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COUPLING IMPEDANCE OF A LONG SLOT AND AN ARRAY OF SLOTS IN A CIRCULAR VACUUM CHAMBER

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Abstract

We find the real part of the longitudinal impedance for both a small hole and a long slot in a beam vacuum chamber with a circular cross section. The slot can be arbitrarily long; the only requirement on the dimensions of the slots is that its width be much smaller than c/ω. Regular array of N slots periodically distributed along the pipe is also considered.

I. INTRODUCTION

Existing theory for the impedance produced by small holes in the wall of a vacuum chamber of an accelerator has been developed in papers by Kurennoy [1] and Gluckstern [2]. Bethe's approach, developed to study diffraction of an electromagnetic wave on a perfectly conducting plane screen with a small hole [3], was applied to the problem of radiation of the beam propagating in a circular pipe with a hole in its wall. The method is based on utilization of small parameters, α_{el}/b^3 and α_{mg}/b^3, where α_{el} is the electric and α_{mg} is the magnetic polarizabilities of the hole, and b is the beam-pipe radius. For circular holes, $\alpha_{mg} \sim |\alpha_{el}| \sim w^3$, where w is the radius of the hole, and these ratios are of the order of $(w/b)^3$. This theory also assumes that the wavelength of the electromagnetic waves radiated by the hole is much larger than the dimensions of the hole. In the first approximation of the perturbation theory, the impedance is expressed in terms of polarizabilities α_{el} and α_{mg} and turns out to be purely imaginary: [1]

$$Z \;=\; -\frac{Z_0 i\omega}{2\pi c b^2}\left(\alpha_{el}+\alpha_{mg}\right)\ . \tag{1}$$

In many cases it is necessary to know the real part of the impedance. In this paper we find ReZ for small holes and slots of arbitrary length l, assuming only that the width of the slot w is much smaller than b and c/ω. We also find the impedance of a regular array of N slots. A more detailed study of relevant issues, including the effect of randomization of the slot positions in the array, can be found in Ref. [4].

II. REAL PART OF THE IMPEDANCE FOR A HOLE

To calculate the longitudinal impedance of a circular beam pipe with a hole, it is convenient to consider an oscillating current traveling with the velocity of light along the axis of the pipe,

$$I\left(z,t\right) \;=\; I_0\ \exp(-i\omega t + i\kappa z)\ , \tag{2}$$

where $\kappa = \omega/c$. The pipe is assumed to have a small hole located at $z = 0$ with characteristic dimensions much less than pipe radius b. Perturbation of the electromagnetic field caused by the hole can be represented as a superposition of the waveguide modes propagating away from the hole.

We choose normalization of the eigenmodes in a circular pipe such that for E modes

$$E_z^{(n,m)} = \frac{\mu_{n,m}^2}{b^2} J_n\left(\mu_m \frac{r}{b}\right)\cos(n\theta) exp(\sigma i\kappa_{n,m} z), \tag{3}$$

and for H modes

$$H_z^{(n,m)} = \frac{\mu_{n,m}'^2}{b^2} J_n\left(\mu_{n,m}' \frac{r}{b}\right)\cos(n\theta) exp(\sigma i\kappa_{n,m}' z), \tag{4}$$

where J_n is the Bessel functions of the nth order, $\mu_{n,m}$ is the mth root of J_n, $\mu_{n,m}'$ is the mth root of the derivative J_n', $\kappa_{n,m} = \sqrt{\omega^2-\omega_{n,m}^2}\Big/c$, $\kappa_{n,m}' = \sqrt{\omega^2-\omega_{n,m}'^2}\Big/c$, $\omega_{n,m} = c\mu_{n,m}/b$, $\omega_{n,m}' = c\mu_{n,m}'/b$, and b is the radius of the waveguide. The variable σ denotes the direction of the propagation of the wave; $\sigma = +1$ corresponds to the waves propagating in the positive direction along the z-axes, and $\sigma = -1$ marks the waves traveling in the opposite direction.

In the first order of the perturbation theory, the electromagnetic field scattered by the hole into the waveguide is characterized by the amplitudes $a_{n,m}\left(\sigma\right)$ such that

$$F = h(z)\sum_{E,\,H}\sum_{n,m} a_{n,m}\left(\sigma=1\right) F^{(n,m)}\left(r,z,\sigma=1\right)$$

$$+h(-z)\sum_{E,\,H}\sum_{n,m} a_{n,m}\left(\sigma=-1\right) F^{(n,m)}\left(r,z,\sigma=-1\right), \tag{5}$$

where $h\left(z\right)$ is the step function and F denotes any of the components E_z, E_r, or H_ϑ. The factors $a_{n,m}$ can be expressed in terms of the electric α_{el} and magnetic α_{mg} polarizabilities of the hole [1]

