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Instanton Calculations Versus Exact Results in 4 Dimensional SUSY Gauge Theories

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We relate the non-perturbative exact results in supersymmetry to perturbation theory using several different methods: instanton calculations at weak or strong coupling, a method using gaugino condensation and another method relating strong and weak coupling. This allows many precise numerical checks of the consistency of these methods, especially the amplitude of instanton effects, and of the network of exact solutions in supersymmetry. However, there remain difficulties with the instanton computations at strong coupling.

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1. Introduction

Instantons have played an important conceptual role in the study of non-perturbative effects in four dimensional gauge theories. However, their role in the quantitative understanding of these theories has remained obscure. In theories of physical interest like QCD, instanton calculations are plagued by infrared divergences. In the case of supersymmetric gauge theories, non-renormalization theorems and improved infrared properties allow one to make definite statements about the effects of instantons. Two main approaches have been taken to study instanton effects in supersymmetric theories. Affleck, Dine and Seiberg studied theories along the flat directions of the scalar potential, where the gauge group is spontaneously broken and the theory is weakly coupled, and computed the effective interactions of the light degrees of freedom [1]. They demonstrated that instantons generate a calculable superpotential in $SU(N_c)$ theories with $N_c - 1$ flavors. This weak coupling approach was adopted by the authors of [2] who extended it to the study of Green functions and the calculation of chiral condensates. Following [1], analytic continuation in holomorphic parameters was used to extend the results obtained at weak coupling to the strong coupling regime [3]. In the other approach, begun in [4] and continued in [5,6], the instanton calculations are performed at the origin of field space in the strong coupling regime. One computes chiral Green functions which vanish in perturbation theory, are independent of position by supersymmetry and are argued to be saturated by instanton contributions; cluster decomposition is then used to extract the values of condensates. Discrepancies arise in the quantitative comparison of the two approaches [7,3,2]; there is a long-standing controversy (section 9 of [6]) over why this is so and which approach is correct.

The new ingredient to this story is the progress over the last two years in obtaining exact results about the low-energy behavior of supersymmetric gauge theories [8-13]. Following [1], this recent progress allowed a greater number of theories to be exactly related to one another. The convenient way that is used to relate two different theories is by expressing some non-perturbative result of one theory in terms of the dynamically generated scale Λ of the other. Considering the results of [1,8-15] as a whole, one sees an impressive web of interdependent results. Sometimes it happens that in the solution of one theory, a non-perturbative effect can directly be related to perturbation theory. After this is done once, each subsequent connection to perturbation theory provides an independent consistency check of the network of exact solutions.

The next step in this framework, which is one purpose of this paper, is to actually perform such checks with perturbation theory. In section 2, we choose a specific perturbative regularization scheme ($\overline{DR}$), define the scale of each theory in this scheme and relate the scales of different theories in perturbation theory at one-loop. To establish a first connection with perturbation theory, we use a method involving an interplay between gluino condensation and the strength of instanton effects in the $SO(4)$ theory with one vector solved in [10]. This connection precisely determines the amplitude of non-perturbative effects for all the theories of the network. By relating strong to weak coupling, the solution of N=2 $SU(2)$ SYM of Seiberg and Witten provides another connection to perturbation theory, fully consistent with the first one. Instanton calculations provide more connections with perturbation theory. In the remaining sections, we compare the results of explicit instanton calculations to the predictions for their amplitude derived throughout section 2. In section 3, we review the instanton calculations of effective lagrangians, correcting some minor errors appearing in the literature, and compare these results to the predictions. In section 4, we compare the predictions to the Green functions calculations in the weak coupling regime. In both cases, we find complete quantitative agreement. Finally, in section 5 we briefly point out some discrepancies when Green functions are computed with instanton methods at strong coupling. As a byproduct of this analysis, we obtain in section 2.4 the precise normalization of the nonperturbative superpotentials for $SU(N)$ and gaugino condensates for $SU(N)$, $SO(N)$ and $Sp(N)$ gauge groups. This result should be useful in practical applications of gaugino condensation, e.g. in string phenomenology.

2. Relating the scale Λ to perturbation theory in $\overline{DR}$

2.1. The $\overline{DR}$ regularization scheme

In asymptotically free gauge theories, a scale Λ is dynamically generated by dimensional transmutation of the gauge coupling and non-perturbative results are naturally expressed in terms of this scale. It is well-known that this scale can be defined in perturbation theory, that it depends on the regularization scheme chosen and that this dependence is exactly given by a one-loop perturbative computation [16]. We will find it convenient to use the scheme of dimensional regularization through dimensional reduction, with modified minimal subtraction, $\overline{DR}$. This scheme was introduced by Siegel [17] and it is standard in supersymmetry. The coordinates and momenta are analytically continued to $4-\epsilon$ dimensions, but it differs from the more common $\overline{MS}$ scheme in that the number of components

of spinor and tensor fields is held fixed. Like the $\overline{MS}$ scheme, $\overline{DR}$ is a valid regularization scheme to all orders in perturbation theory [18], but $\overline{DR}$ has the advantage of keeping supersymmetry manifest. Of course, one could just as well work in any other scheme, and the formulas for translating to other conventional schemes such as $\overline{MS}$ and Pauli-Villars are given below.

