}(\frac{\kappa\omega_y s z^2}{8\omega_b})^m \cos(\omega_b s - \omega_y z + \psi_0). \quad (19)$$

Indeed, substitution of the term $y^{(m)}(s,z)$ in the right hand side (RHS) of Eq. (17) gives the next term $y^{(m)}(s,z)$ provided the small terms of the order of $(\omega_y z/2m)^{-1}$, and $(\omega_b s/2m)^{-1}$ can be neglected. These conditions, compared to the original paper[1], include the additional factor m. From the expansion in Eq. (19), it is easy to see that significant $m \simeq \sqrt{\frac{\kappa\omega_y s z^2}{8\omega_b}}$ are large, and neglected terms are small provided

$$\sqrt{s/s_{eff}} << 1, \quad \frac{1}{s_{eff}} = \frac{\kappa}{2\omega_b\omega_y}, \quad z << \frac{2\omega_b^2}{\kappa}\sqrt{s/s_{eff}}. \quad (20)$$

Under these conditions, Eq. (18) gives the result[1]

$$y_b(s,z) = A_0 I_0(\omega_y z \sqrt{s/s_{eff}}) \cos(\omega_b s - \omega_y z + \psi_0). \quad (21)$$

The solution grows exponentially as $exp[\omega_y z \sqrt{s/s_{eff}}]$. This expression depends only on two parameters: number of bunches in the train n_b, and characteristic time s_c

$$\frac{y_b}{a_0} \simeq e^{(n/n_b)\sqrt{s/s_c}}, \quad \frac{1}{s_c} = \frac{(\omega_y s_b n_b)^2}{s_{eff}}. \quad (22)$$

For larger $s >> s_{eff}$, the solution can be found in series

$$y_b(s,z) = \frac{A_0}{s} \sum_m \frac{1}{m!}(\frac{\kappa s z}{2\omega_b})^m \cos(\omega_b s - \omega_y z + \frac{m\pi}{2}) \quad (23)$$

giving oscillatory dependence on time

$$y_b(s,z) = A_0 \frac{s_{eff}}{s} \cos(\omega_b s - \omega_y z + \frac{\kappa s z}{2\omega_b}). \quad (24)$$

This solution describes oscillations with the amplitude modulated in time s with frequency $2\omega_b/\kappa z$.

The PEP-II HER parameters are: $E = 9$ GeV, pressure 5 nTorr, $\sigma_x = 0.1$ cm, $\sigma_y = 0.02$ cm, bunch spacing $s_b = 1.24$ m, number of bunches $n_b = 1650$, $N_b = 8.3 \times 10^{10}$ (at the average beam current $I_b = 3$ A), and $\omega_b = 0.07\ m^{-1}$ (for $\nu_b = 23.5$). That gives $dN_i/ds = 24.9\ cm^{-1}$, $\omega_y = 0.96 \times 10^{-2}/\ \overline{A}\ cm^{-1}$, $\kappa = 2.1 \times 10^{-15}\ cm^{-3}$, $s_{eff} = 1.2 \times 10^9$ cm for $A = 28$. In this case z_{max} is larger than the ring circumference. Clearly, the regime shown in Eq. (24) can not be achieved, and the exponential growth of Eq. (22) can be stopped only by nonlinear effects.

The nonlinear regime

Exponential growth, as it was noticed in the original paper[1], tends to saturate at large amplitudes. Let us consider this regime in more details. Eq. (17) shows that Y_k grows faster than a bunch centroid y_b. Ions can be cleared out by a gap in the bunch train, but memory is retained in the beam, and the amplitude of ions is determined by the amplitude of bunch centroids. Motion becomes substantially nonlinear when the amplitude growth, Eq. (21), changes the ion frequency $\omega'(a/\sigma_y)^2 \simeq 1$ where the nonlinearity of ion oscillations $\omega' = \partial\omega_y/\partial(a/\sigma_y)^2$ on the order of $\omega' \simeq 0.1\omega_y$. This provides criterion for the time when transition to the nonlinear regime for the last n_b-th bunch takes place:

$$\frac{\overline{s}}{s_c} \equiv (\omega_y s_b n_b) \quad \frac{\overline{s}}{s_{eff}} \simeq \ln(\frac{3\sigma_y}{a_0}). \tag{25}$$

Clearly, the exponential regime can be expected for $s >> s_c$ only if the initial amplitude a_0 is small.

