

ANALYTIC SECOND- AND THIRD-ORDER ACHROMAT DESIGNS*

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I. INTRODUCTION

An achromat is a transport system that carries a beam without distorting its transverse phase space distribution. In this study, we apply the Lie algebraic technique [1-6] to a repetitive FODO array to make it either a second-order or a third-order achromat. (Achromats based on reflection symmetries [7,8] are not studied here.) We consider third-order achromats whose unit FODO cell layout is shown in Fig. 1. The second-order achromat layout is the same, except the octupoles are absent.

For the second-order achromats, correction terms (due to the finite bending of the dipoles) to the well-known formulae for the sextupole strengths are derived. For the third-order achromats, analytic expressions for the five octupole strengths are given. The quadrupole, sextupole and octupole magnets are assumed to be thin-lens elements. The dipoles are assumed to be sector magnets filling the drift spaces. More details of the analysis have been reported elsewhere.[9] We thank Y. Yan, H. Ye, J. Irwin and A. Dragt for their help.

II. ANALYSIS

We first calculate the Lie maps of each of the magnet elements. The map for a magnet element of length L is given by $e^{-L:H}$, where H is the Hamiltonian of the element. For a particle with $\delta = \Delta P/P_0$, we use (we ignore the path-length dynamics)

$$\text{thin quadrupole: } HL = \frac{1}{2F_k} (x^2 - y^2)(1 - \delta + \delta^2)$$

$$\text{thin sextupole: } HL = \frac{S_k}{3} (x^3 - 3xy^2)(1 - \delta)$$

$$\text{thin octupole: } HL = \frac{O_k}{4} (x^4 - 6x^2y^2 + y^4)$$

$$\begin{aligned} \text{sector dipole: } H = & \frac{P_x^2 + P_y^2}{2} + \frac{x^2}{2R^2} - \frac{x\delta}{R} + \frac{x(P_x^2 + P_y^2)}{2R} \\ & - \frac{x^2\delta}{2R^2} + \frac{x\delta^2}{R} + \frac{(P_x^2 + P_y^2)^2}{8} - \frac{x\delta^3}{R} + \frac{x^2\delta^2}{2R^2}, \end{aligned} \quad (1)$$

where R is the bending radius; F_k is the focal length of the k -th quadrupole; S_k and O_k are the k -th integrated sextupole and octupole strengths. Fringe fields are ignored.

Given the Hamiltonian H of an element, we factorize the element map as

$$e^{-L:H} = e^{H_2 + H_3 + H_4 + \dots} = e^{f_2} e^{f_3} e^{f_4} e^{\mathcal{O}(X^5)}, \quad (2)$$

where H_n and f_n are polynomials of order n in the variables $X = (x, P_x, y, P_y, \delta)$. We performed this factorization [3,5]

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and obtained

thin quadrupole:

$$f_3 = \frac{1}{2F_k} (x^2 - y^2)\delta, \quad f_4 = -\frac{1}{2F_k} (x^2 - y^2)\delta^2$$

thin sextupole:

$$f_3 = -\frac{S_k}{3} (x^3 - 3xy^2), \quad f_4 = \frac{S_k}{3} (x^3 - 3xy^2)\delta$$

thin octupole: $f_3 = 0, \quad f_4 = -\frac{O_k}{4} (x^4 - 6x^2y^2 + y^4)$

sector dipole:

