# Renormalization of Two Dimensional $R^2$ -Gravity

## S. ICHINOSE \*

Stanford Linear Accelerator Center Stanford University, Stanford, California 94309<sup>†</sup> and

## Department of Physics, Universuty of Shizuoka, Yada 52-1, Shizuoka 422, Japan<sup>‡</sup>

#### Abstract

Two Dimensional Gravity with  $R^2$ -term is quantized around the  $R^2$ -Liouville solution in the semiclassical way. Renormalization, regularization (infrared,ultra-violet) and a topological term  $(\partial(\varphi\partial\varphi))$  are carefully treated. All (1-loop) divergences are renormalized by the cosmological constant  $(\mu)$  and the  $R^2$ -coupling-constant  $(\beta)$  for the case  $\beta > 0$ . Quantum meaning of the topological term is clarified. The renormalization group beta-functions of the couplings  $\beta$  and  $\mu$  are obtained. It is found that the theory is conformal (i.e. the beta-functions=0) for  $w = (\beta/A) \cdot (16\pi \cdot 48\pi/(26 - c_m)) \ge 1$  (where A is a fixed area) exactly when the coupling constant  $\xi$  of the topological term takes the value of 1. As for 0 < w < 1,  $\beta$  is asymptotically free for  $c_m < 26$ and  $\mu$  is asymptotically non-free.

Submitted to Nuclear Physics B

- 2

<sup>\*</sup>Work supported by Department of Energy contract DE-AC03-76SF00515 <sup>†</sup>the address untill March 12,1995

<sup>&</sup>lt;sup>‡</sup>the parmenent address, E-mail address: ichinose@u-shizuoka-ken.ac.jp

## 1 Introduction

The quantum gravity (QG) is a mysterious world in modern physics. We do not know yet how to quantize the theory of gravity correctly. Three decades have passed since Feynman [1],in 1963, tried to quantize gravity in an analogous way to QED. In the last ten years, however, the understanding of QG has been greatly improved both in conceptually and technically, especially for lower-dimensional Euclidean models. One of important achievement is the result of the 2 dimensional(2d) Euclidean quantum gravity, where some exact or non-perturbative features are obtained by using the conformal field theory. We already know, for the ordinary 2d QG, the form of the partition function Z[A] for  $A \to +\infty$  [2, 3, 4]. It is also known that quantizing the 2d Euclidean manifold leads to the fractal structure of a surface in the conformal limit. Having in mind this recently accumulated precious knowledge, we aim at clarifying some basic problems of 2d QG field theory, such as renormalization, regularization (infra-red and ultra-violet) and the treatment of the global topology. For the analysis we use the standard method of the semiclassical quantization[5].

In 2 dimensions, the Einstein action gives a purely topological quantity and does not give the local dynamics. Therefore the lowest (derivative) order term that describes the local interaction and can play the role of 'kinetic' term is the  $R^2$ -term. This term controls the value of the scalar curvature R. It can be expected that adding this term (with a proper sign) makes the 2d manifold smooth.

The  $R^2$ -gravity has been studied recently by several people. It was first carefully treated by T.Yoneya[6] for the Lorezian (not Euclidean) metric. He took the Hamilton-Jacobi formalism and solved the Wheeler-DeWitt equation. In lattice simulations,  $R^2$ -gravity was studied by [7] in the early stage of the lattice gravity. They regarded  $R^2$ -term as an irrelevent term and expected it does not influence the final critical behaviours. Ref.[8] recently examined the higher-derivative model in the same method with high-statistics. They have clearly found a cross-over phenomenon in the measurement of  $\langle \int d^2x \sqrt{g}R^2 \rangle$ . Ref.[5] has given a theoretical explanation for this at the lowest-order of the semiclassical quantization. (Present paper gives the next-order analysis). The importance of  $R^2$ -term as the ultraviolet regularization was implied in [2]. The renormalizability was shown in [9] using the background-field method. The  $R^2$ -gravity was treated as a conformal theory by Kawai and Nakayama(KN)[10] and the partition function is obtained for the region  $\beta \rightarrow \infty$ . In [5] the comparison between the semiclassical result and KN's is made.

Various approaches are taken , at present, to clarify the dynamics of 2d QG. Using analytical approaches, we have the method of  $2 + \epsilon$  dim gravity [11, 12, 13], the semiclassical method[14, 15, 5] and the matrix model[16, 17, 18]. They have small perturbation parameters  $\epsilon$  (the deviation of dimension number from 2),  $\hbar$  (Planck constant) and 1/N (the inverse of the matrix size) respectively. As for the numerical approach, we have the lattice simulation[18, 19, 20]. It has no perturbation parameter and is a completely non-perturbative approach. Each approach tries to clarify the dynamical meaning of the conformal limit provided by the conformal field theory approach, which is ,by itself, a nonperturbative analytical approach but does not clarify the dynamics very much.

The analysis of renormalization properties is very important to understand the dynamics. It was formally done using the background-field method in [21, 9]. But the connection with the conformal result is not very obvious. The renormalization analysis is obscure in  $2 + \epsilon$  dim gravity[22], and is partially successful in the matrix model[23]. It is still at the primitive stage in the lattice simulation[24]. The corresponding procedure in the conformal approach is essentially the so-called gravitational dressing[2] or  $e^{\kappa\phi}$ -term in [3, 4]. They introduce an additional parameter ,without the dynamical explanation, in order to find a non-trivial conformal solution. The present paper clarifies the renormalization properties in the semiclassical approach, where we take the formalism of [5].

## 2 Semiclassical Quantization

In this section we explain the semiclassical quantization of the  $R^2$ -gravity[5].

### **2.1** General Formalism

Let us take the Euclidean action,

$$S_{tot} = S_{gra} + S_m \quad , \quad S_{gra}[g_{ab}; G, \beta, \mu] = \int d^2x \sqrt{g} (\frac{1}{G}R - \beta R^2 - \mu) \quad ,$$
  
$$S_m[g_{ab}, \Phi; c_m] = -\int d^2x \sqrt{g} (\frac{1}{2} \sum_{i=1}^{c_m} \partial_a \Phi_i \cdot g^{ab} \cdot \partial_b \Phi_i) \quad , \quad (a, b = 1, 2) \quad , \quad (1)$$

under the fixed area condition  $A = \int d^2x \sqrt{g}$ . Here G is the gravitaional coupling constant,  $\mu$  is the cosmological constant,  $\beta$  is the coupling strength for  $R^2$ -term and  $\Phi$  is the  $c_m$ - components scalar matter fields.