In the substantially nonlinear regime when amplitudes of ions and electrons are large, $|y_b - \hat{Y}_k| \geq \sigma_y$, the equations of motion are

$$\frac{\partial^2 y_b(s, z_n)}{\partial s^2} + \omega_b^2 y_b(s, z_n) = - \quad \frac{\overline{\pi}}{2}\kappa\sigma_y s_b \quad {}_k^{n-1} \operatorname{sign}[y_b(s, z_n) - \hat{Y}_k(y_b(s, s - z_n)], \tag{26}$$

$$\frac{\partial^2 \hat{Y}_m(s, z_0)}{\partial s^2} = - \quad \frac{\overline{\pi}}{2}\sigma_y\omega_y^2 \operatorname{sign}[Y_m(s, z_0) - y_b(s, s - z_0)]. \tag{27}$$

The analysis of these equations is complicated. To achieve a qualitative result, we proceed as follows. Take $y_b(s, z_m) = a(z_m)\cos\Phi(s, z_m)$ where, by analogy with the linear case,

$$\Phi(s, z_m) = \omega_b s - (\omega_b + \omega_y)z_m + \mu s. \tag{28}$$

Note, that

$$\Phi(s, s - z_0) = -\Psi(s, z_0) \equiv= -[\omega_y s - (\omega_b + \omega_y)z_0 - \mu s]. \tag{29}$$

Hence, $y_b(s, s - z_0) = a(s - z_0)\cos\Psi(s, z_0)$. The RHS of Eq. (27) is

$$- \quad \frac{\pi}{2}\sigma_y\omega_y^2 \mathrm{sign}[Y_m(s, z_0) - a(s - z_0)\cos\Psi(s, z_0)].$$

We assume

$$Y_m(s, z_0) = y_b(z_0 + z_m, z_m) + A_m(z_0)\cos[\Psi(s, z_0) + \xi_m]\,, \tag{30}$$

where the first term corresponds to the initial condition $Y_m(z_0 + z_m, z_0) = y_b(z_0 + z_m, z_m)$ for ions generated by the m-th bunch at location z_0 of the ring, and the second term describes oscillations of these ions induced by kicks of the following bunches. The argument of the *sign* function oscillates with s with frequency ω_y. The main Fourier harmonics of the *sign* function oscillates with the same frequency, confirming the choice of the solution shown in Eq. (29). From Eq. (27) it follows that $A_m \simeq \sigma_y$, but the phase ξ_m depends on the relation between amplitudes of different bunches.

Substitute now $Y_m(s, s - z_m)$ in Eq. (26) using

$$Y_m(s, s - z_m) = y_b(s, z_m) + \sigma_y\cos[\Phi(s, z_n) - \xi_m]. \tag{31}$$

The RHS of Eq. (26) takes the form

$$- \quad \frac{\pi}{2}\kappa\sigma_y s_b \quad {}^{n-1}_{k} \quad \mathrm{sign}[a(z_n)\cos\Phi(s, z_n) - a(z_k)\cos\Phi(s, z_k) - \sigma_y\cos(\Phi(s, z_n) - \xi_m)]. \tag{32}$$

It oscillates with the frequency ω_b in the resonance with the betatron oscillations.

The solution of Eq. (26) grows linearly with time

$$y_b(s, z_n) = a_n \cos \Phi(s, z_n) - (\frac{\kappa \sigma_y s_b}{2\omega_b}) s d_n \sin \Phi(s, z_n) \tag{33}$$

where coefficients d_n depend again on the relation between the amplitudes of different terms in the sum of Eq. (32). The result, Eq. (33), can be written in the form Eq. (28) with the frequency shift $\mu = \kappa \sigma_y s_b/(2\omega_b)$, making analysis self-consistent. Note that the frequency shift μ kills resonance and stops amplitude growth but it happens only at very large numbers of turns.