$$\begin{aligned} f_3 = & -\frac{1}{6R^2} \sin^3 \frac{L}{R} x^3 - \frac{1}{4R} \sin \frac{L}{R} \sin \frac{2L}{R} x^2 P_x \\ & - \frac{1}{4} \cos \frac{L}{R} \sin \frac{2L}{R} x P_x^2 + \frac{R}{6} (1 - \cos^3 \frac{L}{R}) P_x^3 \\ & - \frac{1}{2} \sin \frac{L}{R} x P_y^2 + \frac{x^2\delta}{2R} \sin \frac{L}{R} (\cos \frac{L}{R} + \sin^2 \frac{L}{R}) \\ & + R \sin^2 \frac{L}{2R} P_x P_y^2 - 2 \sin^2 \frac{L}{2R} \sin^2 \frac{L}{R} x P_x \delta \\ & - \frac{R}{2} \sin^2 \frac{L}{2R} \sin \frac{2L}{R} P_x^2 \delta - \frac{1}{2} (L - R \sin \frac{L}{R}) P_y^2 \delta \\ & - \frac{x\delta^2}{2} (\sin^3 \frac{L}{R} + \sin \frac{2L}{R}) + \frac{R}{2} (2 - \cos \frac{L}{R}) \sin^2 \frac{L}{R} P_x \delta^2 \\ & + \frac{1}{12} (-6L + 2R \sin^3 \frac{L}{R} + 3R \sin \frac{2L}{R}) \delta^3, \\ f_4 = & \left[-\frac{x^2}{8R} \sin^3 \frac{L}{R} - \frac{x P_x}{8} \sin \frac{L}{R} \sin \frac{2L}{R} \right. \\ & \left. - \frac{R}{8} \cos^2 \frac{L}{R} \sin \frac{L}{R} P_x^2 - \frac{R}{8} \sin \frac{L}{R} P_y^2 \right] (P_x^2 + P_y^2) \\ & + \frac{x^3\delta}{12R^2} \sin^3 \frac{L}{R} + \left(\frac{1}{2} + \cos \frac{L}{R} \right) \sin^2 \frac{L}{2R} \sin \frac{L}{R} x P_x^2 \delta \\ & + \left[\frac{R}{12} (3 + 4 \cos \frac{L}{R} + 5 \cos \frac{2L}{R}) P_x^3 \delta \right. \\ & \left. + \frac{x P_y^2 \delta}{4} \sin \frac{2L}{R} + \frac{R}{4} (3 + \cos \frac{2L}{R}) P_x P_y^2 \delta \right] \sin^2 \frac{L}{2R} \\ & - \frac{1}{4R} (\sin^3 \frac{L}{R} + \sin \frac{2L}{R}) x^2 \delta^2 + \frac{1}{2} \sin^2 \frac{L}{R} x P_x \delta^2 \\ & - \frac{R}{4} (1 + 3 \cos \frac{L}{R}) \sin^2 \frac{L}{2R} \sin \frac{L}{R} P_x^2 \delta^2 \\ & + \left[\frac{R}{2} \sin^4 \frac{L}{2R} P_y^2 + (\cos \frac{L}{R} + \frac{1}{4} \sin^2 \frac{L}{R}) x \delta \right] \sin \frac{L}{R} \delta^2 \\ & - \frac{R}{2} \sin^2 \frac{L}{R} P_x \delta^3 + \frac{\delta^4}{12} (6L - R \sin^3 \frac{L}{R} - 3R \sin \frac{2L}{R}). \end{aligned} \quad (3)$$

Having factorized the maps of all magnets, the total map $\mathcal{M}_{\text{cell}}$ of a cell is obtained by multiplying and concatenating the maps of the component elements [3,9]:

$$\mathcal{M}_{\text{cell}} = \prod_{i=1}^N (e^{f_2^i} e^{f_3^i} e^{f_4^i}) = e^{h_2} e^{h_3} e^{h_4} e^{\mathcal{O}(X^5)}, \quad (4)$$

where

$$\begin{aligned} \mathcal{R} &= e^{h_2} = \prod_{i=1}^N e^{f_2^i}, \quad h_3 = \sum_{i=1}^N f_3^i \\ h_4 &= \sum_{i=1}^N \tilde{f}_4^i + \frac{1}{2} \sum_{j>i=1}^N [\tilde{f}_3^i, \tilde{f}_3^j]. \end{aligned} \quad (5)$$

In Eq.(5), \tilde{f}^i means $\tilde{f}^i(X) = f^i(R_{N-i}X)$ with R_{N-i} the linear map from the last element to the i -th element. The map of the N -cell achromat is $\mathcal{M} = \mathcal{M}_{\text{cell}}^N$. The number of cells N is so that $\mu_{x,y}$ (the total phase advances in x are y) are both multiples of 2π , but avoid resonances.

