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STUDY OF BEAM DECOHERENCE IN THE PRESENCE OF HEAD-TAIL INSTABILITY USING A TWO-PARTICLE MODEL*

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Abstract

The decoherence behavior of a beam centroid motion after a kick is studied using a two-particle model. A simple theory based on averaging of the governing equation is developed. The effects of a finite chromaticity and synchrotron motion are taken into account. Increasing the tune spread in the beam, a transition from a head-tail instability to a stable decay of an initial kick is explicitly demonstrated.

I. INTRODUCTION

When a bunched beam is kicked in a storage ring, it executes a betatron oscillation. If there is a spread in the betatron frequencies of the beam particles, it is well-known [1-4] that the centroid motion of the beam will decay in time as a result of decoherence among the oscilations of different particles. The rate of decoherence depends on the spread of the betatron frequencies.

In addition to this decoherence effect, the beam centroid motion after the kick is also affected by the collective effects if the beam is sufficiently intense [4-6]. The interplay between the decoherence and the collective effects was analyzed in Ref. 5, except that it neglected the effect of the head-tail instability by assuming a zero chromaticity. For a coasting beam, a similar problem has been treated in Ref. 6. In this note, we offer a bunched-beam analysis that includes the effect of a finite chromaticity using a simplified two-particle model of the beam. We obtain the time behavior of the beam centroid after the kick as a function of the frequency spread, the wake field strength, and the chromaticity. The results reduce to those of Ref. 5 when the chromaticity is set to zero. It is shown that by an appropriate transformation, the formalism of Ref. 5 for the case with zero chromaticity applies also to the case with finite chromaticity.

Our result demonstrates the transition from Landau-damped oscillations to instability. In particular, it gives explicitly the condition for the collective instability to be Landau damped.

II. GENERAL ANALYSIS

To study the interplay between the decoherence and the collective head-tail effects, we consider a simplified two-particle model in which the beam is modeled as two macroparticles interacting with each other through a wake field according to

$$y''_{1,2} + \frac{\omega_{1,2}^2}{c^2} y_{1,2} = \varepsilon h_{1,2} \bar{y}_{2,1}, \tag{1}$$

where y_1 and y_2 are the transverse offsets for the first and the second macroparticles respectively, and the prime designates the differentiation with respect to the longitudinal coordinate s. The betatron frequencies ω_1 and ω_2 in Eq. (1) for each particle are modulated due to the synchrotron motion,

$$\omega_{1,2} = \omega \left(1 \mp \xi \hat{\delta} \cos\left(\omega_s s/c\right)\right), \tag{2}$$

where ω is the unperturbed betatron frequency and $\hat{\delta} = \hat{z}\omega_s/c\eta$ with ξ designating the chromaticity parameter, ω_s the synchrotron frequency, η the momentum compaction factor and $\hat{z}$ the amplitude of synchrotron oscillations. We assume the two macroparticles execute their synchrotron oscillations according to $z_1 = -z_2 = \hat{z}\sin\left(\omega_s s/c\right)$. On the right hand side of Eq. (1) we have $\varepsilon = Nr_0W_0/2\gamma C$, where N is the number of particles in the bunch (each macroparticle contains $N/2$ particles); r_0 is the classical radius of the particle; W_0 is the wake function at $z = 0$ (in case when $W(z = 0) = 0$, W_0 is some characteristic value of W); γ is the relativistic factor; and C is the accelerator circumference. The function $h_1(s)$ accounts for the time variation of macroparticle positions: it is equal to $W\left(2\hat{z}|\sin\left(\omega_s s/c\right)|\right)/W_0$ for $z_1 < z_2$, and $h_1(s) = 0$ for $z_1 > z_2$. The function $h_2(s)$ differs from $h_1(s)$ in that z_1 is interchanged with z_2. Note that $h_1(s)$ ($h_2(s)$) is nonvanishing only when the first (second) macroparticle trails the other macroparticle.

