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A QUANTUM TREATMENT OF THE LANDAU–POMERANCHUK–MIGDAL EFFECT*

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ABSTRACT

In this letter the high-energy expansion for scattering from extended targets, which the authors previously applied to beamstrahlung radiation and pair production, is applied to the problem of radiation in a medium with multiple scattering. The suppression of the emission of long wave-length photons, the Landau-Pomeranchuk-Migdal effect, is treated and explained in physical terms. This extends previous classical treatments of the problem to the quantum domain and corrects certain approximations made in these earlier works; for example, the effects of finite target thickness can be treated. A model of a random scattering medium is defined that allows a quantum treatment of multiple scattering and the resultant suppression of bremsstrahlung radiation.

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Perhaps the most ubiquitous process occurring in high-energy physics is the bremsstrahlung of photons by a charged particle in the field of an atom first described by Bethe and Heitler [1]. Following experimental confirmation in 1993 of the Landau–Pomeranchuk–Migdal (LPM) effect [2-6], there is renewed interest in extensions of this process, as well as in its strong interaction analogue, gluon radiation at very high energies in heavy nuclei. We describe here the application of eikonal techniques developed for the beamstrahlung process [7] that lead to a simpler, more straightforward, and physically transparent quantum mechanical derivation of the LPM suppression of soft photon radiation from high-energy electrons in dense matter.

This effect was first described by Landau and Pomeranchuk [8] who treated the classical radiation of a high-energy particle in the fluctuating and 'random' field inside an infinitely thick medium. The longitudinal momentum transfer $q_{||}$ of a high-energy electron of momentum p and mass m, radiating a photon of momentum $k \equiv (1-x)p$, has a minimum value given by $q_{||}^{\rm min} = m^2(1-x)/2xp$. The uncertainty principle can be used to define the formation length $l_f = (1/q_{||}^{\rm min})$ which at high energies ($p >> m$) and soft photon emission $(1-x) << 1$ can grow quite large relative to the interatomic spacing.

In their classical derivation, which is appropriate to this kinematics limit, Landau and Pomeranchuk were the first to show that the familiar Bethe-Heitler radiated photon spectrum $dN \sim dk/k$ is modified by the multiple scattering of the electron as it traverses the rapidly varying electric fields of the medium. When the mean free path of the electron αL is comparable or less than the formation length l_f, they found that the spectrum is suppressed, ultimately achieving the form

$$dN \sim dk/(p\sqrt{Lk}) \ . \tag{1}$$

Subsequently, Migdal [9] presented a quantum mechanical derivation of this effect, treating multiple scattering via the Vlasov equation, and including the effects of electron spin and energy loss. His derivation contains a number of approximations that are formally difficult, not very transparent on physical grounds, and numerically not well controlled. These works have been extended by several authors [10].

The approach presented here is a simple application of the eikonal formalism previously developed for high-energy beamstrahlung processes, and has the advantage of greater generality and physical transparency. Aside from providing a simple and intuitive framework for more accurate studies of the LPM effect (including finite target thickness) our motivation is to provide a formalism that may be adapted to other problems, such as radiation by electrons transiting random magnetic domains and non-Abelian gluon radiation by quarks transiting heavy nuclei and undergoing multiple inelastic collisions.

The essential physics input in LPM leading to the behavior in Eq. (1) is the random scattering of the electron while transiting matter. We use the fact that the radiation length L is energy independent at high-energy, given for screened coulomb fields by

$$\frac{1}{L} = 4n\alpha\, r_e^2\, Z^2 \ln\left(\frac{183}{Z^{1/3}}\right) \ , \tag{2}$$

where $r_e = \alpha/m = 2.8\text{f}$ and n is the number density of target particles. The mean free path is of order αL. Hence, in traversing its path, the longitudinal momentum transfer due to multiple scattering of the electron increases to

$$
\begin{aligned}
q_{ms} &= \left[E - \frac{m^2 + (\delta p_\perp)^2}{2p}\right] - \left[k + xE - \frac{m^2 + (\delta p_\perp)^2}{2xp}\right] \\
&= \frac{k}{2xp^2}\ \left[m^2 + (\delta p_\perp)^2\right]
\end{aligned}
\qquad (3)
$$

$$
\text{with} \qquad (\delta p_\perp)^2 \sim \left| \int_0^{z/2} dz' \overrightarrow{E}_\perp(z') \right|^2 ,
$$

where $\overrightarrow{E}_\perp(z')$ is the random (from one electron to the next) atomic electric field that scatters the electron over its path of length $z/2$, the average for both the incident and scattered electron. Since it is of higher order, scattering of the photon due to density effects is excluded. The standard formula for multiple scattering by statistically independent atoms is

$$
(\delta p_\perp)^2 = \frac{1}{2}\, E_s^2\, \frac{z}{L}\ , \qquad E_s^2 = \frac{4\pi m^2}{\alpha} \sim (21 \text{ Mev})^2\ . \qquad (4)
$$