For a generic supersymmetric gauge theory, we define the dynamical scale as

$$\Lambda_{\overline{DR}}^{\mathrm{b}_0} \equiv \mu^{\mathrm{b}_0} e^{-8\pi^2/g^2_{\overline{DR}}(\mu)} \tag{2.1}$$

with the bare coupling defined at a scale μ. Here b_0 is the one-loop coefficient of the β function, given by $\mathrm{b}_0 = \frac{3}{2} I_2(adjoint) - \frac{1}{2}\sum_R I_2(R)$, where I_2 is the second Casimir index and the sum is over the chiral multiplets in a representation R of the group (e.g. $\mathrm{b}_0 = 3N_c - N_f - N_a N_c$ for $SU(N)$ with N_f flavors in the fundamental and N_a chiral multiplets in the adjoint representation). This expression is invariant under the renormalization group of the Wilsonian effective action [19,20] to all orders in perturbation theory. It is not invariant under the more usual renormalization group of the 1PI effective action, however this difference appears at two and higher loops. As stressed above, only a one-loop computation is required for our purposes, and we will ignore this difference. In other words, we will not write down explicitly the powers of g throughout this paper, as they are a higher loop effect.

One more advantage of $\overline{DR}$ is that it realizes the simple threshold relations assumed in [10]. The matching is most easily accomplished using background field methods [21,22]. It involves computing the coefficient of the background field operator $F^2_{\mu\nu}$ at one-loop and was done by Weinberg [23]. When integrating out a set of superfields with the same canonical mass M (e.g. vector bosons and their superpartners under the Higgs mechanism or heavy quarks and squarks which decouple from the low-energy physics), the formula of Weinberg expressed in the $\overline{DR}$ scheme yields the simple relation between the scales of the low and the high energy theories:

$$\left(\frac{\Lambda_L}{M}\right)^{\mathrm{b}_{0\,L}} = \left(\frac{\Lambda_H}{M}\right)^{\mathrm{b}_{0\,H}}. \tag{2.2}$$

That is, we find that the thresholds are unity. In the case where the high-energy gauge group is broken, the smaller group is assumed to be trivially embedded in the following sense: we have chosen its generators to be a subset of the generators of the larger group. In other regularization schemes, such as $\overline{MS}$, the relation (2.2) would be corrected by

finite numerical factors, which one should keep track of. Relating the scale parameters in different schemes is similarly done using the background field method. One obtains a relation between the scales of two different schemes A and B:

$$\Big(\frac{\Lambda^{(A)}}{\Lambda^{(B)}}\Big)^{\mathrm{b}_0} = \exp\Big(\frac{I_2(adjoint)}{2}(a_1^{(A)} - a_1^{(B)}) - \sum_R \frac{I_2(R)}{2}(a_2^{(A)} - a_2^{(B)})\Big). \tag{2.3}$$

In the $\overline{DR}$, Pauli-Villars and ζ function schemes, both a_1 and a_2 vanish, while for $\overline{MS}$, $a_1 = 1/6$ and $a_2 = 0$. We thus have the well-known relations between schemes [22,24,25]:

$$\Lambda^{\mathrm{b}_0}_{\overline{DR}} = \Lambda^{\mathrm{b}_0}_{PV} = \Lambda^{\mathrm{b}_0}_{\zeta} = e^{-I_2(adjoint)/12}\Lambda^{\mathrm{b}_0}_{\overline{MS}}. \tag{2.4}$$

2.2. Matching perturbation theory to the exact solutions

In this section, we will show that the exact solution of the $SO(4) = SU(2)_1 \times SU(2)_2$ theory with matter Q in the vector $(2,2)$ representation provides a direct way to relate the non-perturbative results to perturbation theory. The effective superpotential for this model was exactly determined by Intriligator, Leigh and Seiberg in [10] up to an arbitrary constant c:

$$W = S_1\Big[\log\Big(\frac{c\Lambda_1^5}{S_1^2}\Big) + 1\Big] + S_2\Big[\log\Big(\frac{c\Lambda_2^5}{S_2^2}\Big) + 1\Big] + (S_1 + S_2)\log\Big(\frac{S_1 + S_2}{X}\Big) + mX \tag{2.5}$$

where $X = \det Q = Q_1^\alpha Q_{2\alpha}$ and $S_i = \frac{W_{i\alpha}^{a2}}{32\pi^2}$ are the glueball superfields. The following argument shows that, for consistency of the model, we must have $c = 1$ in the $\overline{DR}$ scheme. For $m \neq 0$, integrating out X leads to two pure $SU(2)$ gauge theories with $W = 2S_1 + 2S_2$. Solving for S_i gives $S_i = \pm\sqrt{c}\tilde{\Lambda}_i^3$ where $\tilde{\Lambda}_i^3$ is the scale of the low-energy $SU(2)_i$ group and $\tilde{\Lambda}_i^6 = m\Lambda_i^5$ from equation (2.2). This shows that with this set of conventions and normalizations, a contribution of $2S$ to the superpotential must appear with $S = \pm\sqrt{c}\Lambda_{0,0}^3$ whenever gaugino condensation occurs. If we now take X to be massless and take $\langle X\rangle \neq 0$, the gauge group is broken to a diagonal subgroup $SU(2)_D$. This is a pure $SU(2)$ SYM theory up to irrelevant interactions with the field X. Therefore we expect that $S_D = \pm\sqrt{c}\Lambda_D^3$, where the matching $\Lambda_D^3 = \Lambda_1^{5/2}\Lambda_2^{5/2}/\langle X\rangle$ was performed using equation (2.2) and recalling that $1/g_1^2 + 1/g_2^2 = 1/g_D^2$. Integrating out S_1 and S_2 from equation (2.5) gives $W_{eff} = c(\Lambda_1^{5/2} \pm \Lambda_2^{5/2})^2/X$. The cross-term $\pm 2c\Lambda_1^{5/2}\Lambda_2^{5/2}/X$ must arise from gluino condensation in $SU(2)_D$. For this to be the case, we must have $\sqrt{c} = c$. Therefore $c = 1$.