We assume a frequency spread within each macroparticle which has a distribution $\rho(\omega)$ normalized so that $\int \rho(\omega)\, d\omega = 1$. The function $\rho(\omega)$ has a maximum at $\omega = \omega_0$ with a characteristic width $\Delta\omega_0 \ll \omega_0$. The functions $y_1 = y_1(s|\omega)$ and $y_2 = y_2(s|\omega)$ in Eq. (1) are, as a matter of fact, functions of two variables, s and ω, and the bar designates averaging over frequency,

$$\bar{y}_{1,2}(s) = \int y_{1,2}(s|\omega)\, \rho(\omega)\, d\omega. \tag{3}$$

We will be looking for solution of Eq. (1) in the following form [7]

$$y_{1,2}(s|\omega) = \tilde{y}_{1,2}(s|\omega) \exp\left(-i\omega_0 s/c \pm i\chi \sin\left(\omega_s s/c\right)\right), \tag{4}$$

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where $\chi = \xi\hat{\delta}\omega_0/\omega_s$ is the head-tail phase, and $\tilde{y}_{1,2}(s|\omega)$ is a slowly varying amplitude. Substituting Eq. (4) into Eq. (1) and neglecting small terms we have

$$-2i\frac{\omega_0}{c}\tilde{y}'_{1,2} + \frac{\omega^2-\omega_0^2}{c^2}\tilde{y}_{1,2} = \varepsilon h_{1,2}\hat{y}_{2,1}e^{\left(\mp 2i\chi\sin\left(\frac{\omega_s s}{c}\right)\right)}, \quad (5)$$

where

$$\hat{y}_{1,2}(s) = \int \tilde{y}_{1,2}(s|\omega)\,\rho(\omega)\,d\omega. \quad (6)$$

Assuming $\chi \ll 1$ we can expand the right hand side of Eq. (5) and average it over s. Also, because the frequency spread is assumed to be small, we have $\omega^2-\omega_0^2 \approx 2\omega_0\Delta\omega$, where $\Delta\omega = \omega-\omega_0$. We than have

$$\tilde{y}'_{1,2} + i\frac{\Delta\omega}{c}\tilde{y}_{1,2} = \frac{ic}{2\omega_0}\varepsilon(\alpha_1 + 2i\alpha_2\chi)\,\hat{y}_{2,1}, \quad (7)$$

where

$$\alpha_1 = \frac{\omega_s}{2\pi W_0 c}\int_0^{c\pi/\omega_s} W\left(2\hat{z}\sin\left(\frac{\omega_s s}{c}\right)\right)ds, \quad (8)$$

$$\alpha_2 = \frac{\omega_s}{2\pi W_0 c}\int_0^{c\pi/\omega_s} W\left(2\hat{z}\sin\left(\frac{\omega_s s}{c}\right)\right)\sin\left(\frac{\omega_s s}{c}\right)ds. \quad (9)$$

For a constant wake, $W(z) \equiv W_0$, we have $\alpha_1 = 1/2$, $\alpha_2 = 1/\pi$.

It is convenient to define the center of mass Y and the relative displacement D of the macroparticles so that

$$Y = \frac{1}{2}(\tilde{y}_1 + \tilde{y}_2), \quad D = \tilde{y}_1 - \tilde{y}_2. \quad (10)$$

This reduces Eq. (7) to a pair of decoupled equations,

$$Y' + i\frac{\Delta\omega}{c}Y = r\hat{Y}, \quad D' + i\frac{\Delta\omega}{c}D = -r\hat{D}, \quad (11)$$

where

$$\left\{\begin{array}{c}\hat{Y}(s)\\ \hat{D}(s)\end{array}\right\} = \int\left\{\begin{array}{c}Y(s|\Delta\omega)\\ D(s|\Delta\omega)\end{array}\right\}\rho(\Delta\omega)\,d\Delta\omega, \quad (12)$$

and $r = ic\varepsilon(\alpha_1 + 2i\alpha_2\chi)/2\omega_0$.