$$a_{n,m}^{(E)} \;=\; \frac{4I_0\left(\kappa\alpha_{mg}+\sigma\kappa_{n,m}\alpha_{el}\right)}{cb^2\kappa_{n,m}\mu_{n,m}J_n'\left(\mu_{m,n}\right)\left(1+\delta_{n,0}\right)}, \tag{6}$$

for an E mode, and

$$a_{n,m}^{(H)} \;=\; -\frac{4nI_0\left(\sigma\kappa_{n,m}'\alpha_{mg}+\kappa\alpha_{el}\right)}{cb^2\kappa_{n,m}'\left(\mu_{n,m}'^2-n^2\right)J_n\left(\mu_{m,n}'\right)}, \tag{7}$$

[1]Our definitions of α_{el} and α_{mg} agree with Bethe's paper [3]; they are two times larger than those used by Kurennoy [1].

for an H mode. Calculating Z using Eqs. (5)–(7) with the help of the following relation,

$$Z = -\frac{1}{I_0} \int_{-\infty}^{\infty} dz\, E_z\left(z, r=0\right) \exp\left(-i\omega z/c\right), \qquad (8)$$

gives Eq. (1).

The real part of the impedance of a hole arises in the second order of the perturbation theory based on the smallness of the parameters α_{mg}/b^3 and α_{el}/b^3. It turns out, however, that we can find the real part of the impedance without going to higher orders if use is made of the following relation between the ReZ and the energy P radiated per unit time by the hole (see Ref. [5]):

$$P = \frac{1}{2}\, I_0^2 \,\mathrm{Re} Z\left(\omega\right) . \qquad (9)$$

The energy flux P in Eq. (9) should include all the waves radiated by the hole, both inside and outside of the waveguide. The outside radiation will depend on the geometry and location of the conducting surfaces in that region and cannot be computed without knowing particular details of the specific design. Here we neglect its contribution, assuming that the thickness of the pipe wall is large enough so that the electromagnetic field does not penetrate through the hole.

Inside the waveguide, we have to take into account the radiation going into all E and H modes. The energy flow in the mode of unit amplitude is equal to

$$P^{(E)}_{n,m} = \frac{1+\delta_{0,n}}{16}\, \omega\kappa_{n,m}\, \mu^2_{n,m}\, J'^2_n\left(\mu_{n,m}\right), \qquad (10)$$

and

$$P^{(H)}_{n,m} = \frac{1+\delta_{n,0}}{16}\, \omega\kappa'_{n,m}\left(\mu'^2_{n,m} - n^2\right) J^2_n\left(\mu'_{n,m}\right), \qquad (11)$$

respectively. The energy flux in each mode radiated by the hole is given by $\left|a_{n,m}\left(\sigma=1\right)\right|^2 P_{n,m}$ and $\left|a_{n,m}\left(\sigma=-1\right)\right|^2 P_{n,m}$ in the forward and backward directions, respectively. It is evident that this radiation occurs only if the frequency ω is larger than the cutoff frequency $\omega_{n,m}$ (or $\omega'_{n,m}$).

The total energy flux P is

$$P = \sum_{E,H} \sum_{n,m} \sum_{\sigma=\pm 1} P_{n,m}\left|a_{n,m}\right|^2, \qquad (12)$$

where the summation is carried out over both directions of propagation, $\sigma = \pm 1$, all possible values of n and m, and over E and H modes. Combining Eqs. (9)–(12) yields the following equation for the contribution of E and H modes into the real part of the impedance:

$$\mathrm{Re}Z^{(E)} = \frac{Z_0}{\pi}\frac{\omega^2}{c^2 b^4} \sum_{n,m} \frac{1}{\left(1+\delta_{n,0}\right)} F^{(E)}\left(\frac{\omega}{\omega_{n,m}}\right), \qquad (13)$$

where

$$F^{(E)}\left(x\right) = \frac{\alpha^2_{mg} x^2 + \alpha^2_{el}\left(x^2-1\right)}{x\sqrt{x^2-1}} \qquad (14)$$

for $x > 1$, and $F^{(E)}\left(x\right) = 0$ for $x < 1$. For the H modes

$$\mathrm{Re}Z^{(H)} = \frac{Z_0}{\pi}\frac{\omega^2}{c^2 b^4} \sum_{n,m} \frac{n^2}{\mu'^2_{n,m} - n^2} F^{(H)}\left(\frac{\omega}{\omega'_{n,m}}\right), \qquad (15)$$

where

$$F^{(H)}\left(x\right) = \frac{\alpha^2_{el} x^2 + \alpha^2_{mg}\left(x^2-1\right)}{x\sqrt{x^2-1}} \qquad (16)$$

for $x > 1$, and $F^{(H)}\left(x\right) = 0$ for $x < 1$.