Depending on the correlation of signs of different terms in Eq. (32), the amplitude squared is proportional to n or to n^2. Such uncertainty translates in the computer simulations to a non-monotonic growth of an amplitude for different bunches with the bunch number: the amplitudes of different bunches vary by several orders of magnitude and a bunch with a larger number does not necessarily have a larger amplitude. Assuming random signs of different terms in Eq. (32), we get the estimate for the time dependence of the amplitude of the n-th bunch when the amplitude is large

$$< [\frac{y_b(s, z_n)}{\sigma_y}]^2 > \simeq (\frac{s}{s_{eff}})^2 (\omega_y s_b)^2 n. \tag{34}$$

As discussed below, this estimate qualitatively agrees with numerical simulations, although it does not predict rather random variation of amplitude with time for individual bunches.

Because full scale modeling is time consuming, a simple code was used to model the instability based on Eqs. (17), (26), and (27). Equation (34) is compared below with the code to demonstrate that the estimate in Eq. (34) is consistent with Eqs. (17), (26), (27). Equation (34) is also compared with Raubenheimer's tracking simulations.

A train of bunches was modeled in the code as a train of macroparticles, a single macroparticle per bunch. Bunches generate a macroparticle representing ions at

all locations around the ring separated by a bunch spacing. Ions are generated by all bunches, which pass a given location and then are retained at this location for a specified number of turns, or until their amplitude is less than a certain aperture ($3 \times 10^3 \sigma_y$ was used in the simulations). The interaction between ions and bunches is described as a kick to ions and bunches proportional to the difference of the positions of the microparticles if the difference is smaller than σ_y, inversely proportional to the difference if the difference is larger than $10\sigma_y \simeq \sigma_x$, and as a constant kick dependent only on the sign of the difference in other cases. Variation of the ion frequencies around the ring and variation of the betatron frequencies of the individual bunches were optional.

The results of the simulations with this code are shown in Figs. 1-5. The train of $n_b = 80$ macroparticles moves in the 130 rf bucket long ring. Bunch spacing is $s_b = 0.42$ m, the ion frequency ω_y corresponds to $\omega_y s_b/c = 1.55$, the interaction is defined by $s_{eff} = 2.35 \times 10^5$ m, and characteristic time s_c, defined by $1/s_c = (\omega_y s_b n_b)^2/s_{eff}$, is $s_c = 15.3$ m. The initial amplitude is not zero only for the first bunch $y_1(0)/\sigma_y = 10^{-6}$.

Under this conditions, the linear theory predicts that the amplitude of the n-th bunch grows to $(a_n/\sigma_y)^2 = 0.1$ after $s = s_n = 2450(n_b/n)^2$ m. Fig. 1 shows the result of the simulations where all ions are cleared out after each turn and no frequency spread is included both for ions and bunches. The results qualitatively reproduce the main features of the exact simulations[1][2]. They give an exponential growth in agreement with linear theory, and show that exponential growth slows down at large amplitudes. The behavior predicted by Eq. (34) for the same bunches is superimposed with the results of the code in the lower part of Fig. 1. The curves obtained without any fitting confirm that the estimate Eq. (34) derived from Eqs. (26) and (27) agrees reasonably well with the code describing the same equations.

Fig. 2 is calculated with the same conditions but ions were retained in 10 turns rather than cleared out after the first turn. Results are similar but behavior of different bunches is much less systematic than in the first case. Quadratic

dependence on s is seen more clearly if results are plotted in the usual (not semi-logarithmic) scale, Fig. 2b. Note that the amplitude of the first bunch grows because the ion are not cleared out completelly by the gap. However, the amplitude of the first bunch remains smaller than that for the last bunch in the train by two orders of magnitude.

The effect of the frequency spread of ions was studied by G. Stupakov[2]. Fig. 3 shows the variation of the amplitude with time for the same parameters but ion frequency is varied periodically five times around the ring with 10% amplitude. The ions are cleared out in one turn.