Identifying $q_{ms} \sim 1/z$ by the uncertainty principle, we obtain

$$
\frac{1}{z} \sim \frac{1}{l_f}\left[1 + \frac{1}{2}\,\frac{E_s^2}{m^2}\,\frac{z}{L}\right] , \qquad (5)
$$

so that in the Bethe-Heitler limit of no multiple scattering, $z_{BH} \sim l_f \propto 1/k$, whereas for strong multiple scattering, $z_{LPM} \sim \sqrt{l_f} \propto 1/\sqrt{k}$.

This simple argument, confirming (1), indicates that it is necessary for the $1/p$ corrections to be included in the eikonal treatment, since they contain the effects of scattering from the incident direction as shown in Ref. [7]. We now turn to some details of the formulation and calculation.

The appropriate eikonal wave functions for a spin-zero electron in the static field denoted by $V(z,b)$ are pure phases (see Ref. [7]), where (z,b) are the parallel and transverse coordinates respectively. These phases for the final (incoming) and initial (outgoing) wave functions are:

$$\Phi^f \;=\; \overrightarrow{p}^{\,f} \cdot \overrightarrow{r} + \tau_0(z,b) + \frac{1}{p^f}\;\tau_1(z,b)\;, \tag{6}$$

$$\text{where} \quad \tau_0(z,b) \;=\; \int\limits_z^\infty dz'\, V(z',b)$$

$$\text{and} \quad \tau_1(z,b) \;=\; \frac{1}{2}\int\limits_z^\infty dz' \left[\left(\overrightarrow{\nabla}_\perp\, \tau_0(z',b)\right)^2 \;+\; 2\, p_\perp^f \cdot \overrightarrow{\nabla}_\perp\, \tau_0(z',b)\right]\;;$$

$$\Phi^i \;=\; \overrightarrow{p}^{\,i} \cdot \overrightarrow{r} - \chi_0(z,b) - \frac{1}{p^i}\;\chi_1(z,b)\;, \tag{7}$$

$$\text{where} \quad \chi_0(z,b) \;=\; \int\limits_{-\infty}^z dz'\, V(z',b)$$

$$\text{and} \quad \chi_1(z,b) \;=\; \frac{1}{2}\int\limits_{-\infty}^z dz' \left[\left(\overrightarrow{\nabla}_\perp\, \chi_0(z',b)\right)^2 \;-\; 2\, p_\perp^i \cdot \overrightarrow{\nabla}_\perp\, \chi_0(z',b)\right]\;.$$

The initial momentum p_i will be oriented along the z axis. The relevant matrix element takes the form

$$M \;=\; ie \int dz \int d^2b \;\; \vec{\epsilon}^{\,*} \cdot \overrightarrow{P}(z,b) \quad \exp[\, i\Phi_{\rm tot}(z,b)\,]\;, \tag{8}$$

where

$$\overrightarrow{P}(z,b) \;=\; \overrightarrow{\nabla}\; (\Phi_i + \Phi_f) \qquad \text{and} \qquad \Phi_{\rm tot}(z,b) \;=\; \Phi_i - \Phi_f - k\cdot r\;.$$

Our model for the random medium is defined by

$$V(z,b) \;=\; -\, b \cdot E_\perp(z) \;, \tag{9}$$

which incorporates the physical assumption that the scattering is independent of the electron's impact parameter b in the medium, but the transverse field $E_\perp(z)$ varies with depth z as in Eq. (3). Each electron enters the surface at a different point, and therefore sees very different atomic electric fields as it traverses the medium. The variations in $E_\perp(z)$ from electron to electron are expressed by the statistical average over the incident electron ensemble [11]

$$\langle E_\perp(z_2) \cdot E_\perp(z_1) \rangle \;=\; \frac{\langle p_\perp^2 \rangle}{L}\, \delta(z_2 - z_1) \;, \tag{10}$$

in accord with the classical result, Eqs. (3) and (4); $\langle p_\perp^2 \rangle$ is the average transverse momentum acquired over the distance L.