To summarize, taking the coupling $g_2 = 0$, we deduce that the gluino condensate of pure $SU(2)$ SYM is

$$\langle S \rangle = \pm \Lambda^3 \tag{2.6}$$

and that the superpotential of the $SU(2)$ theory with one flavor is

$$\frac{\Lambda^5}{X}. \tag{2.7}$$

Thus, the exact solutions predict that the coefficient of the well-known instanton-induced superpotential of Affleck, Dine and Seiberg is 1 when expressed in terms of $\Lambda_{\overline{DR}}$.

2.3. $SU(2)$ model with 2 doublets and a triplet

The solution of this theory by Intriligator and Seiberg [12] relates the non-perturbative results of the $SU(2)$ theories without matter, with one fundamental flavor or with one adjoint. Our purpose is to match this solution to perturbation theory via the connection established in section 2.2. We give some details to illustrate precisely what we mean. This is straightforward and, for other models, we will simply quote in section 2.4 the results of such matchings.

We begin by taking in its entirety the solution of the model in [12]. We will only summarize the features that we need. The basic gauge singlets are $X = Q_1^\alpha Q_{2\alpha}$, $U = \frac{1}{2}\phi^a\phi^a$ and $\vec{Z} = \frac{\sqrt{2}}{2} Q_f^\alpha \phi^b \sigma^b_{\alpha\beta} Q_g^\beta \vec{\sigma}^{fg}$ $(\alpha, \beta, f, g = 1, 2; a, b = 1, 2, 3)$. Their expectation values label the inequivalent vacua. They are constrained classically by $UX^2 = \vec{Z}^2$, but they are independent quantum mechanically. It is enough for our purposes to consider only the subspace $\vec{Z} = 0$. We will use the notation Λ_{N_f, N_a} for the scales of the theories obtained by reducing the number of fundamentals (of mass m) or adjoints (of mass M) by giving masses to them and integrating them out. The matching of the scales is simply $mM^2\Lambda^3_{1,1} = M^2\Lambda^4_{0,1} = m\Lambda^5_{1,0} = \Lambda^6_{0,0}$ using equation (2.2). This model has three phases:

1. For generic VEVs of X, U, the model is in a Higgs-confining phase described by the superpotential (whose form is determined by the symmetries)

$$W = \frac{-XU^2}{4\Lambda^3_{1,1}}. \tag{2.8}$$

It describes the theory everywhere away from the Coulomb phase. The normalization of this superpotential is chosen in order to match the results of the previous section, as will be shown below.

2. On the line labeled by $U \neq 0$, $X = 0$, the theory is in a free Coulomb phase. The quantum moduli space is singular on this line, because the elementary fields Q_f are charged under an unbroken $U(1)$.
3. At $X = U = 0$, the theory is in an interacting non-Abelian Coulomb phase.

Now we consider perturbing this theory by mass terms. Adding MU to the superpotential (2.8) and integrating out U, we find an effective $SU(2)$ theory with one fundamental flavor, with precisely the superpotential (2.7). Thus we have the chosen the correct normalization in (2.8). Adding a mass mX to the superpotential (2.8) and integrating out X, we flow to the N=2 $SU(2)$ SYM model (for more details see section 2.5), at the location of its singularities on its quantum moduli space:

$$\langle U \rangle = \pm 2\Lambda_{0,1}^2. \tag{2.9}$$

Integrating in S, starting from (2.8), leads to $W = S[\log\left(\frac{4\Lambda_{1,1}^3 S}{XU^2}\right) - 1]$. We give a mass to X and integrate it out, to get $W = S\log\left(\frac{4m\Lambda_{1,1}^3}{U^2}\right)$. Giving a mass to U and integrating it out yields $W = S[\log\left(\frac{mM^2\Lambda_{1,1}^3}{S^2}\right) + 2]$: we have flown to a pure $SU(2)$ SYM theory and recovered that gaugino condensation occurs at the location (2.6):

$$\langle S \rangle = \pm \Lambda_{0,0}^3. \tag{2.10}$$

We view this as a consistency check on the matching of the $SU(2) \times SU(2)$ model with a $(2,2)$ to the $SU(2)$ model with one flavor and one adjoint. Along the way, we also got that in the confining phase:

$$m\langle X \rangle = \langle S \rangle \qquad \text{and} \qquad M\langle U \rangle = 2\langle S \rangle. \tag{2.11}$$

These relations between condensates are seen to follow from the exact solution of Intriligator and Seiberg. Alternatively, they are known to arise from the Konishi anomaly [26].