Fig. 4 is calculated under the same conditions but betatron frequency is increased from bunch to bunch monotonically, increasing totally by 5% along the bunch train.

Fig. 5 compares Eq. (34) with the Raubenheimer tracking code. Parameters of the NLC damping ring were used: $E = 2GeV$, $n_b = 90$, $A = 28$, $N_b = 1.5 \times 10^{10}$, pressure 10^{-7} Torr, $s_b = 0.42$ m, $\beta_y = 8$ m, $\sigma_x = 42\mu m$, $\sigma_y = 7.7\mu m$. That gives $s_c = 0.82$ m, $s_{eff} = 1.3 \times 10^4$ m, and $\omega_y s_b = 1.4$. Agreement is reasonable.

Raubenheimer's numerical simulations for the ALS ring ($\omega_y s_b = 0.5$) give $s_{eff} = 1.7 \times 10^4$ m in good agreement with the linear theory for this machine. If this regime continued, the amplitude of the last bunch in the train of 150 bunches would grow to $y_b/\sigma_y \simeq 5 \times 10^9$ at $s = 1500$ m. The simulations show that this quantity is in the range of 0.1 to 10.0 for different bunches in the nonlinear regime after $s = 1500$ m. The estimate Eq. (34) for the last bunch is 0.3. Note that the range of the amplitudes for different bunches agrees with linear dependence on n in Eq. (34).

For parameters of PEP-II, $s_{eff} = 1.2 \times 10^7$ m, $\omega_y s_b = 0.22$, and $n_b = 1658$, equation (34) gives $(y_b/\sigma_y) = 7.5 \times 10^{-7}(s/m) \quad \overline{n/n_b}$, or $y_b = \sigma$ in 4.5 msec for the last $n_b = 1650$ bunch. This rate is smaller than the PEP-II damping time but can be managed with the bunch-by-bunch feedback system.

We can estimate the amplitude, at which the transition from the exponential regime of Eq. (22) to the linear regime of Eq. (34) takes place by calculating s at which both formulas give the same result:

$$\frac{\overline{s}}{s_{eff}}(\omega_y s_b n) = \ln[(\frac{\sigma_y}{a_0})(\frac{s}{s_{eff}})(\omega_y s_b) \quad \overline{n}]. \tag{35}$$

The break point amplitude for the n-th bunch is

$$(\frac{y}{\sigma_y}) \simeq \frac{\ln^2(\sigma_y/a_0)}{4\omega_y s_b n^{3/2}}. \tag{36}$$

It is very small in the tail of the train, indicating that tail of bunches is always described by Eq. (34). In the head of the train, the break point amplitude can be large but time Eq. (35) in this case is large too, and growth can be stopped by a feedback system.

Conclusion

Ion-induced fast transverse instability is constrained by nonlinear effects. Nonlinear effects stop exponential growth of the amplitude at fractions of the rms bunch size. At large s the exponential growth described in the papers[1] [2] is replaced by the much slower linear growth of the amplitude. An estimate of the growth rate in this case is given by Eq. (34). The estimate is qualitatively consistent with numerical simulations. Instability at PEP-II can be controlled by the bunch-by-bunch feedback system with damping time of less than 4.5 ms.

Acknowledgement

I thank T. Raubenheimer for kindly providing me with the results of his simulations. I also thank A. Chao, F. Zimmermann, and G. Stupakov for discussions.