At this point, we note that averaging Eq. (5) actually assumes that the functions $\hat{y}_{1,2}$ vary on the time scale that is larger than the synchrotron period. From Eq. (11) we find that, due to the wake field, Y and D will be modulated with the frequency equal to $c\mathrm{Im}r$. Hence, we have to require $c^2\varepsilon\alpha_1/2\omega_0 \ll \omega_s$, as an applicability condition of our approach. It is worth noting that the ratio $\pi c^2\varepsilon\alpha_1/2\omega_0\omega_s$ is equal to the parameter Υ defined in [7], p. 180 (for a constant wake), that governs the strong head-tail instability. The above condition therefore implies that we are well below the threshhold of the strong head-tail instability.

We will first focus on the behavior of the centroid of the bunch and consider the first of Eq. (11). The analysis follows closely that of Ref. 5. Integrating the equation for Y we find,

$$Y = Y_0 e^{-i\Delta\omega s/c} + r\int_0^s \hat{Y}(s')e^{-i\Delta\omega(s-s')/c}ds', \quad (13)$$

where Y_0 is a constant equal to the initial value of Y at $s=0$. Averaging Eq. (13) according to Eq. (12) yields

$$\hat{Y} = Y_0 K(s) + r\int_0^s \hat{Y}(s')K(s-s')\,ds', \quad (14)$$

where $K(s)$ is the *decoherence function*

$$K(s) = \int e^{-i\Delta\omega s/c}\rho(\Delta\omega)\,d\Delta\omega. \quad (15)$$

Eq. (14) can now be solved by means of a Laplace transform. Defining

$$u(p) = \int_0^\infty \hat{Y}(s)\,e^{-ps}ds, \quad \kappa(p) = \int_0^\infty K(s)\,e^{-ps}ds, \quad (16)$$

the Laplace transform of Eq. (14) is

$$u(p) = Y_0\kappa(p) + r\kappa(p)\,u(p), \quad (17)$$

from which we find $u(p)$, and making inverse Laplace transform yields

$$\hat{Y}(s) = \frac{1}{2\pi i}Y_0\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\kappa(p)}{1-r\kappa(p)}e^{ps}dp. \quad (18)$$

Since Eq. (11) for D differs only by the sign of r, all our results for Y are also applicable to D upon the substitution $r \to -r$.

III. DISTRIBUTION FUNCTIONS AND INSTABILITY

We have thus solved formally the motion of the beam centroid after a kick. The amplitude of the beam centroid motion is described by $\hat{Y}(s)$ of Eq. (18) where Y_0 is the initial kick amplitude. The parameter r contains the wake field and the chromaticity information. The function $\kappa(p)$, given by Eqs. (15) and (16), contains the information of the betatron frequency spectrum of the beam. To proceed, we assume a Gaussian distribution function,

$$\rho(\omega) = \frac{1}{\sqrt{2\pi}\Delta\omega_0}\exp\left(-\frac{\Delta\omega^2}{2\Delta\omega_0^2}\right), \quad (19)$$

where $\Delta\omega_0$ is the rms width of the spectrum. Then

$$K(s) = \exp\left(-\Delta\omega_0^2 s^2/2c^2\right), \quad (20)$$

and

$$\kappa(p) = \frac{c\sqrt{\pi}}{\sqrt{2}\Delta\omega_0}\exp\left(\frac{p^2c^2}{2\Delta\omega_0^2}\right)\left[1-\mathrm{erf}\left(\frac{pc}{\sqrt{2}\Delta\omega}\right)\right], \quad (21)$$

where $\mathrm{erf}(x)$ is the error function. Defining the variable $\zeta = ipc/\Delta\omega_0$ and the function $w(\zeta) = -i\exp\left(-\zeta^2/2\right)\left[1-\mathrm{erf}\left(-i\zeta/\sqrt{2}\right)\right]$, we can rewrite Eq. (18) in the following form

$$\hat{Y}(s) = \frac{\Delta\omega_0}{2\pi rc}Y_0\int_C \frac{w(\zeta)\exp\left(-i\Delta\omega_0\zeta s/c\right)d\zeta}{w(\zeta) + i\sqrt{2}\Delta\omega_0/rc\sqrt{\pi}}, \quad (22)$$