2.4. Other models

Using the result for $SU(2)$ with one flavor obtained in equation (2.7), we fix the normalization of the superpotentials in $SU(N_c)$ theories with $N_f < N_c$ flavors by induction. We start with $SU(N_c)$ with $N_c - 1$ flavors, by giving a large VEV to one flavor, and flow to $SU(N_c - 1)$ with $N_c - 2$ flavors. Repeating this procedure, we are eventually lead back

to $SU(2)$ with one flavor. Having fixed the superpotential for $N_f = N_c - 1$, we then add mass terms to flow to theories with fewer flavors. Using this method [1,9], we obtain

$$W_{eff} = (N_c - N_f)\frac{\Lambda^{\frac{3N_c - N_f}{N_c - N_f}}}{(\det Q\tilde{Q})^{\frac{1}{N_c - N_f}}}, \tag{2.12}$$

for $N_f < N_c$, exactly as in [9], but now Λ has been precisely identified with $\Lambda_{\overline{DR}}$. For the case of $N_f = N_c$, there is no superpotential, but [9] the classical constraints $\langle \det M - B\tilde{B}\rangle = 0$ ($\langle \mathrm{Pf}\ V\rangle = 0$ for $SU(2)$) are modified quantum mechanically to

$$\langle \mathrm{Pf}\ V\rangle = \Lambda^4 \qquad \text{and} \qquad \langle \det M - B\tilde{B}\rangle = \Lambda^{2N_c}. \tag{2.13}$$

Again, we have fixed the normalization by adding a mass and flowing to the $N_f = N_c - 1$ theory, and used the notation $V_{ij} = Q_i Q_j$, $\mathrm{Pf}\ V = \frac{1}{8}\epsilon_{ijkl}V^{ij}V^{kl}$ for $SU(2)$ and $M_{i\tilde{j}} = Q_i\tilde{Q}_{\tilde{j}}$ and $B = Q_1 Q_2 \cdots Q_{N_c}$ for $SU(N_c)$ (with color indices suppressed).

We obtain the gaugino condensates of other groups in a similar way [1,3,27]: and obtain (where we have taken both scales equal for $SO(4)$):

$$\langle \frac{\lambda^a\lambda^a}{32\pi^2}\rangle_{SU(N)} = \Lambda^3, \tag{2.14}$$

$$\langle \frac{\lambda^a\lambda^a}{16\pi^2}\rangle_{SO(N),N\geq 4} = 2^{4/(N-2)}\Lambda^3 \qquad \text{and} \qquad \langle \frac{\lambda^a\lambda^a}{64\pi^2}\rangle_{Sp(2N)} = 2^{-2/(N+1)}\Lambda^3. \tag{2.15}$$

To derive this result for $Sp(2N)$, one has to take into account in using equation (2.2) that when $Sp(2N) \to Sp(2N-2)$ by the VEV v of a flavor of fundamentals, the vector bosons which are singlets under $Sp(2N-2)$ have mass v, while those who transform as the $2N-2$ representation have mass $v/\sqrt{2}$. Although this will not be needed here, other models are easily matched to the model of section 2.2. Namely, all the results of [10] for $SO(5)\times SU(2)$ and $SU(2)\times SU(2)$ models with various matter contents have already the correct normalization for the $\overline{DR}$ scheme. For the $SU(2)$ theory with 2 triplets solved in [12], a rescaling $\Lambda \to 4\Lambda_{\overline{DR}}$ is required and, similarly, for the $N = 2$ $SU(2)$ theories with i matter hypermultiplets solved in [14], the rescaling should be $\Lambda_i^{4-i} \to 4(\Lambda_i^{4-i})_{\overline{DR}}$.

2.5. A second method to relate exact solutions to perturbation theory

In this section we relate the solution of N=2 $SU(2)$ SYM obtained by Seiberg and Witten [11] to perturbation theory. This should be viewed as an alternate method to that of section 2.2. Before jumping into technical details, let us first outline the procedure. In terms of N=1 components, the theory contains a vector multiplet $W^a_\alpha = (\lambda^a_\alpha, F^a_{\mu\nu})$ and a chiral multiplet $\Phi^a = (\phi^a, \psi^a_\alpha)$ in the adjoint representation. N=2 SUSY is unbroken and forbids any superpotential. Thus, the adjoint ϕ^a may obtain a complex VEV $(0, 0, a)$, breaking the $SU(2)$ down to $U(1)$. The low energy degrees of freedom are then the $U(1)$ photon multiplet and a neutral chiral superfield $a = a + \sqrt{2}\theta\psi + \theta^2 F$. (We will use the same notation a, and $U = \frac{1}{2}\phi^a\phi^a$ needed later, for the chiral superfields, their lowest component, and the VEV of their lowest component.) The low-energy effective lagrangian for these fields contains a kinetic term for a and an effective gauge coupling

$$\frac{1}{16\pi} Im \int d^2\theta \tau(a) W^2_\alpha. \tag{2.16}$$

If the VEV a is large, the theory is semiclassical, and the general form of τ follows from the symmetries [28]

$$\tau(a) = \frac{2i}{\pi} \log \frac{C_0 a^2}{\Lambda^2} + \sum_{l=1}^{\infty} \frac{C_l \Lambda^{4l}}{a^{4l}}. \tag{2.17}$$

The logarithmic term is just the perturbative renormalization of the coupling constant and the power corrections are generated by instantons. By relating strong coupling to weak coupling, the solution of Seiberg and Witten gives all the coefficients C_l exactly. The scale Λ can be matched to $\overline{DR}$ by comparing this logarithmic term to that calculated directly in the $\overline{DR}$ scheme using equation (2.2). This requires that we extract the constant C_0 from the exact solution for τ. We will also extract the coefficient C_1 of the one-instanton term for comparison with an explicit instanton calculation.