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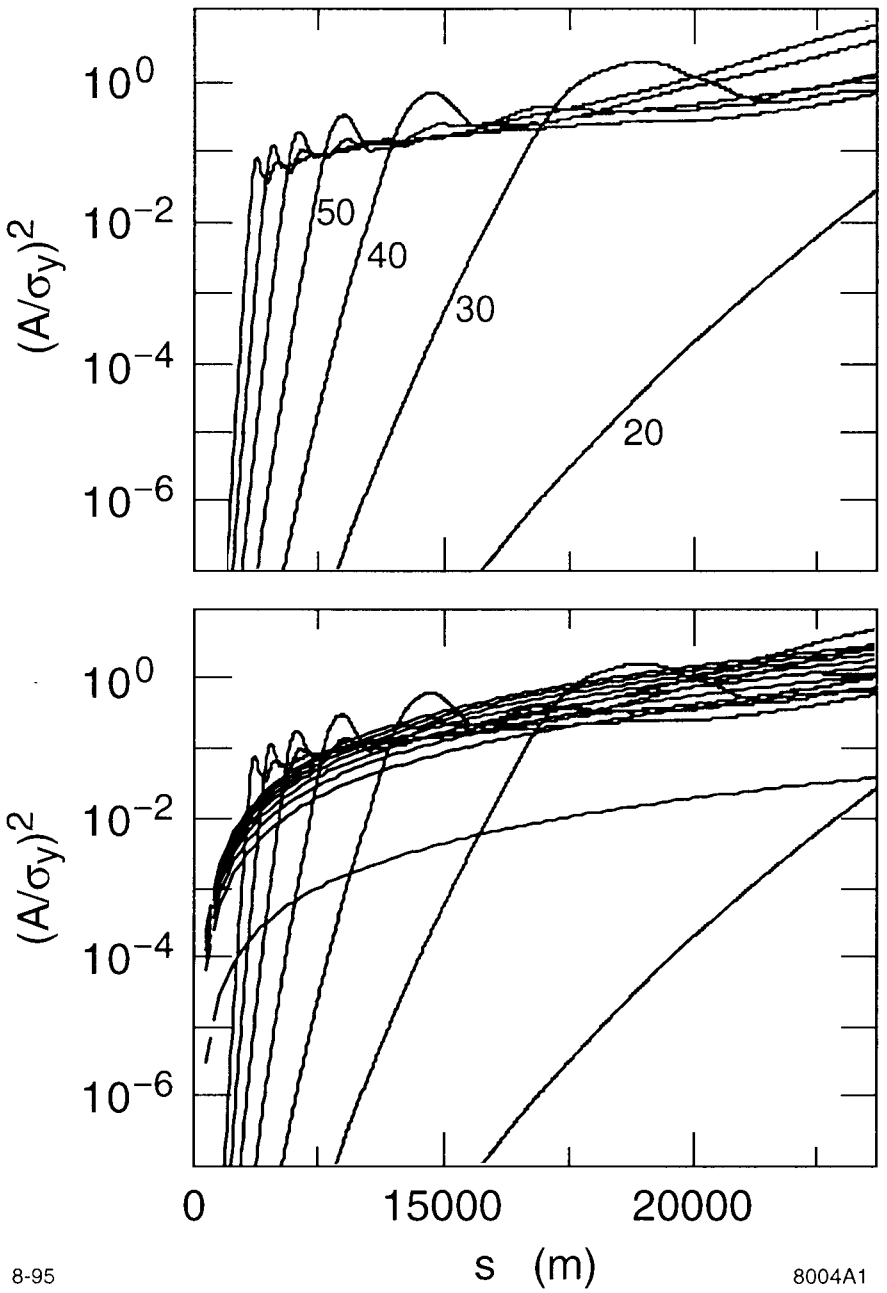