Equations (13) and (15) apply also for short slots such that $l \ll b$ and $l\kappa \ll 1$. For a large aspect ratio, $l \gg w$, we have $\alpha_{mg} \approx -\alpha_{el}$, and $F^{(E)}\left(x\right) = F^{(H)}\left(x\right)$. In this case, the plot of the $\mathrm{Re}\left(Z^{(E)} + Z^{(H)}\right)$ measured in units $\alpha^2_{mg} Z_0/\pi b^6$ as a function of $\omega b/c$ is shown in Fig. 1.

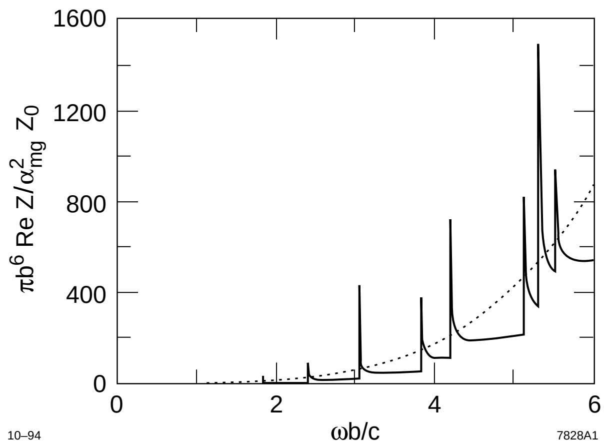

Figure 1. Real part of the impedance of a short large-aspect ratio slot as a function of the frequency (solid curve), and a high-frequency approximation given by Eq. (17) (dotted curve).

Because the functions $F^{(E)}\left(x\right)$ and $F^{(H)}\left(x\right)$ go to infinity when $x \to 1$, ReZ has singularities at the cutoff frequencies $\omega_{n,m}$ and $\omega'_{n,m}$. Formally, this happens because the amplitude of the radiated waves given by Eqs. (6) and (7) scales as $\kappa^{-1}_{n,m}$ when ω approaches a cutoff frequency. The actual height of the cutoff peaks will be determined by higher order corrections of the theory and finite conductivity of the walls.

In the limit $\omega \gg c/b$, a large number of harmonics is involved in the sums (13) and (15). By considering them to be continuous variables, it is possible to integrate over n and m instead of summing. This integration yields

$$\mathrm{Re}Z = \frac{2}{3\pi}\, Z_0\, \frac{\omega^4 \alpha^2_{mg}}{c^4 b^2} . \qquad (17)$$

This function is also plotted in Fig. 1; it gives a good approximation of the averaged dependence of the ReZ, even for small frequencies.

III. REAL PART OF THE IMPEDANCE FOR A LONG SLOT

To find the real part of the impedance of a long slot for which l is comparable or larger than b and/or κ^{-1}, we consider the long slot as a distributed system of magnetic and electric dipoles. The field radiated by the slot consists of the waves coming from different elements of the slots with a relative phase advance between them. For two infinitesimal elements located at distance z, the phase advance is composed of two parts. The first part is due to the change of phase of the driving field of the beam, and is equal to κz. The second part is caused by the relative phase shift of the two radiated waves, and is equal to $-\sigma\kappa_{n,m}z$, where $\sigma = \pm 1$ for the forward and backward propagating waves. The total phase exponent, $\exp(i\kappa z - i\sigma\kappa_{n,m}z)$ should be integrated over the length of the slot, yielding the factor

$$f_{n,m}(\sigma) = \frac{1}{l}\int_0^l \exp(i(\kappa - \sigma\kappa_{n,m})z)\, dz = \frac{1}{il(\kappa - \sigma\kappa_{n,m})}\left[\exp(i(\kappa - \sigma\kappa_{n,m})l) - 1\right] \qquad (18)$$

for the E modes, and a similar factor $f'_{n,m}(\sigma)$, for which $\kappa_{n,m} \to \kappa'_{n,m}$ in Eq. (18), for the H modes. These factors multiply the amplitudes $a^{(E)}_{n,m}$ and $a^{(H)}_{n,m}$ in Eqs. (6) and (7). Combining all these changes, and taking into account that for a long slot, $\alpha_{el} = -\alpha_{mg}$, results in the following modifications of the functions $F^{(E)}$ and $F^{(H)}$ in Eqs. (13) and (15):