We now give a brief description of the exact solution and the steps needed to extract this information. The VEV a is not a good global coordinate on the quantum moduli space, and is traded for U defined above, which is. The low energy effective lagrangian is expressed in terms of two functions of U, $a = a(U)$ and $a_D(U)$

$$L = \frac{1}{8\pi} Im \int d^4\theta a_D \bar{a} + \frac{1}{16\pi} Im \int d^2\theta \tau(U) W^2_\alpha \tag{2.18}$$

with $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{e^2} = a'_D/a'$, and $\theta(U)$ and $e(U)$ the $U(1)$ effective couplings (note that we are using for the normalization of τ the conventions of [14] which differ from [11] by a factor of 2). The exact solution gives $a(U)$ and $a_D(U)$ as periods of the elliptic curve

$$y^2 = x^3 - Ux^2 + \Lambda^4 x, \tag{2.19}$$

with the same conventions as in the previous sections. This formula is the definition of Λ. The curve is singular when $U = \pm 2\Lambda^2$ and is branched at $x = \infty$, $x_0 = 0$ and $x_\pm = \frac{1}{2}(U \pm \sqrt{U^2 - 4\Lambda^4})$. Seiberg and Witten found that

$$a'_D = \frac{da_D}{dU} = c_1 \int_{x_-}^{x_+} \frac{dx}{y} \qquad a' = \frac{da}{dU} = c_1 \int_{x_0}^{x_-} \frac{dx}{y} \tag{2.20}$$

with c_1 a constant, which leads to the correct monodromies

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \text{ at } U \sim 2\Lambda^2 \text{ and } \qquad \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \text{ at large U.} \tag{2.21}$$

acting on the vector (a'_D, a'). Equations (2.20) have the explicit solution

$$a'_D = i\frac{c_2}{\sqrt{x_+}} K(k') \qquad a' = \frac{c_2}{\sqrt{x_+}} K(k) \tag{2.22}$$

where K is the elliptic function of the first kind, $k = \sqrt{x_-/x_+}$ and $k' = \sqrt{1-k^2}$ and c_2 is a constant. The limit $U \to \infty$ corresponds to $k \to 0$. We make use of the expansion

$$K(k) = \begin{cases} \frac{\pi}{2}(1 + \frac{k^2}{4} + \mathcal{O}(k^4)), & k \to 0; \\ (1 + \frac{k'^2}{4}) \log(\frac{4}{k'}) - \frac{k'^2}{4} + \mathcal{O}(k'^4), & k \to 1. \end{cases} \tag{2.23}$$

At large U, the theory is semiclassical and we find

$$\tau = a'_D/a' \cong \frac{2i}{\pi} \log \frac{4U}{\Lambda^2}. \tag{2.24}$$

For the matching between $SU(2)$ and $U(1)$, with the mass of the heavy bosons $W_\pm$ being $2\sqrt{U}$ classically, equation (2.2) yields $\frac{-8\pi^2}{e^2} = \frac{-8\pi^2}{g^2(2\sqrt{U})} = -4\log(\frac{2\sqrt{U}}{\Lambda_{\overline{DR}}})$ where g is the $SU(2)$ coupling. But from equation (2.24), $Re\ [i\pi\tau] = \frac{-8\pi^2}{e^2} = -2\log(\frac{4U}{\Lambda^2})$. Thus we identify Λ with $\Lambda_{\overline{DR}}$ which is one result of this section. Moreover, the location of the singularities $U = \pm 2\Lambda^2$ in $\overline{DR}$, matches correctly with equation (2.9).

More work is needed to extract the one-instanton predictions of the effective lagrangian (2.18). We will be interested in the four-fermi interaction $\psi\psi\lambda\lambda$ predicted by the photon kinetic term. To the one-instanton approximation,

$$U = \frac{a^2}{2}(1 + 2\frac{\Lambda^4}{a^4} + \mathcal{O}(\frac{\Lambda^8}{a^8})), \tag{2.25}$$

obtained by integrating a' with respect to U. One gets different formulas for τ depending on whether one expresses it as a function of a or of U. For comparison with instanton calculations, it is appropriate to express τ as a function of a;

$$\tau(a) = \frac{2i}{\pi}\Big(\log\frac{2a^2}{\Lambda^2} - \frac{3\Lambda^4}{a^4} + \mathcal{O}(\frac{\Lambda^8}{a^8})\Big). \tag{2.26}$$

To get the four-fermion interaction, remember that a above is a chiral superfield, so that

$$a^{-4}(\theta) = a^{-4}(1 + \frac{\sqrt{2}\theta\psi}{a})^{-4} = a^{-4}(1 - 10\frac{\theta^2\psi^2}{a^2}) + other\ terms.$$

Therefore, we get the prediction expressed in terms of $\tau(a)$:

$$S_{eff} \supset \frac{1}{16\pi} Im \int \tau(a) W_\alpha^2 d^2\theta d^4x = \frac{15\Lambda^4}{8\pi^2 a^6}\int d^4x \psi^2\lambda^2 + h.c. + other\ interactions. \tag{2.27}$$

3. Instanton Calculations of L_{eff}

For $SU(N_c)$ with $N_c - 1$ flavors, the superpotential (2.12) leads to a mass term for a classically massless fermion. Affleck, Dine and Seiberg [1] showed that this dynamical mass is generated by instantons. The numerical coefficient was computed by Cordes in [29]. The effective lagrangian (2.27) in the N=2 theory contains a four-fermi interaction of massless fermions, also generated by an instanton amplitude; Seiberg [28] showed that the coefficient is non-zero. We will show that both calculations agree quantitatively with the exact predictions.