By taking the conformal-flat gauge,  $g_{ab} = e^{\varphi} \delta_{ab}$ , the action (1) gives us, after integrating out the matter fields and Faddeev-Popov ghost, the following partition function[25]:

$$\bar{Z}[A] \equiv \int \frac{\mathcal{D}_g \mathcal{D}\Phi}{V_{GC}} \{ exp\frac{1}{\hbar} S_{tot} \} \ \delta(\int d^2 x \sqrt{g} - A) = exp\frac{1}{\hbar} (\frac{8\pi(1-\hbar)}{G} - \mu A) \times Z[A] \quad ,$$

$$Z[A] \equiv \int \mathcal{D}\varphi \ e^{+\frac{1}{\hbar} \mathcal{S}_0[\varphi]} \ \delta(\int d^2x \ e^{\varphi} - A) \quad , \tag{2}$$

$$S_0[\varphi] = \int d^2x \left( \frac{1}{2\gamma} \varphi \partial^2 \varphi - \beta \ e^{-\varphi} (\partial^2 \varphi)^2 + \frac{\xi}{2\gamma} \partial_a(\varphi \partial_a \varphi) \right) \quad , \quad \frac{1}{\gamma} = \frac{1}{48\pi} (26 - c_m) \ , (3)$$

where we have used the following relations for the Einstein term and the cosmological term:  $\int d^2x \sqrt{g}R = 8\pi(1-h)$ , h = number of handles,  $\int d^2x \sqrt{g} = A^{-1}$ .  $V_{GC}$  is the gauge volume due to the general coordinate invariance.  $\xi$  is a free parameter. The

<sup>&</sup>lt;sup>1</sup>The sign for the action is different from the usual convention as seen in (2).

total derivative term (we call it topological term) generally appears when integrating out the anomaly equation  $\delta S_{ind}[\varphi]/\delta \varphi = \frac{1}{\gamma} \partial^2 \varphi$ . This term controls the global condition and turns out to be very important<sup>2</sup>. We consider the manifold with the fixed topology of a sphere h = 0, and with a finite area A. Furthermore we consider the case  $\gamma > 0$  ( $c_m < 26$ ).  $\hbar$  is the Planck constant.<sup>3</sup>

The Laplace transform of Z[A],(2), is written as

$$\hat{Z}[\lambda] = \int_0^\infty Z[A] e^{-\lambda A/\hbar} \, dA = \int \mathcal{D}\varphi \, \exp[\,+\frac{1}{\hbar} S_\lambda[\varphi]] \quad ,$$
$$S_\lambda[\varphi] \equiv S_o[\varphi] - \lambda \, \int d^2x \, e^\varphi \quad . \tag{4}$$

Conversely, Z[A] (the micro-canonical partiton function) can be obtained from  $\hat{Z}[\lambda]$  (the grand-canonical partition function) by the inverse Laplace transformation,

$$Z[A] = \int \frac{d\lambda}{\hbar} \hat{Z}[\lambda] \ e^{+\lambda A/\hbar} = \int \frac{d\lambda}{\hbar} \ Y[A,\lambda] \quad ,$$
  
$$Y[A,\lambda] \equiv \int \mathcal{D}\varphi \ exp \ \frac{1}{\hbar} [\ S_0[\varphi] - \lambda (\int d^2 x e^{\varphi} - A)] \quad .$$
(5)

The integral should be carried out along an appropriate contour parallel to the imaginary axis in the complex  $\lambda$ -plane.

Here we need the renormalization procedure, but we ignore it, for simplicity, in this subsection. We will consider the procedure in detail in the improved formalism of sec.4.2. The integral  $\hat{Z}[\lambda]$  can be calculated loop-wise by the semiclassical expansion around a solution of the classical field equation.

$$\varphi(x) = \varphi_c(x;\lambda) + \sqrt{\hbar} \psi(x) \quad ,$$
  

$$\text{COND.1} \quad \frac{\delta}{\delta\varphi} S_{\lambda}[\varphi]\Big|_{\varphi_c} = 0 \quad . \tag{6}$$

Then we have

Ę

$$\hat{Z}[\lambda] = exp \frac{1}{\hbar} S_{\lambda}[\varphi_{c}] \times \hat{Z}^{loop}[\lambda] , 
\hat{Z}^{loop}[\lambda] = \int \mathcal{D}\psi \ exp\{S_{\lambda}[\varphi_{c} + \sqrt{\hbar}\psi] - \frac{\delta S_{\lambda}}{\delta\varphi}\Big|_{\varphi_{c}}\psi - S_{\lambda}[\varphi_{c}]\}$$

$$= \int \mathcal{D}\psi \ exp\{\frac{1}{2}\frac{\delta^{2}S_{\lambda}}{\delta\varphi^{2}}\Big|_{\varphi_{c}}\psi\psi + O(\sqrt{\hbar})\} ,$$
(7)

where the quantum effect up to 1-loop is explicitly written in the last expression.

The stationary point  $\lambda_c$  defined by

$$Y[A,\lambda] \equiv exp \ \frac{1}{\hbar} \Gamma^{eff}[A,\lambda] \quad ,$$
  
COND.2  $\frac{d}{d\lambda} \Gamma^{eff}[A,\lambda]|_{\lambda_c} = 0 \quad , \quad \lambda_c = \lambda_c^0 + \hbar \lambda_c^1 + \cdots ,$  (8)

<sup>2</sup>The uniqueness of this term, among all possible total derivatives, is shown in [5].

<sup>&</sup>lt;sup>3</sup>In this subsection only, we explicitly write  $\hbar$  (Planck constant) in order to show the perturbation structure clearly.

will give the dominant contribution to the contour integral (5).

$$Z[A] \approx \frac{1}{\hbar} Y[A, \lambda_c] \quad , \quad Y[A, \lambda_c] = exp \; \frac{1}{\hbar} \Gamma^{eff}[A, \lambda_c] \quad . \tag{9}$$

In the following, we will evaluate  $\Gamma^{eff}[A, \lambda_c]$  up to the 1-loop order (order of  $\hbar^1$ ).

