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The Application of Lie Algebra Methods to PEP-II Design*

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Abstract

Lie algebraic methods are described which were used to optimize the design of the PEP-II lattices. These methods include calculation of the nonlinear Hamiltonian for lattice modules, dimensionless normalized representation of resonance basis coefficients, correlation of coefficients with dynamic aperture determination, and techniques for coefficient control and compensation.

1 INTRODUCTION

Simple lattice tracking for dynamic aperture determination is essential but limited by the fact that information is obtained at only one working point and one set of lattice parameters. Furthermore, inadvertent errors in the lattice and control files can remain undetected. To supplement tracking and to enhance our understanding we have used Lie algebraic mapping methods for the design of PEP-II lattices. Computer codes have been developed, which guarantee a faithful correspondence between tracking and mapping [1]. There exists a unique polynomial Lie generator for each map. The coefficients of the polynomials determine resonance and tune-shift strength, and uniquely specify the lattice. Since most of these coefficients depend only weakly on operating tunes, very general information is obtained.

Comparison of tracking results with map coefficients has been used to facilitate PEP-II lattice design process. The methods are described in the following sections.

2 THE ONE-TURN MAP

To obtain global information for a lattice we first extract a one-turn map at a suitable observation position as a Taylor expansion about the on-momentum closed orbit. In general, we include all lattice nonlinearities. However, we can concentrate on a particular lattice module by inserting a linear lattice for the rest of the ring. We usually consider 2-dimensional maps with a parameter δ representing the off-momentum dp/p. Thus, the Taylor map can be expressed as

$$\vec{Z} = \vec{U}(\vec{z},\delta) + \emptyset(N+1), \qquad (1)$$

where $\emptyset(N+1)$ indicates that the Taylor map is truncated at an order of N, $\vec{z} = (x, p_x, y, p_y)$ is the global or initial phase-space coordinates and $\vec{Z} = (X, P_x, Y, P_y)$ is the phase-space coordinates after one turn.

Once the one-turn Taylor map is obtained, we make a Floquet transformation such that

$$\vec{Z} = \mathcal{A}^{-1}(\vec{z},\delta)\mathcal{R}(\vec{z})e^{:f(\vec{z},\delta):}\mathcal{A}(\vec{z},\delta)\vec{z} + \emptyset'(N+1), \qquad (2)$$

where $\mathcal{R}(\vec{z})$ is one-turn pure rotational map in the 4-dimensional transverse canonical phase-space, $\mathcal{A}(\vec{z},\delta)$ and its inverse $\mathcal{A}^{-1}(\vec{z},\delta)$ are the 4-by-5 matrices that generate the Floquet transformation. The dispersion, η, and the Courant-Snyder parameters, α, β, and γ are all included in $\mathcal{A}(\vec{z},\delta)$ and $\mathcal{A}^{-1}(\vec{z},\delta)$. Making the Floquet transformation $\vec{z}_F = \mathcal{A}^{-1}(\vec{z},\delta)\vec{z}$ and then dropping the subscript F for convenience, one obtains a one-turn map

$$\vec{Z} = \mathcal{R}(\vec{z})e^{:f(\vec{z},\delta):}\vec{z}. \qquad (3)$$

The polynomial $f(\vec{z},\delta)$ of the Lie transformation in Eq. 3 can be decomposed in a complete basis consists of the rotational eigen-modes, $\hat{x}_\pm = x \mp ip_x = \sqrt{2J_x}e^{\pm i\theta_x}$, $\hat{y}_\pm = x \mp ip_y = \sqrt{2J_y}e^{\pm i\theta_y}$, where J_x, J_y, θ_x, and θ_y are action-angle variables. One may obtain

$$f(\vec{z},\delta) = \sum_{\vec{n}\vec{m}p} a_{\vec{n}\vec{m}p}(2J_x)^{\frac{n_x}{2}}(2J_y)^{\frac{n_y}{2}}\delta^p cos(m_x\theta_x + m_y\theta_y + \phi_{\vec{n}\vec{m}p}), \qquad (4)$$

where the terms with $m_x = m_y = 0$ are the tune shift terms. For convenience, all these tune shift terms are grouped together and represented by $h_T(J_x, J_y, \delta)$. The remaining terms, all with angular variable dependence, are also grouped and represented by $h_R(J_x, J_y, \theta_x, \theta_y, \delta)$. Thus, the one-turn map given by Eq. 3 can be written as

$$\vec{Z} = e^{:-\mu_x J_x - \mu_y J_y:}e^{:-h_T(J_x,J_y,\delta) - h_R(\theta_x,J_x,\theta_y,J_y,\delta):}\vec{z}, \qquad (5)$$

where we have replaced the rotation $\mathcal{R}(\vec{z})$ with its Lie representation $e^{:-\mu_x J_x - \mu_y J_y:}$, where μ_x and μ_y are the working tunes of the lattice.