where the integration goes along a straight horizontal line in the upper half plane of the complex variable ζ, above the singularities of the integrand. The function $w(\zeta)$ is an analytic function in the upper half plane of the complex variable ζ. To perform the integration we can shift the integration path down to the real axis of ζ. However, if the denominator in Eq. (22) has a root in the upper half plane, $\zeta = \zeta_0$, the integration path will have to encircle the corresponding pole, and the residue from the pole will give a contribution to $\hat{Y}(s)$ with the time dependence $\propto \exp(-i\Delta\omega_0\zeta_0 s/c)$. This implies an instability with the growth rate equal to $\Delta\omega_0 \mathrm{Im}\zeta_0$.

We can easily find the root of the denominator in Eq. (22) and obtain the condition for the stability assuming $|\Delta\omega_0/r| \ll 1$. In this limit, a solution to the equation $w(\zeta) = -i\sqrt{2}\Delta\omega_0/rc\sqrt{\pi}$ is [5]

$$\zeta = -\frac{\varepsilon\alpha_1 c^2}{2\omega_0\Delta\omega_0} - \frac{i\varepsilon\chi\alpha_2 c^2}{\omega_0\Delta\omega_0} - \frac{i\sqrt{\pi}\varepsilon^2\alpha_1^2 c^4}{\sqrt{2}\omega_0^2\Delta\omega_0^2}\exp\left(-\frac{\varepsilon^2\alpha_1^2 c^4}{2\omega_0^2\Delta\omega_0^2}\right). \tag{23}$$

For small $\Delta\omega_0$, the last term is exponentially small, and we can neglect it. The result will be a head-tail instability in the system of two macroparticles for $\chi < 0$ with the growth rate $\gamma_{inst} = \varepsilon\alpha_2 c^2 |\chi|/\omega_0$. The last term in Eq. (23) accounts for the Landau damping effect. It overcomes the second term and suppresses the instability if

$$|\chi|\,\alpha_2 < \frac{\sqrt{\pi}\varepsilon\alpha_1^2 c^2}{\sqrt{2}\omega_0\Delta\omega_0}\exp\left(-\frac{\varepsilon^2\alpha_1^2 c^4}{2\omega_0^2\Delta\omega_0^2}\right). \tag{24}$$

Eq. (22) has been integrated numerically in Ref. [5] for both stable and unstable regimes. The relevant plots can be found in that paper.

We will also consider the case when the tune spread is associated with the lattice nonlinearity so that the tune is $\nu = \nu_0 - \mu a^2$, where a is the ratio of the amplitude of the betatron oscillations to the rms width of the beam, and μ is a nonlinearity parameter. In this case, for a small amplitude oscillations of the centroid, the decoherence function has a form [1]:

$$K(s) = \frac{1}{(1 - is\Delta\omega_0/c)^2}, \tag{25}$$

where $\Delta\omega_0 = 2\mu\omega_{rev}$, and ω_{rev} is the revolution frequency. The Laplace transform of Eq. (25) yields

$$\kappa(p) = \frac{ic}{\Delta\omega_0}\exp\left(\frac{ipc}{\Delta\omega_0}\right)E_2\left(\frac{ipc}{\Delta\omega_0}\right), \tag{26}$$

where $E_2(x)$ is the exponential integral function [8]. Using the variable ζ we can rewrite Eq. (18) in the following form

$$\hat{Y}(s) = \frac{\Delta\omega_0}{2\pi rc}\int_C d\zeta \frac{\exp(\zeta - i\Delta\omega_0 s\zeta/c)\,E_2(\zeta)}{i\Delta\omega_0/rc - \exp(\zeta)\,E_2(\zeta)}. \tag{27}$$

This equation is similar to Eq. (22) in that it exhibits stabilization effect for sufficiently large $\Delta\omega_0$ due to Landau damping. We will demonstrate this in the next section for a particular example considered in Ref. 4.