## 2.2 Classical Configuration of R<sup>2</sup>-Gravity

In this subsection we discuss the lowest order  $(\hbar^0)$ . The classical equation  $\frac{\delta S_{\lambda}[\varphi]}{\delta \varphi} = 0$ , is explicitly written as

$$\frac{\delta S_{\lambda}[\varphi]}{\delta \varphi} = \frac{1}{\gamma} \partial^2 \varphi + \beta \{ e^{-\varphi} (\partial^2 \varphi)^2 - 2\partial^2 (e^{-\varphi} \partial^2 \varphi) \} - \lambda e^{\varphi} = 0 \quad . \tag{10}$$

We take the constant curvature solution(Liouville solution). <sup>4</sup>

$$-R|_{\varphi_c} = e^{-\varphi_c} \partial^2 \varphi_c = \text{const} \equiv \frac{-\alpha}{A} \quad , \tag{11}$$

where  $\alpha$  is a dimensionless constant which should satisfy

COND.1 
$$\alpha^2 \beta' - \frac{1}{\gamma} \alpha - \lambda A = 0$$
,  $\beta' \equiv \frac{\beta}{A}$ , (12)

as the consequence of classical field equation (10).

In this paper we consider only the case of the positive curvature:  $\alpha > 0$ . The spherically symmetric <sup>5</sup> solution of (11) is known to be (cf.[28, 29, 15]),

$$\varphi_c(r;\alpha) = -ln \left\{ \frac{\alpha}{8} (1 + \frac{r^2}{A})^2 \right\} , \quad r^2 = (x)^2 + (y)^2 .$$
 (13)

It gives  $\int d^2x \sqrt{g} R \Big|_{\varphi_c} = -\int d^2x \ \partial^2 \varphi_c = 8\pi$ , which implies that the manifold described by the solution (13) has the topology of a sphere. The area,  $\int d^2x \sqrt{g} \Big|_{\varphi_c} = \int d^2x \ e^{\varphi_c} = \frac{8\pi}{\alpha}A$ , can be interpreted as the effective area covered by the classical solution. The equations (12-13) constitute a solution of (10).

 $S_{\lambda}[\varphi_c]$  is then given as

$$S_{\lambda}[\varphi_{c}] = (1+\xi)\frac{4\pi}{\gamma} \ln \frac{\alpha}{8} - 16\pi\alpha\beta' + C(A) \quad ,$$
  
$$C(A) = \frac{8\pi(2+\xi)}{\gamma} + \frac{8\pi\xi}{\gamma} \{ \ln(1+L^{2}/A) - (L^{2}/A)/(1+(L^{2}/A)) \} \quad , \qquad (14)$$

<sup>5</sup> in the (x, y)-plane

.

<sup>&</sup>lt;sup>4</sup>The importance of the constant-curvature configuration in 2d QG has been pointed out by [26, 27].

where C(A) does not depend on  $\beta$  and  $\alpha$ , while L is the infrared cut-off  $(r^2 \leq L^2)$  introduced for the divergent volume intgral of the total derivative term. See Fig.1. The eq. (8) at the classical level is written as,

$$\frac{dS_{\lambda}[\varphi_c]}{d\lambda} + A = \left\{\frac{4\pi}{\gamma}\frac{1}{\alpha}(1+\xi) - (16\pi\beta' + \frac{1}{\gamma}) + 2\beta'\alpha\right\}\frac{d\alpha}{d\lambda} = 0 \quad , \tag{15}$$

where we have used a relation :  $1 = \frac{d\lambda}{d\alpha}\frac{d\alpha}{d\lambda} = \frac{1}{A}(2\alpha\beta' - \frac{1}{\gamma})\frac{d\alpha}{d\lambda}$ , which is derived from (12). This equation fixes the stationary point which dominates in the contour integral (5):

COND.2 
$$2\beta'\alpha^2 - (16\pi\beta' + \frac{1}{\gamma})\alpha + (1+\xi)\frac{4\pi}{\gamma} = 0$$
, (16)

which has two real solutions:

$$\alpha_{c}^{\pm} = \frac{1}{4\beta'} \{ 16\pi\beta' + \frac{1}{\gamma} \pm \sqrt{(16\pi\beta')^{2} + \frac{1}{\gamma^{2}} - \xi \frac{32\pi}{\gamma}\beta'} \} \quad , \tag{17}$$

when the condition  $D \equiv (16\pi\beta')^2 + \frac{1}{\gamma^2} - \xi \frac{32\pi\beta'}{\gamma} = (16\pi\beta' - \frac{\xi}{\gamma})^2 + \frac{1-\xi^2}{\gamma^2} \ge 0$  is satisfied. The relation (12) then determines  $\lambda_c^{\pm}(\beta) \equiv \lambda(\beta, \alpha_c^{\pm}(\beta))$ . Note that the determinant of the above quadratic equation is positive definite for all real  $\beta$  if we take  $\xi$  to be in the region  $: -1 \le \xi \le +1$ . In the following, we consider this case and use  $w \equiv 16\pi\beta'\gamma$  instead of  $\beta$  or  $\beta'$ . We depict the graphs of the above solutions in Fig.2a and Fig.2b. For the choice  $\xi = 1$ , we have

$$\lim_{\xi \to 1-0} \alpha_c^- = \begin{cases} \frac{8\pi}{w} & w \ge 1\\ 8\pi & w \le 1 \end{cases} , \quad \lim_{\xi \to 1-0} \alpha_c^+ = \begin{cases} 8\pi & w \ge 1\\ \frac{8\pi}{w} & w \le 1 \end{cases} .$$
(18)

In summary two unknown parameters  $\alpha$  and  $\lambda_c$  are fixed by two conditions COND.1 and 2; they are expressed by three physical parameters  $\beta$ ,  $\gamma$ , A and one free parameter  $\xi$ . In [5] the behaviours of the above solutions are analysed closely.  $\alpha_c^-$ -solution explain the simulation data very well. In particular three phases appear for  $w \equiv 16\pi\beta'\gamma \ll -1$ ,  $|w| \ll 1$ , and  $w \gg 1$ . The behaviour of  $\alpha_c^+$ solution contradicts the results of numerical simulation[5]. Therefore we consider quantization only around  $\alpha_c^-$ -solution in the following. ( $\alpha_c^+$ -solution is discussed in [5] in relation to the result of [10]). In [5], the free parameter  $\xi$  was chosen to be  $\xi = 1$  in order to adjust the classical prediction of Z[A] with the known conformal result for  $c_m \to -\infty$ . In the present paper the meaning of  $\xi = 1$  will be clarified in the standpoint of the renormalization group.