By inserting the unit transformation $e^{:-h_T:}e^{:h_T:}$ into Eq. 5 and using the Baker-Campbell-Hausdorf (BCH) theorem, we obtain

$$\vec{Z} = e^{:-H_T(J_x,J_y,\delta):}e^{:-h'_R(\theta_x,J_x,\theta_y,J_y,\delta):}\vec{z}, \qquad (6)$$

where the resonance terms, h'_R, are slightly modified from h_R and $H_T(J_x, J_y, \delta) = \mu_x J_x + \mu_y J_y + h_T(J_x, J_y, \delta)$ is still a pure rotation but with tune advance dependent on momentum and transverse amplitudes.

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3 DIMENSIONLESS SCALING

It should be noted that H_T and h'_R, and the action coordinates, J_x and J_y in Eq. 6 have a dimension of the emittance while θ_x, θ_y, and δ are dimensionless. Therefore, the coefficients in the polynomials, H_T and h'_R, have different dimensions. For convenience in directly using these coefficients for calculating and comparing the tune shift and resonance strength of different orders, we introduce a scale transformation so that $H_T = \epsilon_x \hat{H}_T$, $h'_R = \epsilon_x \hat{h}_R$, $J_x = \epsilon_x \hat{J}_x$, and $J_y = \epsilon_x \hat{J}_y$ to obtain the dimensionless one-turn map which, after dropping the symbol $\hat{}$ is again given by Eq. 6 except with modified coefficient values. Note that ϵ_x is the horizontal emittance which in PEP-II equals to 48 nm-rad and 64 nm-rad for the High-Energy Ring (HER) and the Low-Energy Ring (LER) respectively.

In our numerical studies, we set $\epsilon_y = \frac{1}{2}\epsilon_x$ to obtain the required vertical injection aperture and sufficient aperture for vertical blow-up from the beam-beam interaction. Most often we calculate the resonance strength and tune shift along the ellipse $r_x^2 + \frac{\epsilon_x}{\epsilon_y} r_y^2 = N^2$ with $\frac{\epsilon_x}{\epsilon_y} = 2$ and $N = 10$, where $r_x = \sqrt{2J_x}$, and $r_y = \sqrt{sJ_y}$ are radius in the two-dimensional phase-space planes.

4 TUNE SHIFT

Use of the Hamilton's Equation and the effective dimensionless tune Hamiltonian, H_T, in Eq. 6, one can obtain both horizontal (x) and vertical (y) tunes as explicit polynomials of the dimensionless geometric invariants, J_x and J_y and the chromatic amplitude, δ given by $\nu_x(J_x, J_y, \delta) = \frac{1}{2\pi}\partial H_T(J_x, J_y, \delta)/\partial J_x$ and $\nu_y(J_x, J_y, \delta) = \frac{1}{2\pi}\partial H_T(J_x, J_y, \delta)/\partial J_y$.

In Table 1, maximum tune shifts along the ellipse $r_x^2 + r_y^2 = 10$ (10 times the nominal beam size) may be compared for two HER bare (error free) lattices. In one lattice, $\beta_y^* = 2.0cm$, in the other, $\beta_y^* = 1.5cm$. We also typically make a plot of the tune footprint on a tune space.

The transverse dynamic aperture of the HER $\beta_y^* = 2cm$ lattice is well above 10 σ_t (transverse beam size) with an amplitude of 10 σ_l (longitudinal beam size) synchrotron oscillation and is not sensitive to the nonlinear errors. The dynamic aperture of the $\beta_y^* = 1.5cm$ lattice is very good if there are no nonlinear errors (it is still a nonlinear lattice due to the chromaticity sextupole correctors). However, it is sensitive to the nonlinear errors once the amplitude of the synchrotron oscillation is over $8\sigma_l$. Nevertheless, the dynamic aperture with a full set of errors is above $10\sigma_t$ if the the synchrotron oscillation amplitude is under $8\sigma_l$.

As shown in Table 1, for the lattice with $\beta_y^* = 2cm$, the chromatic tune shifts are pretty good while the geometrical tune shifts are typical except the crossing terms (bold faced) are larger.