We now make a canonical coordinate transformation from (x, P_x, y, P_y) to $(\phi_x, A_x, \phi_y, A_y)$ by $x = \sqrt{2A_x\beta_x} \sin \phi_x + \eta\delta$, $P_x = \sqrt{\frac{2A_x}{\beta_x}}(\cos \phi_x - \alpha_x \sin \phi_x) + \eta'\delta$, and similarly for y and P_y without the η and η' terms, where $\beta_{x,y}, \alpha_{x,y}$ and η, η' are the Courant-Snyder and the dispersion functions [10]. The linear map generator h_2 becomes $h_2 = -\mu_x A_x - \mu_y A_y - \frac{1}{2}\bar{\alpha}_c \delta^2$ where and $\bar{\alpha}_c$ is the momentum compaction factor. We then decompose h_n in terms of the eigenmodes of $:h_2:$ as [5]

$$\begin{aligned} h_n &= \sum_{a+b+c+d+e=n} C_{abcd,e}^n |abcd,e\rangle, \\ |abcd,e\rangle &\equiv .A_x^{(a+b)/2} A_y^{(c+d)/2} e^{i(a-b)\phi_x} e^{i(c-d)\phi_y} \delta^e \end{aligned} \quad (6)$$

To reduce a nonlinear map to its normal form, it can be shown [11] that (in the absence of resonances) [2] all the non-secular terms can be transformed away via a symplectic similarity transformation leaving only terms with $a = b$ and $c = d$; i.e., terms depending on A_x, A_y and δ only. In particular, we have

$$\begin{aligned} h_3 &= C_{1100,1}^3 A_x \delta + C_{0011,1}^3 A_y \delta + C_{0000,3}^3 \delta^3, \\ h_4 &= C_{2200,0}^4 A_x^2 + C_{0022,0}^4 A_y^2 + C_{1111,0}^4 A_x A_y \\ &\quad + C_{1100,2}^4 A_x \delta^2 + C_{0011,2}^4 A_y \delta^2 + C_{0000,4}^4 \delta^4. \end{aligned} \quad (7)$$

III. SECOND-ORDER ACHROMATS

For a second-order achromat, we follow Eqs. (6-7) and find the normal form of the unit cell is given by h_3 of Eq. (7) where

$$\begin{aligned} C_{1100,1}^3 &= \sum_{k=1,2}^{\text{quads}} \left[\frac{1}{2F_k} - \lambda_k \eta(k) \right] \beta_x(k) + w_x, \\ C_{0011,1}^3 &= - \sum_{k=1,2}^{\text{quads}} \left[\frac{1}{2F_k} - \lambda_k \eta(k) \right] \beta_y(k) + w_y, \end{aligned} \quad (8)$$

and

$$\begin{aligned} w_x &= \sum_{k=1,2}^{\text{dipoles}} \frac{1}{2} \sin^2\left(\frac{L}{R}\right) \left\{ \frac{\beta_x(k)}{R} \left[\sin \frac{L}{R} + \cot \frac{L}{R} \right. \right. \\ &\quad \left. \left. - \frac{\eta(k)}{R} \sin \frac{L}{R} - \eta'(k) \cos \frac{L}{R} \right] + 2\alpha_x(k) \left[1 - \cos \frac{L}{R} \right. \right. \end{aligned}$$

$$\begin{aligned} &\quad \left. \left. + \frac{\eta(k)}{R} \cos \frac{L}{R} + \eta'(s) \cos \frac{L}{R} \cot \frac{L}{R} \right] \right\} \\ &+ \gamma_x(k) R \left[- \cos \frac{L}{R} \tan \frac{L}{2R} - \frac{\eta(k)}{R} \cos \frac{L}{R} \cot \frac{L}{R} \right. \\ &\quad \left. + \left(\cos \frac{L}{R} + \frac{1}{2} \sec^2 \frac{L}{2R} \right) \eta'(k) \right], \\ w_y &= \sum_{k=1,2}^{\text{dipoles}} \frac{1}{2} \gamma_y(k) R \left[\sin \frac{L}{R} - \frac{L}{R} - \eta(k) \sin \frac{L}{R} \right. \\ &\quad \left. + \eta'(k) \left(1 - \cos \frac{L}{R} \right) \right]. \end{aligned} \quad (9)$$

The lattice functions are evaluated at the two quadrupoles in Eq. (8) and at the ends of the two dipoles in Eq. (9). In the limit of weak bending with $\epsilon_1 = \frac{L}{R} \ll 1$, we have

$$\begin{aligned} w_x &\simeq \epsilon_1 \sum_s^D \alpha_x(s) \eta'(s) + \frac{1}{4} \gamma_x(s) (3L\eta'(s) - 2\eta(s)), \\ w_y &\simeq \epsilon_1 \sum_s^D \frac{1}{4} \gamma_y(s) (L\eta'(s) - 2\eta(s)). \end{aligned} \quad (10)$$