# **3** Quantization on the Sphere and Ultra-Violet -Regularization

Let us evaluate the (1-loop) quantum correction. It is given, from (7), as

$$\hat{Z}[\lambda]^{1-loop} = \int \mathcal{D}\psi exp S_2$$
,

$$S_{2} \equiv \frac{1}{2} \left. \frac{\delta^{2} S_{\lambda}}{\delta \varphi^{2}} \right|_{\varphi_{c}} \psi \psi = \int d^{2}x \{ (\frac{1}{2\gamma} + 2\beta e^{-\varphi_{c}} \partial^{2} \varphi_{c}) \psi \partial^{2} \psi + \frac{\xi}{2\gamma} \partial_{a} (\psi \partial_{a} \psi) - \frac{1}{2} (\beta e^{-\varphi_{c}} (\partial^{2} \varphi_{c})^{2} + \lambda e^{\varphi_{c}}) \psi^{2} - \beta e^{-\varphi_{c}} \partial^{2} \psi \partial^{2} \psi \}$$

$$= \int d^{2}x \{ -\frac{A}{2} \Theta F(r) \partial^{2} \psi \partial^{2} \psi + \frac{1}{2} \Xi \psi \partial^{2} \psi - \frac{1}{2A} \Omega F(r)^{-1} \psi^{2} + \frac{\xi}{2\gamma} \partial_{a} (\psi \partial_{a} \psi) \} , (19)$$

$$\Theta = \frac{\alpha \beta'}{4} , \Xi = \frac{1}{\gamma} - 4\alpha \beta' , \Omega = 8(\alpha \beta' + \frac{\lambda A}{\alpha}) , F(r) = (1 + \frac{r^{2}}{A})^{2} ,$$

$$8\Theta + \Xi + \frac{1}{8} \Omega = 0 , \qquad (20)$$

where  $\alpha = \alpha_c^-(\beta')$ ,  $\lambda = \lambda_c^-(\beta')$  (eqs.(12,17)). The last relation (20) comes from (12). In order to evaluate (19) we change the coordinate from plane coordinates (x, y) to polar coordinates  $(\theta, \phi)$ .(See Fig.3).

$$\begin{aligned} x &= \sqrt{A} \frac{\cos \phi}{\tan \frac{\theta}{2}} \quad , \quad y = \sqrt{A} \frac{\sin \phi}{\tan \frac{\theta}{2}} \quad , \\ (\frac{y}{x} = \tan \phi \quad , \quad r^2 = x^2 + y^2 = A \cot^2 \frac{\theta}{2} \quad ,) \\ ds^2 &= e^{\varphi_c} ((dx)^2 + (dy)^2) = \frac{8}{\alpha} \frac{1}{(1 + \frac{r^2}{A})^2} (dx^2 + dy^2) = \frac{2A}{\alpha} (d\theta^2 + \sin^2 \theta d\phi^2) \quad . \quad (21) \end{aligned}$$

By use of the relations

$$F(r)\partial^{2} = \frac{4}{A}(-\vec{L}^{2}) \quad , \quad -\vec{L}^{2} = \frac{1}{\sin^{2}\theta}\frac{\partial^{2}}{\partial\phi^{2}} + \frac{\partial^{2}}{\partial\theta^{2}} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} \quad ,$$
$$\frac{d^{2}x}{F(r)} = \frac{A}{4}\sin\theta \ d\phi \ d\theta \quad , \quad F(r) = (\sin\frac{\theta}{2})^{-4} \quad , \tag{22}$$

we can obtain

$$S_{2} = \frac{1}{4} \int d\phi d\theta \sin\theta \left\{ -8\Theta (\vec{L}^{2}\psi)^{2} - 2\Xi\psi \vec{L}^{2}\psi - \frac{1}{2}\Omega\psi^{2} \right\} \quad , \tag{23}$$

where  $\vec{L}^2$  is the square of the angular momentum operator. The topological term is dropped because we will soon see the term does really vanish. Now let us *define* the path-integral. As the general quantum variation  $\psi$ , we take the series expanded by the complete and orthonormal set  $\{Y_{l,m}\}$  on the sphere.

$$\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{l,m} Y_{l,m}(\theta, \phi) \quad , \quad \vec{L}^2 Y_{l,m}(\theta, \phi) = l(l+1) Y_{l,m}(\theta, \phi) \quad , \quad (24)$$

where  $Y_{l,m}(\theta, \phi)$  are spherical harmonics. We can confirm the topological term in (19),  $\int d^2x \{\frac{\xi}{2\gamma} \partial_a(\psi \partial_a \psi)\}$ , does vanish for the above general variation. This fact must be compared with the fact that the topological term at the classical level,  $\int d^2x \{\frac{\xi}{2\gamma} \partial_a(\varphi_c \partial_a \varphi_c)\}$ , which appeared in the evaluation (14), gives a divergent value. We notice the Liouville solution  $\varphi_c = ln \{\frac{8}{\alpha} \sin^4 \frac{\theta}{2}\}$  cannot be expressed as (24) with convergent coefficients. The path-integral (19) is defined by

$$S_{2} = -\sum_{l,m} (c_{l,m})^{2} \varepsilon_{l} ,$$
  

$$\varepsilon_{l}(w) = \frac{1}{4} \{8\Theta l^{2}(l+1)^{2} + 2\Xi l(l+1) + \frac{1}{2}\Omega\}$$
  