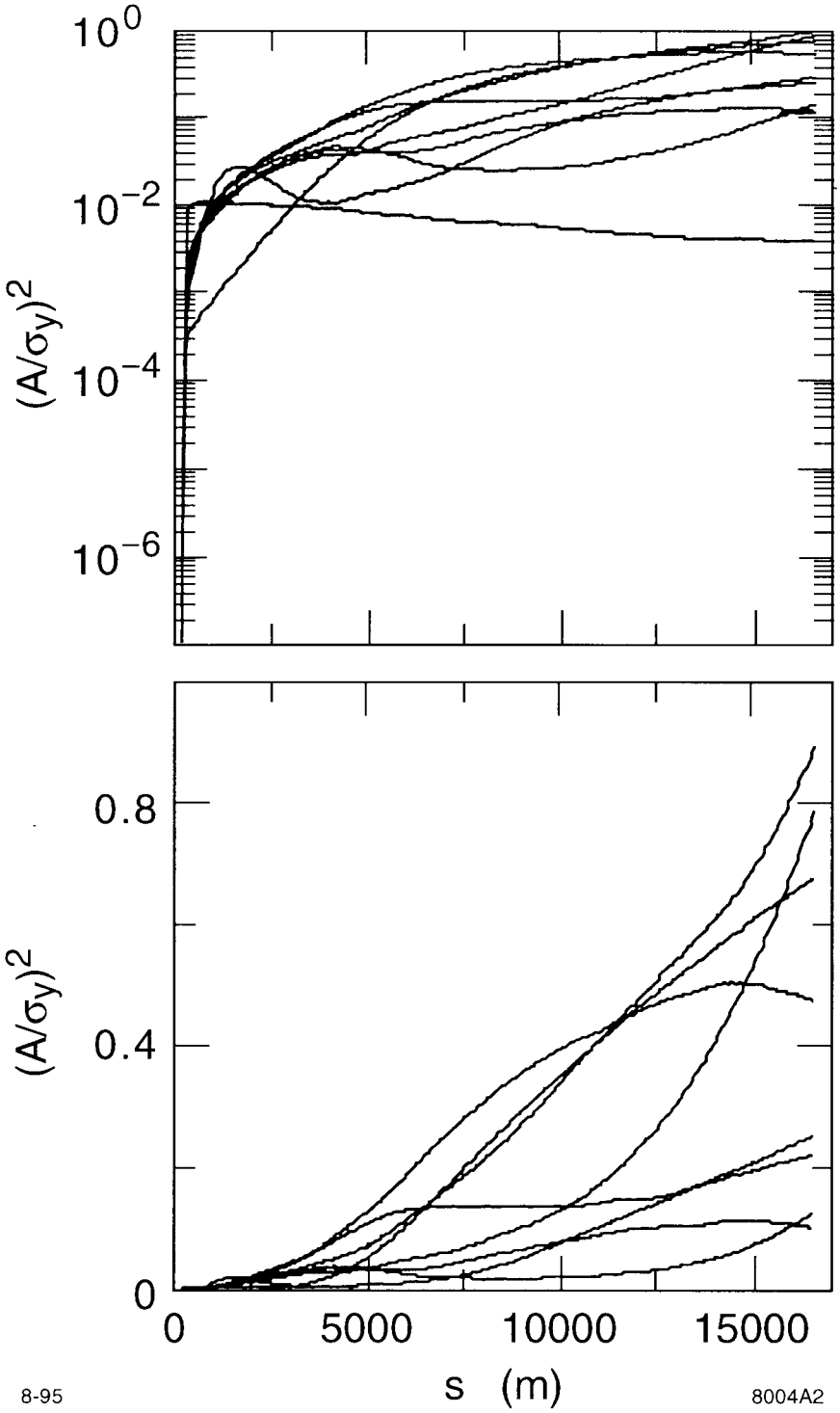
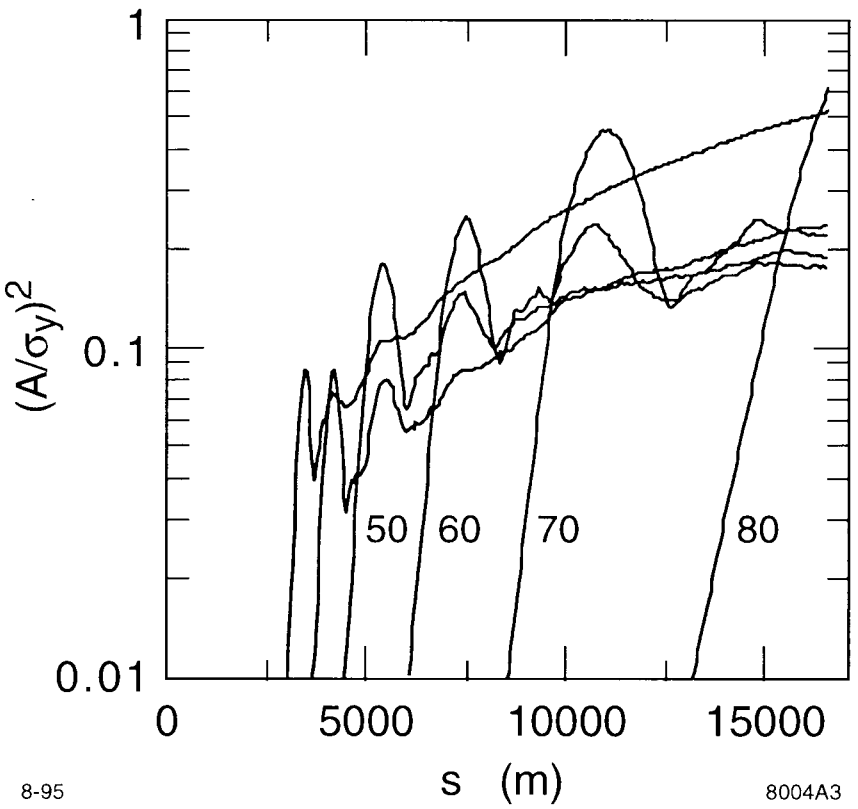