Because of supersymmetry, the nontrivial eigenvalues which enter the instanton determinant cancel between boson and fermion sectors. Thus, such instanton calculations reduce to tree level perturbation theory and combinatorics of fermion zero modes. Both these aspects are considerably simplified in the superfield formalism of [4,2]. Here we will work in components. We assume much familiarity with standard instanton notation [21] and we note that our normalization of the instanton amplitude agrees entirely with the corrected calculational framework summarized in [30].

3.1. *The computation of W_{eff} in $SU(2)$ with one flavor*

We present a simplified version of the calculation of Cordes in [29], specific to the case of $SU(2)$. We are concerned with getting precisely the numerical coefficient that determines the instanton amplitude. This is determined by one-loop fluctuations about the instanton. This computation assumes a large vacuum expectation value $\langle Q_f^i \rangle = v\delta_f^i$ along the flat direction of the classical potential for the two matter doublets $Q_f = Q_f + \sqrt{2}\theta\psi_f + \theta^2 F_f$, ($i$ is the color index). In the instanton background, this becomes

$$Q_f^i = \frac{\sigma_f^{\mu i} x_\mu v}{\sqrt{x^2+\rho^2}}.$$

This VEV provides an infrared cutoff, which suppresses large instantons exponentially, since the classical action is now $8\pi^2/g^2 + 4\pi^2\rho^2 v^2/g^2$. Along this flat direction there is a classically massless fermion $\chi_\alpha = \frac{1}{\sqrt{2}}(\psi_1^1 + \psi_2^2)_\alpha$, ($\alpha$ is the spin index) which obtains a mass from the superpotential (2.7). The considerations of the previous section predict this mass. Expanding the superpotential

$$\frac{\Lambda^5}{Q_1 Q_2} = -\frac{\Lambda^5}{v^4}\left(\frac{3}{4}(\psi_1^1+\psi_2^2)^2 + \frac{1}{4}(\psi_1^1-\psi_2^2)^2\right) + \cdots = \frac{3}{2}\frac{\Lambda^5}{v^4}\chi\chi + \textit{other terms}. \tag{3.1}$$

This gives $m_\chi = 3\Lambda^5/v^4$.

We now do the instanton calculation of m_χ. The bosonic zero modes and the classical action give a standard factor (see, e.g., [21,6]):

$$2^{10}\pi^6 \int \frac{d^4x_0 d\rho}{\rho^5} e^{-8\pi^2/g^2(\mu)} (\mu\rho)^8 e^{-4\pi^2\rho^2 v^2} \tag{3.2}$$

The precise definition of $g^2(\mu)$ in the exponential depends upon how one chooses to regulate the instanton determinants. It is easiest to use ζ-function or alternatively Pauli-Villars [21] regularization. As explained in section 2.1, for either choice the resulting Λ scale can be identified with $\Lambda_{\overline{DR}}$.

At zeroth order in ρv, there are 6 fermion zero modes. They are normalized according to $\Sigma_k \int \psi_k^*(x)\psi_k(x) d^4x = 1$, with k representing color and Lorentz indices. The zero modes form three pairs:

$$\lambda_\alpha^{SSa[\beta]} = \frac{\sqrt{2}\rho^2}{\pi}\frac{\sigma_\alpha^{a\beta}}{(x^2+\rho^2)^2} \qquad \lambda_\alpha^{SCa[\dot\beta]} = \frac{\rho}{\pi}\frac{x_\beta^{\dot\beta}\sigma_\alpha^{a\beta}}{(x^2+\rho^2)^2} \qquad \psi_{\alpha f}^{i[n]} = \frac{\rho}{\pi}\frac{\delta_f^n \delta_\alpha^i}{(x^2+\rho^2)^{3/2}}. \tag{3.3}$$

The indices in brackets label the two members 1,2 of a pair; the explicit indices are α for spin, i, a for color and f for the two doublets.

The Yukawa coupling $\lambda\psi Q^\dagger$ perturbs this picture qualitatively. First it lifts the superconformal and matter fermion zero modes (for details, see [29]), leaving the factor

$$\mu^{-2}\int d\xi_1 d\xi_2 d\tau_1 d\tau_2 \exp(i\sqrt{2}\int d^4x \xi\lambda^a T^a Q^\dagger\psi\tau) = \frac{v^2}{2\mu^2}. \tag{3.4}$$

Second, it mixes the massless fermion with the supersymmetric zero modes. The classical equations are $\not{D}\lambda = 0$ and $\not{D}^\dagger\psi^\dagger = \sqrt{2}Q^\dagger T^a\lambda^a$ which have the solution (up to negligible higher order corrections in $(\rho v)^2$)

$$\lambda = \lambda_{SS} \qquad \psi^{\dagger i[\beta]}_{\dot\alpha f} = \frac{1}{4\pi}(\not{D}^\beta_{\dot\alpha})^i_j Q^{\dagger j}_f. \tag{3.5}$$