$$= \frac{1}{4} \{l(l+1) - 2\} \{8\Theta l(l+1) - \frac{1}{4}\Omega\} ,$$
  

$$\hat{Z}[\lambda]^{1-loop} = \int \mathcal{D}\psi e^{S_{2}} \equiv \prod_{l,m} \int dc_{l,m} exp\{-(c_{l,m})^{2} \varepsilon_{l}\} ,$$
(25)

where the relation:  $8\Theta + \Xi + \frac{1}{8}\Omega = 0$  is used. In the summation of (25), we may omit the modes l = 0 and l = 1. The l = 0 mode is a constant term in (24) which can be absorbed into the freedom of the global Weyl transformation of the background metric  $(g_{cl})_{ab} = \delta_{ab}e^{\varphi_c}$  (the eq.(10) is invariant under the constant translation:  $\varphi \to \varphi + c_{0,0}$ ,  $\beta \to \beta e^{c_{0,0}}$ ,  $\lambda \to \lambda e^{-c_{0,0}}$ ). The l = 1 mode does identically vanish in the sum of (25) due to the relation:  $\varepsilon_1 = 8\Theta + \Xi + \frac{1}{8}\Omega = 0$ .

If the classical solution is stable against the quantum fluctuation, the energy eigenvalue  $\varepsilon_l$  should satisfy  $\varepsilon_l > 0$  for all  $l \ge 2$ . In this case our semiclassical approximation is valid. It will soon be verified for an appropriate range of w. Thus we have

$$\hat{Z}[\lambda]^{1-loop} = \int d^2x \mathcal{D}\psi e^{S_2} = \prod_{l \ge 2} (\sqrt{\frac{\pi}{\epsilon_l}})^{2l+1}$$
$$= const \times exp \left[ -\frac{1}{2} \sum_{l=2}^{\infty} (2l+1) \ln \epsilon_l \right] \quad .$$
(26)

Now we introduce the *ultra-violet cut-off length*: a, or equivalently the *cut-off number* N for the sum in the exponent of the above equation. Then the summation in the exponent of (26) may be *regularized* as

$$\hat{Z}[\lambda]_{N}^{1-loop} \equiv e^{+Q(N)} , \quad Q(N) \equiv -\frac{1}{2} \sum_{l=2}^{N} (2l+1) \ln \varepsilon_{l} ,$$
  
$$\sum_{l=0}^{N} (2l+1) = N^{2} + 2N + 1 = \frac{A}{a^{2}} , \qquad (27)$$

where  $\sum_{l=0}^{N} (2l+1)$  is the degree of freedom of the regularized system.  $a^2$  is regarded as the unit area of a triagle when the 2d manifold is approximated by the traiangulated surface.

## 4 Renormalization

The quantity Q(N) of (27) is divergent as  $N \to +\infty$   $(a \to +0)$  (ultra-violet divergence). Following the famous calculation by 't Hooft[30], where the 1-loop quantum effect around the instanton is skillfully and beautifully evaluated in (4 dim) QCD, we perform two procedures to define a finite quantity meaningfully: i) Normalization of  $\hat{Z}[\lambda]_N^{1-loop}$ , ii) Renormalization of the theory.

#### 4.1 Normalization

We normalize  $\hat{Z}[\lambda]_N^{1-loop}$  as

$$\tilde{Z}[\lambda]_N^{1-loop} \equiv \frac{\hat{Z}[\lambda]_N^{1-loop}}{K} \quad , \quad K = \hat{Z}[\lambda]_N^{1-loop}|_{\beta'=+\infty} \quad . \tag{28}$$

Here we normalize the partition function in such a way that the quantum effect vanishes at the asymptotic region  $\beta' = +\infty$ . (This procedure corresponds to,in

ordinary field theory, the removal of vacuum bubble diagrams:  $\tilde{Z}[J] = Z[J]/Z[0]$ , where J is the external source in Schwinger's formalism). The normalization factor is obtained by considering the extreme case:  $\beta' \to +\infty$  in (20).

$$\begin{aligned} \Xi(+\infty) &= -\frac{\xi}{\gamma} \quad , \quad \Omega(+\infty) = -\frac{4(1-\xi)}{\gamma} \quad , \quad \Theta(+\infty) = +\frac{1+\xi}{16\gamma} \quad ,\\ \varepsilon_l(+\infty) &= \frac{1}{8\gamma} \{ (1+\xi)l(l+1) + 2(1-\xi) \} (l+2)(l-1) > 0 \quad \text{for} \quad l \ge 2 \quad ,\\ \tilde{Z}[\lambda]_N^{1-loop} &\equiv exp \; \tilde{Q}(N) \quad , \quad \tilde{Q}(N) = -\frac{1}{2} \sum_{l=2}^N (2l+1) \; ln \frac{\varepsilon_l(w)}{\varepsilon_l(+\infty)} \quad . \end{aligned}$$
(29)

The normalization factor  $K = exp[-\frac{1}{2}\sum_{l=2}^{N}(2l+1) \ln \varepsilon_l(+\infty)]$  depends only on  $\gamma$  and  $\xi$  and does not affect the dynamics. K can be essentially regarded as an (infinite) constant.

### 4.2 Renormalization

The 1-loop effective action  $\tilde{Q}(N)$ ,(29), is still divergent at  $N \to +\infty$  (or  $a \to +0$ ). We must therefore renormalize the theory.

We present ,in this paragraph (between eq.(30) and eq.(33)), the formalism for the renormalization. Collecting the contents of sect.2.1 up to eq.(7), we already have the following formula before renormalization.

$$\begin{split} \bar{Z}[A] &= \int d\lambda \ e^{+\frac{\lambda A}{\hbar}} \times exp_{\hbar}^{1} (\frac{8\pi(1-h)}{G} - \mu A + S_{\lambda}[\varphi_{c};\beta]) \times \hat{Z}^{loop}[\lambda;\beta] \quad , \\ \hat{Z}^{loop}[\lambda;\beta] &= \int \mathcal{D}\psi \ exp\{S_{\lambda}[\varphi_{c} + \sqrt{\hbar}\psi;\beta] - \frac{\delta S_{\lambda}[\varphi;\beta]}{\delta\varphi}\Big|_{\varphi_{c}} \psi - S_{\lambda}[\varphi_{c};\beta]\} \quad , \quad (30) \\ \text{COND.1} \quad \frac{\delta}{\delta\varphi}S_{\lambda}[\varphi;\beta]\Big|_{\varphi_{c}} = 0 \quad , \end{split}$$

where  $\beta$ -dependence is explicitly written for the immediate use. Generally the quantum effect  $\hat{Z}^{loop}$  is divergent. We absorb those divergences into the ambiguity of the bare cosmological constant  $\mu_b = \mu + \Delta \mu$  and the bare higher-derivative coupling  $\beta_b = \beta + \Delta \beta$ . (We will soon see renormalization for G,  $\gamma$ ,  $\xi$  and  $\varphi_c$  is not necessary.)