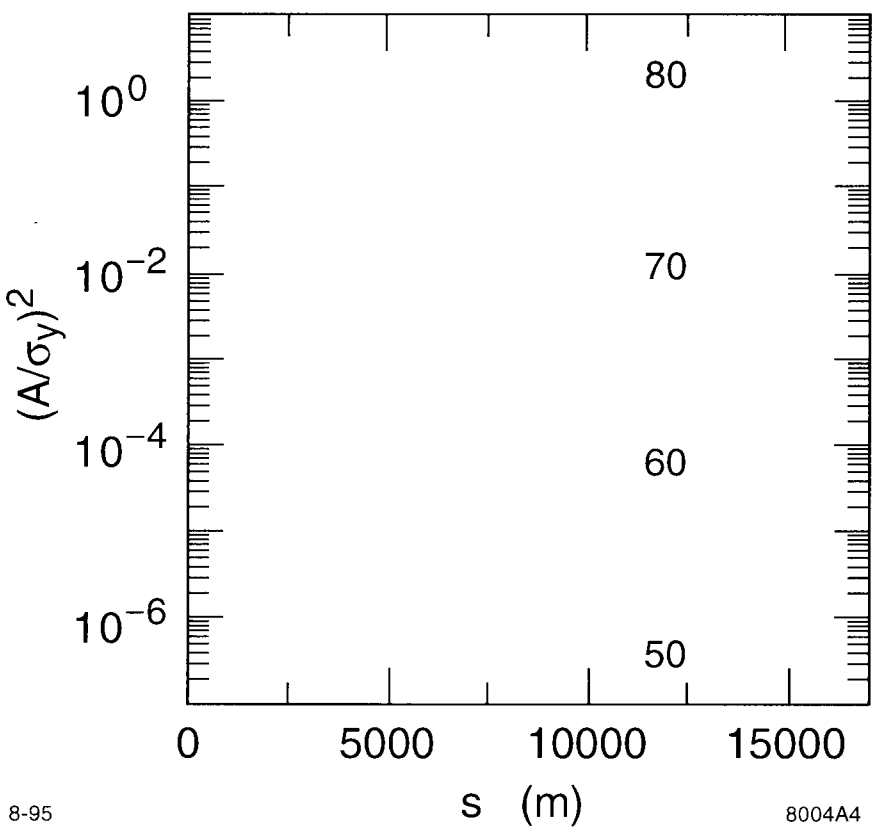
1) Comparison of the results of simulations (above) with Eq. (34). Lines are for different bunch numbers. All ions are cleared out after each turn and no frequency spread is included both for ions and bunches. Below: results of Eq. (34) are added, no additional parameters are used.