IV. DECOHERENCE EFFECTS

As an example, we assume one set of parameters considered in Ref. 4: $N = 3\times10^{10}$, $\sigma_z = 6$ mm, $\nu_\beta = 8.18$, $\beta_x = 3$ m, $\gamma = 2350$, and a linear wake function,

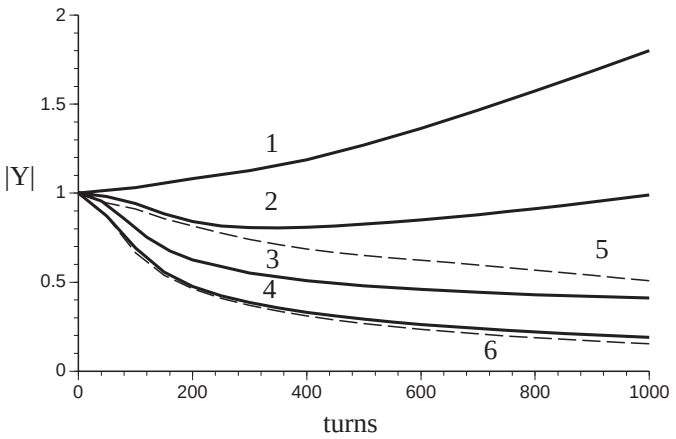

Figure 1. Plot of $|\hat{Y}(s)|$ for $\xi = -1$ (solid curves *1-4*) and for $\xi = 1$ (dashed curves *5* and *6*). The values of the μ are: *1* and *5* – $\mu = 5.3\times10^{-5}$, *2* – $\mu = 1.6\times10^{-4}$, *3* and *6* – $\mu = 2.7\times10^{-4}$, *4* – $\mu = 3.7\times10^{-4}$.

$W(z) = W_0 z$, with $W_0 = 2\times10^7\,\mathrm{m}^{-3}$. The amplitude of the synchrotron oscillation $\hat{z}$ is assumed to be $\hat{z} = \sqrt{2}\sigma_z$, and the betatron frequency $\omega_\beta = c/\beta_x$. This gives for the factor r, $r = 3.0\times10^{-5}(0.64i + 0.08\xi)$. The parameter $\Delta\omega_0$ can be expressed in terms of μ, $\Delta\omega_0/c = 8.14\times10^{-2}\mu\,\mathrm{m}^{-1}$.

The amplitude $|\hat{Y}(s)|$ of the beam calculated with the use of Eq. (27) is plotted in Fig. 1 for various values of μ for the unstable case $\xi = -1$. The critical value for μ that stabilizes the head-tail instability is 2.2×10^{-4}.

Figure 1 shows also a stable case, $\xi = 1$. In this case, increasing μ causes a faster decay of the initial kick.

As mentioned in Sec. II, the time behavior of $\hat{D}(s)$ is governed by the same equations as $\hat{Y}(s)$ with r substituted by $-r$. That means that $\xi = -1$ correspondes to stable oscillations of $\hat{D}(s)$, and $\xi = 1$ leads to the head-tail instability in the absence of the tune spread. Calculations show that for the parameters listed above, Landau damping stabilizes the instability of $\hat{D}(s)$ when $\mu > 6\times10^{-5}$.

References

[1] R.E. Meller, A.W. Chao, J.M. Peterson, S.G. Peggs, and M. Furman. "Decoherence of Kicked Beams," SSC–N–360, 1987.

[2] S.Y. Lee, "Decoherence of Kicked Beams II," SSCL–N–749, 1991.

[3] Ian C. Hsu, *Part. Acc.*, v. **34**, p. 43, 1990.

[4] M. Minty, W. Spence, and A. Chao, these proceedings.

[5] G.V. Stupakov and A.W. Chao, SSCL–621, (1993).

[6] M. Blaskiewicz. *Proc. 1993 Part. Accel. Conf.*, v. **5**, p. 3321, 1993.

[7] For example, see Alexander W. Chao. *Physics of Collective Beam Instabilities in High Energy Accelerators*. New York: Wiley, 1993.

[8] M. Abramovitz and I.A Stegun, *Handbook of Mathematical functions*, New York: Dover, 1968.