$$\begin{split} \bar{Z}^{r}[A] &= \int d\lambda \ e^{\frac{i}{\hbar} \frac{\lambda A}{\hbar}} \times exp\frac{1}{\hbar} \{ \ \frac{8\pi(1-h)}{G} - (\mu + \Delta\mu)A + S_{\lambda}[\varphi_{c};\beta + \Delta\beta] \} \\ &\times \frac{1}{K_{1}} \hat{Z}^{loop}[\lambda;\beta + \Delta\beta] \\ &= exp\frac{1}{\hbar} \{ \ \frac{8\pi(1-h)}{G} - (\mu + \Delta\mu)A \ \} \times \int d\lambda \frac{1}{K_{1}} Y[A,\lambda;\beta + \Delta\beta] \ , \end{split}$$
(31)  
$$\begin{aligned} &\text{COND.1'} \quad \frac{\delta}{\delta\varphi} S_{\lambda}[\varphi;\beta + \Delta\beta] \Big|_{\varphi_{c}} = 0 \quad . \end{split}$$

 $K_1$  is the normalization factor which is, by itself, a divergent constant and is defined below. The  $\lambda$ -integral is approximated by the saddle point  $\lambda_c^r$  defined below.

$$\bar{Z}^{r}[A] \approx exp_{\bar{h}}^{1} \left\{ \frac{8\pi(1-h)}{G} - (\mu + \Delta\mu)A \right\} \times \frac{1}{K_{1}}Y[A, \lambda_{c}^{r}; \beta + \Delta\beta] , \qquad (32)$$

$$COND.2^{r} \quad \frac{\delta}{\delta\lambda} ln Y[A, \lambda; \beta + \Delta\beta] \Big|_{\lambda_{c}^{r}} = 0 ,$$

$$\bar{K}_{1} \equiv \bar{Z}^{loop}[\lambda_{c}^{r}(\beta + \Delta\beta); \beta + \Delta\beta] \Big|_{\beta'=+\infty} . \qquad (33)$$

 $\Delta \mu$  and  $\Delta \beta$  can be fixed perturbatively (with respect to  $\hbar$ ) in such a way that the quantity (32) becomes finite.

We now obtain the explicit forms of  $\Delta \mu$  and  $\Delta \beta$ .

i) w > 0

We consider first the case w > 0. Using the Euler-Maclaurin formula,

$$\sum_{l=2}^{N} g(l) = \int_{1}^{N} g(x) dx + \left[ \frac{1}{2} g(x) + \frac{1}{12} g'(x) - \frac{1}{720} g'''(x) + \cdots \right]_{1}^{N} , \qquad (34)$$

we can evaluate  $\tilde{Q}(N)$  as

$$\tilde{Q}(N) = -\frac{1}{2} \{ (ln \frac{16\gamma\Theta}{1+\xi}) \times \frac{A}{a^2} + (-\frac{1}{32} \frac{\Omega}{\Theta} - \frac{2(1-\xi)}{1+\xi}) \times ln \frac{A}{a^2} \}$$
  
+FiniteTerm +  $O(\frac{1}{N})$ , (35)

where the relation  $(N+1)^2 = A/a^2$  is used.

In the above formalism of renormalization, we have considered the general order case. At the lowest order,  $\Delta \mu$  and  $\Delta \beta$  can be obtained in a simple way: First shift the parameters  $\mu$  and  $\beta$  (  $\mu$   $\rightarrow$   $\mu$  +  $\Delta\mu,\beta$   $\rightarrow$   $\beta$  +  $\Delta\beta$  ) in the lowest-order 'total' effective action:  $\bar{\Gamma}^{eff}[\mu,\beta] \equiv \frac{8\pi}{G} - \mu A + \Gamma^{eff}|_{\hbar^0} = \frac{8\pi}{G} - \mu A + \frac{4\pi}{\gamma}(1+\xi) \ln \frac{\alpha_c^-(w)}{8} - \mu A + \frac{4\pi}{\gamma}(1+\xi) + \frac{4\pi}{\gamma}(1+\xi) \ln \frac{\alpha_c^-(w)}{8}$  $\frac{w+1}{\gamma}\alpha_c^-(w) + \frac{w}{16\pi\gamma}(\alpha_c^-(w))^2 + C(A)$  and then fix  $\Delta\mu$  and  $\Delta\beta$  in such a way that the variation due to the shift cancelles the divergent parts of  $\tilde{Q}(N)$ .

$$\Delta \mu = M(w,\xi) \times \frac{1}{a^2} , \quad \Delta w (= 16\pi\gamma \frac{\Delta\beta}{A}) = \gamma \ W(w,\xi) \ \ln \frac{A}{a^2} ,$$
  
$$M(w,\xi) = -\frac{1}{2} \ \ln \frac{\alpha_c^- \cdot w}{4\pi(1+\xi)} , \quad W(w,\xi) = \frac{2}{\alpha_c^-} (\frac{1}{1+\xi} - \frac{4\pi}{\alpha_c^- w}) / (1 - \frac{\alpha_c^-}{16\pi}) . \quad (36)$$

We need not do renormalization for the Einstein coupling G, the matter-parameter

 $\gamma$  , the topological coupling  $\xi$  and the gravitational field  $\varphi_c$ . <sup>6</sup> The renormalization group beta-function,  $B_w$ , for the  $R^2$ -coupling is given as

$$w_b(a) = w + \Delta w \quad ,$$
  
$$B_w = \lim_{t \to +0} \frac{w_b(ae^{-t}) - w_b(a)}{t} = 2\gamma W(w, \xi) \quad . \tag{37}$$