$$F^{(E)}(x) = \frac{2b^2}{\mu^2_{n,m}}\left(\frac{\alpha_{mg}}{l}\right)^2 \frac{1}{x\sqrt{x^2-1}}\left\{\sin^2\left[\frac{l\mu_{n,m}}{2b}\left(x - \sqrt{x^2-1}\right)\right] + \sin^2\left[\frac{l\mu_{n,m}}{2b}\left(x + \sqrt{x^2-1}\right)\right]\right\}, \qquad (19)$$

and $F^{(H)}$ given by the same expression with $\mu_{n,m}$ substituted by $\mu'_{n,m}$. In the limit $l \gg |\kappa - \sigma\kappa_{n,m}|^{-1}$, the effective length of the slot that contributes to the real part of the impedance turns out to be equal to $|\kappa - \sigma\kappa_{n,m}|^{-1}$, which means that $\mathrm{Re}Z(\omega)$ does not depend on l in the limit $l \gg \kappa^{-1}$ (but $\kappa^{-1} \gg w$).

IV. REGULAR ARRAY OF SLOTS

Consider an array of N identical slots distributed along the beam pipe such that the distance between the slots is equal to d_1. The system has a period $d = l + d_1$. The electromagnetic field scattered by the array is the sum of the fields of individual slots. In the first approximation of the perturbation theory, the impedance is equal to NZ, where Z is given by Eq. (1). However, since the energy radiated by the array of slots is a quadratic function of the amplitude of the waves, it will be shown below that, at resonant frequencies, there is a strong amplification in $\mathrm{Re}Z$ that scales as N^2.

To find the radiation from N slots, it is necessary to sum their fields, taking into account the relative phase advance between the fields of different slots. As shown in the previous section, the phase advance between two adjacent slots is equal to $\exp(i\kappa d - i\sigma\kappa_{n,m}d)$. For N slots, the amplitude of (n,m) E mode should be multiplied by the following factor:

$$g_{n,m}(\sigma) = \sum_{j=0}^{N-1}\exp\left[idj(\kappa - \sigma\kappa_{n,m})\right] = \frac{1 - \exp\left[idN(\kappa - \sigma\kappa_{n,m})\right]}{1 - \exp\left[id(\kappa - \sigma\kappa_{n,m})\right]}. \qquad (20)$$

The square of the absolute value of $g_{n,m}(\sigma)$, multiplies each sine term in Eq. (19) modifying the function $F^{(E)}$ into the following expression:

$$F^{(E)}(x) = \frac{2b^2}{\mu^2_{n,m}}\left(\frac{\alpha_{mg}}{l}\right)^2 \frac{1}{x\sqrt{x^2-1}}\left\{\sin^2\left[\frac{l\mu_{n,m}}{2b}\left(x - \sqrt{x^2-1}\right)\right]\frac{\sin^2\left[\frac{dN\mu_{n,m}}{2b}\left(x - \sqrt{x^2-1}\right)\right]}{\sin^2\left[\frac{d\mu_{n,m}}{2b}\left(x - \sqrt{x^2-1}\right)\right]} + \frac{\sin^2\left[\frac{l\mu_{n,m}}{2b}\left(x + \sqrt{x^2-1}\right)\right]\sin^2\left[\frac{dN\mu_{n,m}}{2b}\left(x + \sqrt{x^2-1}\right)\right]}{\sin^2\left[\frac{d\mu_{n,m}}{2b}\left(x + \sqrt{x^2-1}\right)\right]}\right\}.$$

For the H modes, the function $F^{(H)}(x)$ contains $\mu'_{n,m}$ instead of $\mu_{n,m}$.

The maximum value of $|g_{n,m}|^2$ in Eq. (20) is equal to N^2 and is attained when the following condition holds:

$$d(\kappa - \sigma\kappa_{n,m}) = 2q\pi, \qquad (21)$$

where q is an integer. For large N, Eq. (20) represents narrow peaks with a width at half height $\Delta\omega/\omega \approx 1/(2qN)$ at the resonant frequencies. This implies that the Q factor for these resonances can be estimated as $Q \approx qN$.

If $d/b = 2\pi q/\mu_{n,m}$ (or $d/b = 2\pi q/\mu'_{n,m}$), Eq. (21) is satisfied by the cutoff frequency $\omega_{n,m}$ (or $\omega'_{n,m}$). In this case, the height of the resonant peaks will be strongly amplified because of the superposition of the cutoff singularity for a single peak with a maximum of the $|g_{n,m}|^2$ function.

In the limit of very large N, $N \to \infty$, the width of the resonances becomes so narrow that it will actually be determined by the finite conductivity of the walls σ. The transition to this regime occurs when Q becomes comparable to b/δ, where δ is the skin depth at the resonant frequency. Previously, this regime has been studied in detail for an infinitely long periodic bellow in Ref. [6], where the resonance conditions (21) have also been found.

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