To form a second-order achromat, we set the two C -coefficients to zero, and obtain

$$\begin{aligned} S_1 &= \frac{1}{2\eta(1)F_1} + \frac{\beta_y(2)w_x + \beta_x(2)w_y}{\eta(1)[\beta_x(1)\beta_y(2) - \beta_x(2)\beta_y(1)]}, \\ S_2 &= \frac{1}{2\eta(2)F_2} - \frac{\beta_y(1)w_x + \beta_x(1)w_y}{\eta(2)[\beta_x(1)\beta_y(2) - \beta_x(2)\beta_y(1)]}. \end{aligned} \quad (11)$$

The first terms usually dominate and give the well known results. The correction terms with w_x and w_y are normally, but not always, small.

IV. THIRD-ORDER ACHROMATS

We also studied the case of a third-order achromat. An algebraic program using Mathematica was developed to do the analysis. Here, we only report our results. The normal form of the third-order generator for a unit cell is given by Eq.(9) with

$$\begin{aligned} C_{2200,0}^4 &= -\frac{3}{8} \sum_{k=1}^5 \beta_x(k)^2 O_k + w_{xx}, \\ C_{1111,0}^4 &= \frac{3}{2} \sum_{k=1}^5 \beta_x(k) \beta_y(k) O_k + w_{xy}, \\ C_{0022,0}^4 &= -\frac{3}{8} \sum_{k=1}^5 \beta_y(k)^2 O_k + w_{yy}, \\ C_{1100,2}^4 &= -\frac{3}{2} \sum_{k=1}^5 \beta_x(k) \eta(k) O_k + w_{xd}, \\ C_{0011,2}^4 &= \frac{3}{2} \sum_{k=1}^5 \beta_y(k) \eta(k) O_k + w_{yd}, \end{aligned} \quad (12)$$

and (when $\epsilon_1 = \frac{L}{R} \ll 1$)

$$w_{xx} \simeq \csc \frac{3\mu_x}{2} (2 + 3 \cos \mu_x) \prod_s^S \frac{S_s}{4} \beta_x(s)^{\frac{3}{2}} - \frac{3L}{16} \sum_s^D \gamma_x(s)^2$$

$$\begin{aligned}
& + \frac{1}{8} \csc \frac{3\mu_x}{2} (3 \cos \frac{\mu_x}{2} + 2 \cos \frac{3\mu_x}{2}) \sum_s^S S_s^2 \beta_x(s)^3 , \\
w_{xy} & \simeq -\frac{L}{4} \sum_s^D \gamma_x(s) \gamma_y(s) - \frac{1}{2} \cot \frac{\mu_x}{2} \sum_s^S S_s^2 \beta_x(s)^2 \beta_y(s) \\
& - \csc(\frac{\mu_x}{2} + \mu_y) \csc(\frac{\mu_x}{2} - \mu_y) \sin 2\mu_y \sum_s^S \frac{S_s^2}{4} \beta_x(s) \beta_y(s)^2 \\
& + \left[\csc(\frac{\mu_x}{2} + \mu_y) - \csc(\frac{\mu_x}{2} - \mu_y) - \csc \frac{\mu_x}{2} \sum_s^S \frac{\beta_x(s)}{\beta_y(s)} \right] \\
& \times \frac{1}{2} \prod_s^S S_s \sqrt{\beta_x(s)} \beta_y(s) , \\
w_{yy} & \simeq -\frac{3L}{16} \sum_s^D \gamma_y^2(s) + \frac{1}{16} \sum_s^S S_s^2 \beta_x(s) \beta_y(s)^2 \\
& \times \left[4 \cot \frac{\mu_x}{2} + \sin \mu_x \csc(\frac{\mu_x}{2} + \mu_y) \csc(\frac{\mu_x}{2} - \mu_y) \right] \\
& + \frac{1}{8} \left[4 \csc \frac{\mu_x}{2} + \csc(\frac{\mu_x}{2} + \mu_y) + \csc(\frac{\mu_x}{2} - \mu_y) \right] \\
& \times \prod_s^S S_s \sqrt{\beta_x(s)} \beta_y(s) , \\
w_{xd} & \simeq -\frac{3L}{4} \sum_s^D \gamma_x(s) \eta'(s)^2 - \sum_s^S \beta_x(s) \left(\frac{1}{2F_s} - S_s \beta_x(s) \right) \\
& + \frac{1}{2} \cot \mu_x \sum_s^S \left[\beta_x(s) \left(\frac{1}{2F_s} - S_s \beta_x(s) \right) \right]^2 \\
& + \csc \mu_x \prod_s^S \beta_x(s) \left(\frac{1}{2F_s} - S_s \beta_x(s) \right) \\
& + \csc \frac{\mu_x}{2} \sum_{1,2}^S \frac{S_2}{2} \beta_x(2) \eta(1) (S_1 \eta(1) - \frac{1}{F_1}) \sqrt{\beta_x(1) \beta_x(2)} \\
& + \frac{1}{2} \cot \frac{\mu_x}{2} \sum_s^S S_s \eta(s) (S_s \eta(s) - \frac{1}{F_s}) \beta_x(s)^2 , \\
w_{yd} & \simeq -\frac{L}{4} \sum_s^D \gamma_y(s) \eta'(s)^2 + \sum_s^S \beta_y(s) \left(\frac{1}{2F_s} - S_s \beta_x(s) \right) \\
& + \frac{1}{2} \cot \mu_y \sum_s^S \left[\beta_y(s) \left(\frac{1}{2F_s} - S_s \beta_x(s) \right) \right]^2 \\
& + \csc \mu_y \prod_s^S \beta_y(s) \left(\frac{1}{2F_s} - S_s \beta_x(s) \right) \\
& - \csc \frac{\mu_x}{2} \sum_{1,2}^S \frac{S_2}{2} \beta_y(2) \eta(1) (S_1 \eta(1) - \frac{1}{F_1}) \sqrt{\beta_x(1) \beta(2)} \\
& - \frac{1}{2} \cot \frac{\mu_x}{2} \sum_s^S S_s \eta(s) (S_s \eta(s) - \frac{1}{F_s}) \beta_x(s) \beta_y(s) .
\end{aligned} \tag{13}$$