From (3.5), we get the classical wave-function of $\chi^{\dagger[\beta]}_{\dot\alpha}$, with an index $[\beta]$ telling which supersymmetric zero mode it is attached to. We restore the instanton location x_0, and take $|x - x_0| \to \infty$. In this limit

$$\chi^{\dagger[\beta]}_{\dot\alpha} = \frac{1}{4\pi\sqrt{2}}((\not{D}^\beta_{\dot\alpha})^1_j Q^{\dagger j}_1 + (\not{D}^\beta_{\dot\alpha})^2_j Q^{\dagger j}_2) \simeq \frac{v\rho^2}{4\sqrt{2}\pi}\not{\partial}^\beta_{\dot\alpha}\frac{1}{(x-x_0)^2} = \frac{\pi\rho^2 v}{\sqrt{2}}S^\beta_{F\dot\alpha}(x, x_0) \tag{3.6}$$

where $S^\beta_{F\dot\alpha}$ is the standard free massless Feynman propagator:

$$4\pi^2 S^\beta_{F\dot\alpha}(x, x_0) \equiv \not{\partial}^\beta_{\dot\alpha}\frac{1}{(x-x_0)^2}.$$

Finally, collecting the above factors, we compute a Green function for large $|x - y|$:

$$\begin{aligned}\langle\chi^\dagger_{\dot\alpha}(x)\chi^{\dagger\dot\alpha}(y)\rangle =& 2^{10}\pi^6\int d^4x_0 d\rho\rho^3\Lambda^5\left(\frac{v^2}{2}\right)\left(\frac{\pi\rho^2 v}{\sqrt{2}}\right)^2 e^{-4\pi^2v^2\rho^2}S^{\dot\beta}_{F\alpha}(x, x_0)S^\alpha_{F\dot\beta}(x_0, y)\\ =& 3\frac{\Lambda^5}{v^4}\int d^4x_0 S^{\dot\beta}_{F\alpha}(x, x_0)S^\alpha_{F\dot\beta}(x_0, y).\end{aligned} \tag{3.7}$$

Comparing with the amplitude generated by a mass term

$$\langle\chi^\dagger_{\dot\alpha}(x)\chi^{\dagger\dot\alpha}(y)\rangle = m_\chi\int d^4x_0 S^{\dot\beta}_{F\alpha}(x, x_0)S^\alpha_{F\dot\beta}(x_0, y). \tag{3.8}$$

we find complete agreement with the prediction equation (3.1).

This calculation was carried out by Cordes for any number of colors. It involves a nontrivial integration over the orientation of the instanton. Note that the result in [29]

disagrees with the prediction (2.12) by a factor of 4 as the result of some minor errors[1] appearing in that paper. Correcting for these, the result for arbitrary $SU(N)$ with $N-1$ flavors is

$$W_{SU(N)} = \frac{\Lambda^{2N+1}}{\det Q\tilde{Q}} \tag{3.9}$$

in precise agreement with the prediction (2.12).

3.2. The four-fermi interaction, equation (2.27)

The calculation of the four-fermi interaction in the N=2 $SU(2)$ SYM theory is very similar to that of the previous section [28] and we will be very brief. The gauge group is $SU(2)$ and there is a chiral superfield in the adjoint $\phi^a = \phi^a + \sqrt{2}\theta\psi^a + \theta^2 F^a$. The scalar zero mode is:

$$\phi^a_{instanton} = \frac{a x_\mu x_\nu \eta^a_{\mu\lambda}\bar{\eta}^b_{\nu\lambda} V^b}{x^2+\rho^2} \tag{3.10}$$

where $V^b = (0,0,1)$ is a chosen direction and a is the VEV. The group is broken to $U(1)$, leaving a massless $N=2$ vector multiplet. There are now two classically massless fermions, which we label $\psi = \psi^3$ and $\lambda = \lambda^3$. There are 8 zero modes; the 4 gaugino zero modes of (3.3), and identical supersymmetry and superconformal zero modes associated with ψ^a. The Yukawa coupling $\sqrt{2}\epsilon^{abc}\psi^a\lambda^b\phi^{\dagger c}$ raises the 4 superconformal zero modes, leaving a factor of $a^2/2\mu^2$ and mixes the massless fermion $\lambda^\dagger$ with ψ_{SS} and mixes $\psi^\dagger$ with λ_{SS}. Far from the instanton location, the classical wavefunctions of the massless fermions are $\psi^{\dagger\dot\alpha}_{[\alpha]}, \lambda^{\dagger\dot\alpha}_{[\alpha]} = \pi\rho^2 a S^{\dot\alpha}_{F\alpha}$. The four-fermion amplitude computed at large separation is

$$\begin{aligned}\langle\psi^{\dagger\dot\alpha}(x_1)\psi^\dagger_{\dot\alpha}(x_2)\lambda^{\dagger\dot\beta}(x_3)\lambda^\dagger_{\dot\beta}(x_4)\rangle =& 2^{10}\pi^6\int d^4x_0 d\rho\rho^3\Lambda^4\Big(\frac{a^2}{2}\Big)(\pi\rho^2 a)^4 e^{-4\pi^2\rho^2a^2} \\ & S^{\dot\alpha}_{F\alpha}(x_1,x_0)S^{\alpha}_{F\dot\alpha}(x_0,x_1)S^{\dot\beta}_{F\beta}(x_3,x_0)S^{\beta}_{F\dot\beta}(x_0,x_4) \\ =& \frac{15\Lambda^4}{2\pi^2a^6}\int d^4x_0 S^{\dot\alpha}_{F\alpha}(x_1,x_0)S^{\alpha}_{F\dot\alpha}(x_0,x_1)S^{\dot\beta}_{F\beta}(x_3,x_0)S^{\beta}_{F\dot\beta}(x_0,x_4)\end{aligned} \tag{3.11}$$

which is exactly the amplitude predicted by the effective interaction (2.27).