For convenience, we introduce the transverse momentum acquired in traversing the medium from z_1 to z_2, up to z, and the total

$$\Delta_\perp(z_2, z_1) \;=\; \int_{z_1}^{z_2} dz' E_\perp(z') \;, \quad \Delta_\perp(z) \;\equiv\; \Delta_\perp(z, -\infty) \;, \quad \text{and} \quad \Delta_\perp \;=\; \Delta_\perp(\infty) \;. \tag{11}$$

Combining Eqs. (7) through (11), and writing the total momentum transfer to the electron as $q_\perp = p_\perp^f + k_\perp$, we find $q_\perp = \Delta_\perp(\infty)$ and

$$M \;=\; ie(2\pi)^2\, \delta^2(q_\perp - \Delta_\perp) \int dz\; \vec{\epsilon}^{\,*} \cdot \overrightarrow{P}(z, 0)\; \exp[i\Phi_{\rm tot}(z, 0)] \;. \tag{12}$$

It now follows that

$$\sum_{\text{pol}} |M|^2 \;=\; 4\pi\alpha A(2\pi)^2\delta^2(q_\perp - \Delta_\perp) \int_{-\infty}^{\infty} dz_1 dz_2 S(z_1,z_2)\; \exp\left[i \int_{z_1}^{z_2} dz\; \frac{d\Phi_{\text{tot}}(z,0)}{dz} \right] ,$$

$$S(z_2,z_1) \;=\; \frac{4}{(1-x)^2}\; [k_\perp - (1-x)(\Delta_\perp(z_2)] \cdot [k_\perp - (1-x)(\Delta_\perp(z_1)] \; ,$$

$$\frac{d\Phi_{\text{tot}}(z,0)}{dz} \;=\; \frac{1}{2x(1-x)p}\; \{m^2(1-x)^2 + [k_\perp - (1-x)(\Delta_\perp(z)]^2\} \; . \tag{13}$$

The frontal area of the target is denoted by A. The factor $[k_\perp - (1-x)(\Delta_\perp(z)]$ is proportional to the change in the local transverse velocity of the electron at z, the point of radiation of the photon, as seen by computing

$$v^i_\perp - v^f_\perp \;=\; \frac{1}{p}\,\Delta_\perp(z) - \frac{1}{xp}\, \left\{ p^f_\perp - \left[\Delta_\perp - \Delta^i_\perp(z)\right] \right\} \;=\; \frac{1}{xp}\; \left[k_\perp - (1-x)\Delta^i_\perp(z)\right] \; .$$

Finally, the probability that an electron incident upon the target will emit a photon of energy $k = (1-x)p$ is given from the above by

$$\frac{dP(x)}{dx} \;=\; \frac{1}{A}\,\frac{d\sigma}{dx} \;=\; \frac{1}{16\pi A p^2 x(1-x)} \int \frac{d^2k_\perp}{(2\pi)^2} \int \frac{d^2q_\perp}{(2\pi)^2} \sum_{\text{pol}} |M|^2 \; .$$

$$\;=\; \frac{\alpha}{2p^2x(1-x)} \int \frac{d^2k_\perp}{(2\pi)^2} \int_{-\infty}^{\infty} dz_2 \int_{-\infty}^{z_2} dz_1\, S(z_2,z_1)\; \cos\left[\int_{z_1}^{z_2} dz\, \frac{d\Phi_{\text{tot}}(z,0)}{dz} \right] . \tag{14}$$

It is a straightforward exercise in quadratures to evaluate Eq. (14); however some care is required in interchanging orders of integration and in dealing with the infinite limits on the two z integrations. Notice that Eq. (14) vanishes in the limit of no scattering in the target because $\Phi_{\text{tot}}(z,0)$ is linear in z, and the resultant delta functions cannot be satisfied. It is then straightforward to regulate the full integrals by subtracting this zero from Eq. (14). Since essentially all of the contribution to the emission probability arises from within the target, we can introduce the implied smooth cutoff factors in the z integrations, $\exp(-\epsilon|z|)$, and let $\epsilon \to 0$ at the end. A useful change of variables

$$K_\perp \; = k_\perp - (1-x)\left[\Delta_\perp(z_1) + \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} dz \Delta_\perp(z, z_1)\right] \tag{15}$$

recasts (14) into a function of only the magnitude $K_\perp^2$, which can then be readily integrated. Further interpretation of these variables, their physical significance, and calculational details will be given in a more complete write-up now in preparation.

The last step in evaluating (14) is to perform the statistical averages over the fluctuating fields in the media. We adopt the same approximation as Landau and Pomeranchuk by applying (10) separately to the quadratic forms in the field strengths in (13) and (14). The result for thick targets with $l >> l_f, \alpha L$ is given by

$$\frac{dP(x)}{dx} \;=\; \frac{2\alpha x l}{\pi(1-x)l_f} \int_0^\infty dy \; \frac{\exp(-\epsilon y)}{y} \; \left\{[1 + 3wyr(x)]\sin(y + wy^2) - \sin(y)\right\} \;, \tag{16}$$

where $w \;=\; \left\langle p_\perp^2 \right\rangle l_f/(6m^2 L)$, and the factor $r(x) \;=\; (1+x^2)/(2x)$ has been introduced to include the effect of the spin of a Dirac electron.