Fig.4 shows the behaviour of  $W(w,\xi)$ . We see the negative (semi)definiteness of  $B_w$ , which means the R<sup>2</sup>-coupling is asymptotically free for w > 0<sup>7</sup>. (Note that we consider the case  $\gamma > 0$   $(c_m < 26)$  ). The asymptotic behaviour, for the general  $\xi$  (-1  $\leq \xi \leq 1$ ), is given as

$$W(w,\xi) = \begin{cases} -\frac{1}{4\pi} \frac{1-\xi}{(1+\xi)^2} + O(\frac{1}{w}) &, & w \gg 1\\ -\frac{2}{\pi(1+\xi)^2(3-\xi)} \frac{1}{w} (1+O(w)) &, & 0 < w \ll 1 \end{cases}$$
(38)

<sup>&</sup>lt;sup>6</sup>No wave-function renormalization means we need not introduce the so-called gravitational dressing:  $e^{\varphi} \rightarrow e^{\kappa \varphi}$ , at least in the perturbative approach.

<sup>&</sup>lt;sup>7</sup>This means the surface becomes smoother as we 'average' the small-scale configurations because the  $R^2$ -coupling controls the smoothness of the surface[5]. This outcome is natural.

For the special case  $\xi = 1$ , we obtain

$$w \ge 1 \quad , \qquad \lim_{\xi \to 1-0} W(w,\xi) = 0 \quad , 0 < w \le 1 \quad , \quad \lim_{\xi \to 1-0} W(w,\xi) = -\frac{1}{4\pi} (\frac{1}{w} - 1) \quad .$$
(39)

This clearly shows the choice  $\xi = 1$  guarantees the conformal symmetry for  $w \ge 1^{-8}$ .

The renormalization group behaviour of the cosmological constant is given as

$$\mu_b(a) = \mu + \Delta \mu \quad ,$$
  

$$B_{\mu a^2} = \lim_{t \to +0} \frac{\mu_b(ae^{-t}) - \mu_b(a)}{t} \times a^2 = 2M(w, \xi) \quad .$$
(40)

Fig.5 shows the behaviour of  $M(w,\xi)$ . We see the positive (semi)definiteness of  $B_{\mu a^2}$ , which means the cosmological constant is asymptotically non-free for w > 0. This fact implies the cosmological constant decreases as the scale-size becomes larger. The asymptotic behaviour of  $M(w,\xi)$ , for the general  $\xi$  ( $-1 \le \xi \le 1$ ), is given as

$$M(w,\xi) = \begin{cases} \frac{1-\xi}{4} \frac{1}{w} + O(\frac{1}{w^2}) &, & w \gg 1\\ -\frac{1}{2} \ln w + \frac{1-\xi}{4} w + O(w^2) &, & 0 < w \ll 1 \end{cases}$$
(41)

For the special case:  $\xi = 1$ , we see

$$w > 1 , \qquad \lim_{\xi \to 1-0} M(w,\xi) = 0 , 0 < w < 1 , \qquad \lim_{\xi \to 1-0} M(w,\xi) = -\frac{1}{2} \ln w > 0 .$$
(42)

This result again shows the choice  $\xi = 1$  guarantees the conformal symmetry for  $w \ge 1$ .

ii) 
$$w = 0$$

 $\tilde{Q}(N)$  is evaluated as

$$\tilde{Q}(N) = -\frac{1}{2} \sum_{l=2}^{N} (2l+1) \ln \frac{\varepsilon_{l}(0)}{\varepsilon_{l}(+\infty)} 
= -\frac{1}{2} \sum_{l=2}^{N} (2l+1) \{ \ln \frac{4}{1+\xi} - \ln(l^{2}+l+\frac{2(1-\xi)}{1+\xi}) \} 
= +\frac{1}{2} \frac{A}{a^{2}} \ln \frac{A}{a^{2}} + \frac{1}{2} (\frac{2(1-\xi)}{1+\xi} - \frac{1}{3}) \times \ln \frac{A}{a^{2}} 
-\frac{1}{2} (\ln \frac{4}{1+\xi} + 1) \times \frac{A}{a^{2}} + \text{FiniteTerm} + O(\frac{1}{N}) \quad .$$
(43)

Although the second and third term of the last expression can be renormalized into  $\beta$  and  $\mu$  as in the case (i), the first term cannot be. We conclude, for the case w = 0, the system is stable against the quantum excitation ( $\varepsilon_l > 0$  for  $l \ge 2$ ) but the ultra-violet divergence cannot be renormalized. This is simply because the higher-derivative term  $R^2$  does not contribute in this case.

<sup>&</sup>lt;sup>8</sup>This result reminds us of the similar situation in the combined system of chiral-model+Wess-Zumino-term . In this case the system becomes conformal(WZNW-model) for the special choice of the coupling of WZ-term[31]. The topological term in the present paper plays the similar role to the WZ-term.

iii) w < 0

In this case, the situation is worse than (ii).  $\varepsilon_l(w)$  becomes negative for  $l \ge l^*$ where  $l^*(l^* + 1) \approx \Omega/(32\Theta) = 2 - 16\pi/(\alpha w)$ . The system is unstable against the quantum fluctuation.

## **5** Discussions

We have presented the quantum analysis of  $2d R^2$ -gravity with the topology of a sphere. As far as the  $R^2$ -coupling is positive ( $\beta > 0$ ), the quantum theory is well-defined in a perturbative way. The infrared regularization L and the ultra-violet regularization a are introduced in order to treat those divergences unambiguously. The topological term is carefully treated and its importance in the quantum level is confirmed as in the classical analysis of [5]. The general renormalization procedure is presented for the present model of quantum gravity. The 1-loop explicit calculation is done and the renormalization group beta-functions for the  $R^2$ -coupling ( $B_w$ ) and for the cosmological term ( $B_{\mu a^2}$ ) are obtained. Generally they are asymptotically free and asymptotically non-free respectively. The topological term controls the conformal behaviour. The special choice  $\xi = 1$  guarantees  $B_w = 0$  and  $B_{\mu a^2} = 0$  for  $w \ge 1$ . This clearly shows the choice makes the theory conformal for  $w \ge 1$ . The case  $B_{\mu a^2} > 0$  means that the cosmological constant decreases as the scale-size becomes larger. This fact suggests that, in the real world, the origin of the vanishing cosmological constant could come from such a renormalization effect.