On the other hand, for the lattice with $\beta_y^* = 1.5cm$, the geometric tune shift property is superior compared to the $\beta_y^* = 2cm$ one. In fact it is too good. Since the variation of the tune with energy through the δ^3 term causes the tune footprint to move across resonance lines, a small tune shift with amplitude will result in broad resonance islands. Indeed, when Octupole were introduced in the lattice to increase the tune shift with amplitude terms to similar values of the 2.0cm lattice, the dynamic aperture was acceptable for $10\sigma_l$ synchrotron oscillation with a complete standard set of errors.

Table 1: HER Tune Shifts at 10σ, $2J_x + 2(2J_y) = 10^2$ and $\delta = 10\sigma_\delta = 10 * 0.00061$.

$J_x^i J_y^j \delta^k$	$2cm$	$1.5cm$	$2cm$	$1.5cm$
i, j, k	ν_x	ν_x	ν_y	ν_y
0, 0, 0	.57E-00	.57E-00	.64E-00	.64E-00
0, 0, 1	.40E-06	.23E-06	-.56E-06	.28E-06
1, 0, 0	-.20E-03	-.10E-02	**-.94E-02**	-.36E-02
0, 1, 0	**-.47E-02**	-.18E-02	-.13E-05	-.38E-03
2, 0, 0	.15E-04	-.11E-05	.35E-03	-.14E-04
1, 1, 0	-.88E-04	-.35E-05	-.80E-05	.15E-05
0, 2, 0	-.80E-05	.15E-05	-.37E-04	-.90E-06
1, 0, 1	.42E-03	-.34E-03	.12E-04	-.60E-04
0, 1, 1	.62E-05	-.30E-04	.77E-03	-.49E-03
1, 0, 2	.92E-03	.10E-04	-.55E-03	-.20E-03
0, 1, 2	-.28E-03	-.10E-03	.83E-03	.43E-04
0, 0, 1	.40E-06	.23E-06	-.56E-06	.28E-06
0, 0, 2	-.53E-03	.40E-03	-.58E-03	-.40E-02
0, 0, 3*	-.49E-03	-.61E-02	-.14E-02	-.67E-02
0, 0, 4	.92E-04	.19E-02	-.61E-04	.47E-03

5 RESONANCES

We wish to find a lattice whose dynamic aperture does not depend strongly on the choice of the working tunes, so long as they are not on or near low order resonance lines. If the observation position for obtaining the one-turn map does not break any symmetric property, such as the "-I" between two chromaticity sextupoles, the resonance coefficients as given by the polynomial $h'_R(\theta_x, J_x, \theta_y, J_y)$ will not be changed much if one simply chooses a different working tune. This is not true for the resonance coefficients in a normal-form Hamiltonian, which has resonance denominators depending strongly on working tune location. Therefore, we prefer to study the resonance coefficients of Eq. 6. Since the tune shift information as given in the last Section can be easily plotted in a tune diagram for a chosen pair of working tunes, one could concentrate on analyzing a few resonances suggested by the diagram and investigate how harmful each resonance is to the lattice performance.

Since resonance terms (in h'_R) of higher orders have larger derivatives, thereby causing larger step-sizes in phase space, we prefer to measure the strength of a resonance term by taking its Poisson bracket (PB) with respect to phase space coordinates J_x, J_y, θ_x, and θ_y. From these PBs we compute the phase-space step

$$|\Delta \vec{z}| = \sqrt{[(r_x \Delta\theta_x)^2 + (\Delta r_x)^2] + \frac{\epsilon_x}{\epsilon_y}[(r_y \Delta\theta_y)^2 + (\Delta r_y)^2]}.$$

The ratio $\frac{\epsilon_x}{\epsilon_y}$ is essential to weight the y step-sizes accord-