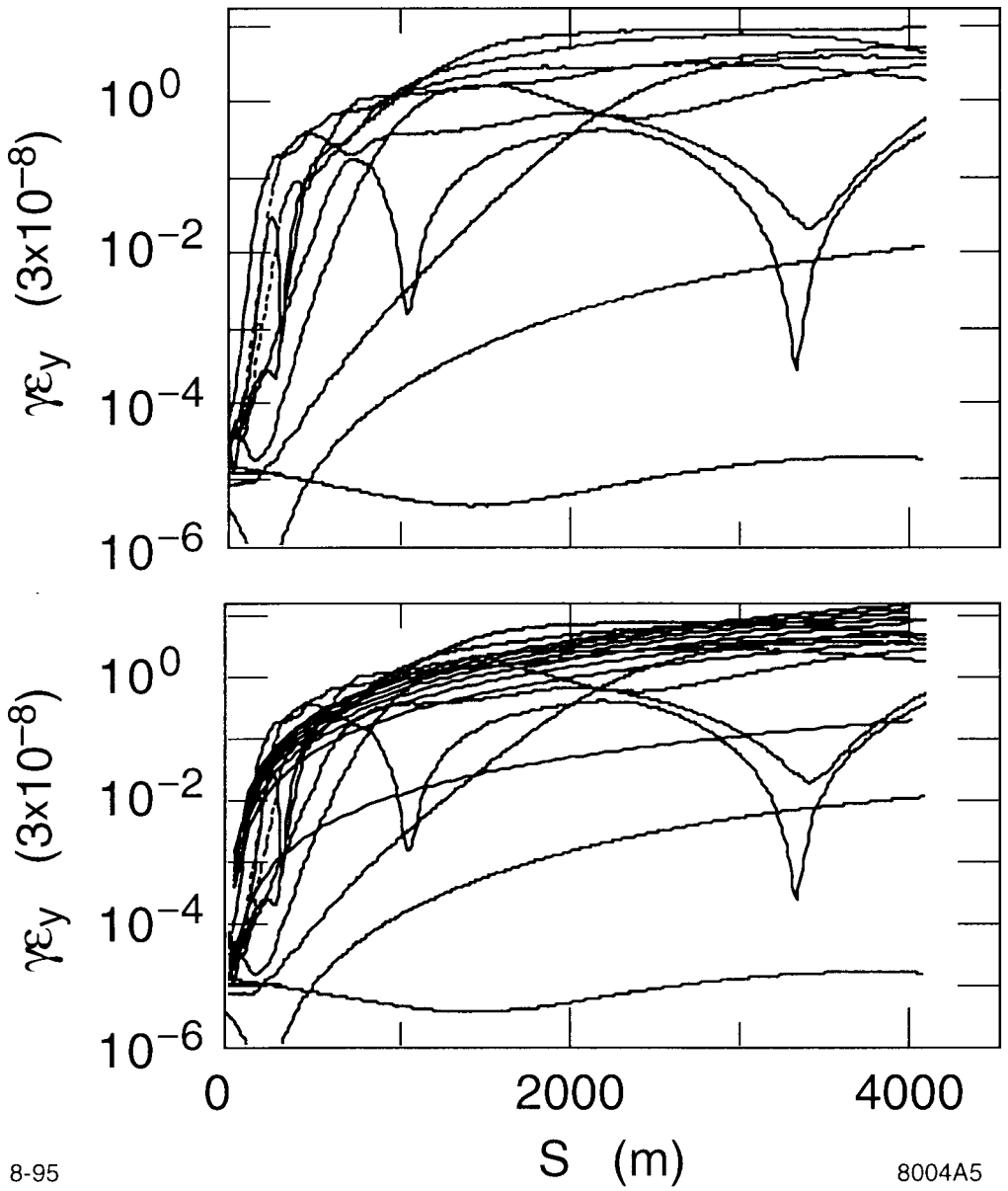
2) The same as Fig. 1, but ions are retained for 10 turns. The same data are plotted in logarithmic (above) and linear (below) scales.



3) Effect of ion frequency variation. The ions are cleared out in one turn.



4) Betatron frequency variation is included.



5) Comparison of Eq. (34) with the Raubenheimer's tracking code.