For  $w \leq 0$ , the quantum effect cannot be evaluated using this method. Since the classical vacuum explains the simulation data very well including the negative  $\beta$  (or w)[5], the quantum effect should be small. For the region  $|w| \ll 1$  the surface is strongly fluctuating (fractal surface)[5], so the simple separation of quantum and classical, as in the semiclassical approach, might not be valid. For the region w < -1, the instability due to the term  $\beta R^2$  is not be controled by the present approach. The evaluation of the quantum effect for the case  $w \leq 0$  is an open question at present.

#### Acknowledgement

The author thanks Prof. T.Yukawa, Dr. A.Fujitsu and Dr. N.Tsuda for stimulating discussions and for sharing the latest knowledge of the present problem, especially their original data of the numerical simulation. He thanks all members at KEK, Theory Group, for kind hospitality during his stays and visits. Especially the discussion and comment by Prof. H.Kawai and Prof. N.Ishibashi were valuable. This work is completed during the stay at SLAC. The author thanks its all members for hospitality. Especially he thanks Prof.M.Peskin for discussions and Dr.D.Atwood for carefully reading the manuscript. Finally the author thanks the financial support by KEK.

## References

-

- [1] R.P.Feynman, Acta Phys.Polonica24(1963)697
- [2] V.G.Knizhnik, A.M.Polyakov A.B.Zamolodchikov, Mod.Phys.Lett. A3(1988)819

and

- [3] J.Distler and H.Kawai, Nucl. Phys. **B321**(1989)509
- [4] F.David, Mod. Phys. Lett. A3(1988)1651
- [5] S.Ichinose, N.Tsuda and T.Yukawa, Preprint of Univ. of Shizuoka, US-94-03(1994), 'Classical Solution of Two Dimensional R<sup>2</sup>-Gravity and Cross-Over Phenomenon'
- [6] T.Yoneya, Phys.Lett.**B149**(1984)111
- [7] D.V.Boulatov and V.A.Kazakov, Phys.Lett. B184(1987)247
- [8] N.Tsuda and T.Yukawa, Phys. Lett. B305(1993)223
  - [9] S.Ichinose, Proc. of Workshop on 'Study of Superstring' (Oct.19-21,1992,YITP,Kyoto Univ.,Japan), page D24-D33. "Role of R<sup>2</sup>-term in 2 Dimensional Quantum Gravity"
- [10] H.Kawai and R.Nakayama, Phys. Lett. 306B(1993)224
- [11] S.Weinberg, in General Relativity: An Einstein Centenary Survey, eds. S.W.Hawking and W.Israel(Cambridge University Press, 1979), 'Ultraviolet divergences in quantum theories of gravitation'
- [12] H.Kawai and M.Ninomiya, Nucl. Phys. B336(1990)115
- [13] H.Kawai,Y.Kitazawa and M.Ninomiya,Nucl.Phys.**B393**(1993)280; Nucl.Phys.**B404**(1993)684
- [14] E.D'Hoker and R.Jackiw, Phys. Rev. D26(1982)3517
- [15] N.Seiberg, Prog. Theor. Phys. Suppl. 102(1990)319
- [16] F.David, Nucl. Phys. B311(1985)45,543

- [17] V.Kazakov, Phys. Lett. **B150**(1985)28
- [18] J.Ambjørn, B.Durhuus and J.Fröhlich, Nucl. Phys. B257 [FS13] (1985)433
- [19] F.David, Nucl. Phys. B257 [FS14] (1985) 543
- [20] V.A.Kazakov, I.K.Kostov and A.A.Migdal, Phys.Lett. B157 (1985)295
- [21] S.Ichinose, Int. Jour. Mod. Phys. A8(1993)2735
- [22] I.Jack and D.R.T.Jones, Nucl. Phys. B358(1991)695
- [23] Brezin and Zinn-Justin, Phys. Lett. B288(1992)54
- [24] J.Nishimura, N.Tsuda and T.Yukawa, Prog. Theor. Phys. Supp. 114 (1993) 19
- [25] A.M.Polyakov, Phys. Lett. **103B**(1981)207
- [26] R.Jackiw, in Quantum theory of gravity,ed. S.Christensen (Adam Hilger,Bristol,1984)p.403
- [27] A.H.Chamseddine, Phys.Lett.**256B**(1991)379; Phys.Lett.**258B**(1991)97; Nucl.Phys.**B368**(1992)98
- [28] E.Onofri and M.Virasoro, preprint TH3233
- [29] A.B.Zamolodchikov, Phys. Lett. **117B**(1982)87
- [30] G. 'tHooft, Phys. Rev. D14(1976)3432

- \_

[31] E.Witten, Comm. Math. Phys. 92(1984)455

## Figure Captions

Fig.1 Infra-red cut-off L in the flat coordinates and the sphere manifold. For simplicity, the picture is for  $\alpha = 8$ . For general  $\alpha$ ,  $(x, y, r, \sqrt{A}, L)$  is substituted by  $\sqrt{8/\alpha} \times (x, y, r, \sqrt{A}, L)$ .

Fig.2a Curvature for the solutions  $\alpha_c^{\pm} = A \times R^{\pm}$  for  $\xi = 0$ .  $w = 16\pi\beta'\gamma$ .

Fig.2b Curvature for the solutions  $\alpha_c^{\pm} = A \times R^{\pm}$  for  $\xi = 0.99$ .  $w = 16\pi\beta'\gamma$ .

Fig.3 Plane coordinate and polar coordinate for a sphere. The same note as in the caption of Fig.1.

Fig.4 Behaviour of  $B_w/(2\gamma) = W(w,\xi)$ .

Fig.5 Behaviour of  $B_{\mu a^2}/2 = M(w,\xi)$ .

÷

- -



÷

- -









Ę.



 $\frac{1}{\sqrt{A}} = \frac{P(\theta, \phi)}{\sqrt{A}} = \frac{(x, y) - plane}{(x, y) - plane}$ 

- -







v