Exact expressions of the w -coefficients are too lengthy to be included here.

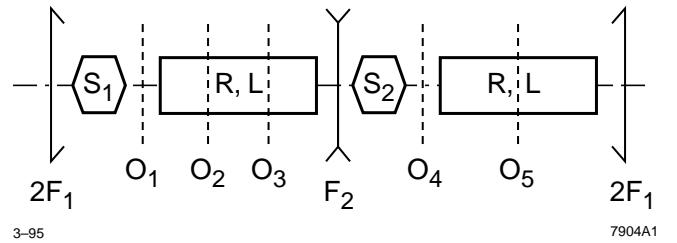


Figure 1. Unit cell of an achromat layout.

The required octupole strengths are such that the five C -coefficients in Eq. (12) are equal to zero. For the case when two of the octupoles are located next to the two sextupoles and the other three are at the $\frac{1}{3}$, $\frac{2}{3}$, and the $\frac{1}{2}$ locations of the two bending magnets, we find

$$\begin{aligned}
O_1 & \simeq \frac{a+b}{6f^3 D}, \quad O_2 \simeq \frac{81(c+d)}{2fD}, \quad O_3 \simeq \frac{81(c-d)}{2fD} \\
O_4 & \simeq \frac{a-b}{6f^3 D}, \quad O_5 \simeq \frac{128e}{3(2f^2-1)D}, \\
a & = 2f(1360 - 22846f^2 - 74476f^4 + 695809f^6 \\
& \quad - 1438146f^8 + 1200096f^{10} - 326592f^{12}), \\
b & = -352 - 3360f^2 + 233290f^4 - 1070910f^6 \\
& \quad + 1917603f^8 - 1364850f^{10} + 361584f^{12}, \\
c & = 6f(-42 + 1076f^2 - 7409f^4 + 16306f^6 - 14368f^8 \\
& \quad + 4032f^{10}), \\
d & = 8 - 394f^2 + 5322f^4 - 16907f^6 + 14866f^8 - 4464f^{10}, \\
e & = -368 + 10536f^2 - 92342f^4 + 307222f^6 - 470547f^8 \\
& \quad + 330642f^{10} - 81648f^{12}, \\
D & = (4f^2 - 1)^2(3f^2 - 4)(10 - 173f^2 - 261f^4 + 324f^6)L^3 \epsilon_1^2 .
\end{aligned} \tag{14}$$

We have defined the dimensionless parameter $f = \frac{2F_1}{L}$ and have assumed that $\epsilon_1 = \frac{L}{R} \ll 1$ and $|\frac{F_1+F_2}{L}| \ll 1$.

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