In the Bethe-Heitler limit, defined by $w \to 0$, this gives

$$\frac{dP_{BH}(x)}{dx} \;=\; \frac{2\alpha}{3\pi}\;\frac{x}{(1-x)}\;\frac{\left\langle p_\perp^2 \right\rangle l}{m^2 L}\;\left[\frac{3}{2}\,r(x) - \frac{1}{2}\right] \;, \tag{17}$$

which has the familiar form for radiation from a charge scattered through a squared transverse momentum transfer by a single scatterer, multiplied by the number of statistically independent scatterings, and including a spin factor that goes to one in the elastic limit. It can be verified that for scattering from a single thin electric field slab, i.e., with the replacement $V(z,b) = -b \cdot Q_\perp \delta(z)$, where $Q_\perp$ is the total transverse momentum imparted to a (spinless) charge at arbitrary energy, our procedure as outlined above leads directly to the familiar result

$$\frac{dP_{\rm single}(x)}{dx} = \frac{2\alpha}{3\pi} \frac{x}{(1-x)} \frac{Q_\perp^2}{m^2} \, . \tag{18}$$

The transition to the LPM regime occurs for w around 1; in the extreme LPM limit, where $w >> 1$,

$$\frac{dP_{LPM}(x)}{dx} = \frac{3\alpha x l}{2(1-x)} r(x) \sqrt{\frac{\langle p_\perp^2 \rangle}{3\pi m^2} \frac{1}{L l_f}} = l \, r(x) \sqrt{\frac{3\alpha x m^2}{4(1-x)pL}} \, , \tag{19}$$

which indeed is proportional to $1/\sqrt{k}$ in the soft photon limit, as first shown by Landau and Pomeranchuk [8].

We can compare our results in the soft photon limit $x \to 1$ with the original classical result of LP. We find that Eq. (17) is larger by a factor of 2 in the B–H limit. We have identified the source of this difference in an erroneous approximation made in that paper [12]. Equation (19) differs by a factor of $3/\sqrt{2}$ in the LPM limit from that given in Ref. [8].

In comparing their calculation with the standard analysis of the average radiation loss of electrons passing through matter, Landau and Pomeranchuk used Eq. (2) and set $\langle p_\perp^2 \rangle = 4\pi m^2/\alpha$, as is appropriate for a screened coulomb field. This gave them the well known result for the probability of photon emission

$$\frac{dP(x)}{dx} = \frac{4}{3} \frac{x}{(1-x)} \frac{l}{L} \, . \tag{20}$$

Equation (17) shows that when calculating the radiative loss as a result of a statistical ensemble of independent scatterers without identifying the one *hard* scattering (the Bethe-Heitler event), the appropriate choice for our model is $\langle p_\perp^2 \rangle = 2\pi m^2/\alpha$, in order to obtain the correct value (20).

The situation is somewhat more complicated in attempting to compare with Migdal's work because of the nature of his various approximations and the range of uncertainties in his final answer. He arrives at the same result as Eq. (20) in the B–H limit. In the LPM limit, we are in close agreement if a parameter that Migdal [5] estimates to lie between 1 and 2 is set to unity. With that choice, our answer is only slightly larger, by a factor of $\sqrt{3\pi/8} \sim 1.085$. Our advantage is in using modern eikonal techniques to describe the quantum process, and our main approximation is in the splitting of the statistical averaging that leads from Eq. (14) to Eq. (16).

We are completing a manuscript for publication that describes our calculation in more detail; in particular, it fills in the steps between Eqs. (14) and (16). This manuscript also includes extension of the results presented here to targets of finite thickness. When the target thickness l and l_f are comparable, we can still apply statistical averaging over random fields, as in Eq. (10), as long as $l > \alpha L$, the multiple scattering length or mean free path as defined in Eq. (2) [13]. The LPM limit will still be relevant as long as $w > 1$ or $l_f > \alpha L$. Experiments studying the effect of finite l have been performed [6]. Currently, further extensions of these techniques to the non-Abelian quark-gluon problem are in progress, including their relation to recent work by M. Gyulassy et al. [14] and R. Baier et al. [15].

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11. We have derived a generalization of this result for a randomly sited collection of Gaussian potentials. This extended formula permits a full quantum treatment of multiple scattering and introduces a target correlation length related to the range of the potential (manuscript in preparation).

12. In carrying out their classical evaluation, LP neglected certain derivative terms. These terms seem to be of higher order in the inverse of the formation length and should therefore be negligible; however, this is not the case, since the leading terms cancel. In the single thin slab problem, the LP formula yields one-half of the well-known result, Eq. (18).

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