Table 2: Normalized Resonance Coefficients for HER $\beta_y^* = 1.5cm$ Lattices

m_x	m_y	n_x	n_y	p	Bare	with Errors
3	-1	3	1	2	0.0	.16E-03
3	2	3	2	0	0.33E-06	.20E-3
3	2	5	2	0	0.11E-06	.11E-3
3	2	3	2	1	0.22E-05	.14E-3
1	-4	1	4	0	0.21E-04	.14E-3
1	-4	3	4	0	0.14E-06	.75E-4
1	-4	1	6	0	0.58E-07	.14E-3
1	-4	1	4	1	0.88E-05	.15E-3
4	-2	4	2	0	0.26E-06	.32E-4

ing to the vertical emittance. We then compute the maximum value of $|\Delta\vec{z}|$ for all possible values of θ_x, θ_y, J_x, and J_y with the constraint $r_x^2 + \frac{\epsilon_x}{\epsilon_y} r_y^2 = N^2$. This maximum is what we call the normalized resonance basis coefficient. For any $\vec{n}, \vec{m}, p$, there is a positive number $B(\vec{n}, \vec{m}, p)$ such that

$$a_{\vec{n}\vec{m}p}^{NORM} = B(\vec{n}, \vec{m}, p)|a_{\vec{n}\vec{m}p}|.$$

$a_{\vec{n}\vec{m}p}^{NORM} = 1$ means that this resonance can at the maximum cause a phase-space motion of $1\sigma_t$ in one turn for a particle on the $10\sigma_t$ boundary,

Table 2 compares these normalized coefficients for two HER 1.5 cm lattice, one bare and the other with complete errors. Here we only print the resonances specifically chosen by reading the tune diagram with the nominal pair of working tunes, $\nu_x = 0.57$ and $\nu_y = 0.64$. The large chromatic tune shifts of this lattice (as discussed in the last Section) causes those resonances in Table 2 to be encountered. Resonances with strengths less than 10^{-3} are expected to be of no significance because even on this resonance the maximum total movement in 10^3 turns would be 1σ. It has been our expectation that if a particle is not lost in 10^3 turns without damping, then it will survive undoubtfully with damping present. For example, the sum resonance $3\nu_x + 2\nu_y$ listed in Table 2 could cause at most an approximate deviation of $.2\sigma$ in 1000 turns of a particle initially at 10σ.

In our regular studies, we normally print out all normalized resonance coefficients. We can also plot the resonance basis coefficients in a histogram. A sample plot is shown in Figure 1.

6 NON-LINEAR COMPENSATION

To thoroughly probe the effects of a single resonance it is possible to introduce at the end of the lattice a symplectic map which changes the strength of a single resonance while leaving the rest of the map unchanged. The lattice is tracked with the map appended to it. We have practiced this compensation technique primarily with tune-shift terms and have found optimal values for these terms. Studies for varying resonance strengths are in process.

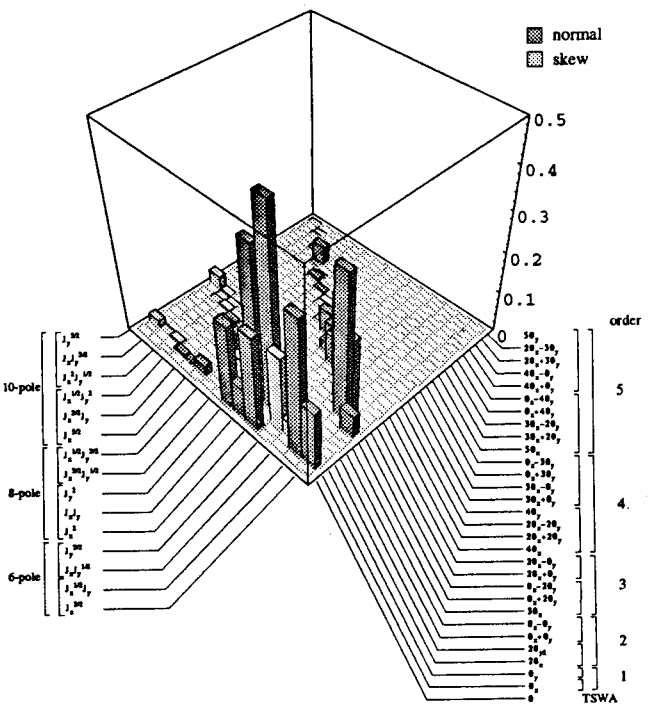

Figure 1. Histogram for normalized resonance basis coefficients.

7 SUMMARY

By tabulating normalized tune shift and resonance coefficients of the Lie generator of one-turn maps, we have been able to i) identify resonant terms needing improvement, ii) locate inadvertent lattice preparation errors, iii) determine optimal values for tune shift parameters, and iv) direct lattice development efforts. We have been altering these coefficients using numerical methods and/or introducing lattice modifications. These methods are also being used to study the impact of lattice parameters on beam-beam tails.

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9 REFERENCES

[1] E. Forest, "Despot - a program for tracking and extraction of maps".