[1] In particular, the extraction of the mass term, equation 6.9, and multiplicative errors appearing in equations 5.19 and 6.7.

4. Green functions at weak coupling

Another fruitful approach to studying SUSY theories has been the instanton computation of chiral Green functions. Supersymmetry guarantees that correlation functions of lowest components of chiral superfields, which vanish in perturbation theory, will be independent of position. If the Green functions can be computed reliably, then clustering can be used to extract the values of various gauge invariant condensates. In this section, we compare the calculations done at weak coupling in the formalism of [2,7] with the predictions. We differ from the literature [6,7,31] which uses zero modes normalized according to $\int \mathrm{tr}\lambda\lambda = 1$ for fermions in the adjoint representation, whereas we take $\int \sum_a \lambda^a \lambda^a = 2\int \mathrm{tr}\lambda\lambda = 1$ in accord with (3.3). Results quoted in the literature must therefore be corrected by factors of 1/2 for every pair of zero modes in the adjoint representation.

In the $SU(2)$ theory with one massless flavor, assuming a large VEV for X classically, one computes:

$$\langle S(x_1)X(x_2)\rangle = \Lambda^5. \tag{4.1}$$

To compare this with the exact solution, we use the effective superpotential of equation (2.7), and add a mass term mX. Integrating out X, we find $\langle X\rangle = (\Lambda^5/m)^{\frac{1}{2}}$. Using the relation $\langle S\rangle = m\langle X\rangle$ of equation (2.11), we get $\langle SX\rangle = \langle S\rangle\langle X\rangle = \Lambda^5$, which agrees with the explicit calculation (4.1).

In the $SU(2)$ theory with 2 massless flavors, one computes at weak coupling (with a large VEV for V_{12} for example):

$$\langle \frac{1}{8}\epsilon^{ijkl}V_{ij}(x_1)V_{kl}(x_2)\rangle = \Lambda^4 \tag{4.2}$$

which agrees with the prediction (2.13).

Finally, in the N=2 SU(2) SYM theory, Green function methods at weak coupling allow us to compute the quantum correction to the classical relation $\langle U\rangle \cong \frac{1}{2}\langle a^2\rangle$ at large a. One computes,

$$\langle U(x)\rangle = \frac{1}{2}a^2 + \frac{\Lambda^4}{a^2} + \mathcal{O}(\frac{\Lambda^8}{a^6}) \tag{4.3}$$

in perfect agreement with the prediction (2.25), and

$$\langle U(x_1)U(x_2)\rangle = \frac{1}{4}a^4 + \Lambda^4 + \mathcal{O}(\frac{\Lambda^8}{a^4}), \tag{4.4}$$

in agreement with the clustering property.

5. Green functions at strong coupling

In another approach to instanton calculations, summarized in [6], Green functions of gauge invariant, chiral operators are computed directly in the confining phase, without vacuum expectation values of the elementary fields. Instantons are expected to saturate the amplitude and the integration over the instanton size is convergent, due to the finite separation between the operators inserted. Using the position independence guaranteed by supersymmetry, and cluster decomposition, the values of the condensates can be extracted.

In the case of pure $SU(2)$, one computes in the formalism of [6,7]:

$$\langle S(x_1)S(x_2)\rangle = \frac{4}{5}\Lambda^6. \tag{5.1}$$

From this, one extracts $\langle S\rangle = \pm\sqrt{4/5}\Lambda^3$, which disagrees with equation (2.10). For pure $SU(N)$ SYM, the discrepancy worsens. One computes:

$$\langle S(x_1)S(x_2)\dots S(x_N)\rangle = \frac{2^N\Lambda^{3N}}{(N-1)!(3N-1)}. \tag{5.2}$$

For large N, $\langle S\rangle \sim \frac{\Lambda^3}{N}$, in disagreement with equation (2.14).

However, some computations do yield results that agree with the predictions. In the $SU(2)$ theory with 2 massless flavors, we find:

$$\langle \frac{1}{8}\epsilon^{ijkl}V_{ij}(x_1)V_{kl}(x_2)\rangle = \Lambda^4 \tag{5.3}$$

and for the $SU(N)$ theory with N massless flavors, we compute using the formalism of [31,6] $\langle \frac{1}{N!}\epsilon^{i_1 i_2\dots i_N}\epsilon^{\tilde{\imath}_1\tilde{\imath}_2\dots\tilde{\imath}_N}M_{i_1\tilde{\imath}_1}(x_1)M_{i_2\tilde{\imath}_2}(x_2)\cdots M_{i_N\tilde{\imath}_N}(x_N)\rangle = \frac{N}{2N-1}\Lambda^{2N}$ and $\langle B(x_1)\tilde{B}(x_2)\rangle = -\frac{N-1}{2N-1}\Lambda^{2N}$, which we add to get:

$$\langle \det M - B\tilde{B}\rangle = \Lambda^{2N}. \tag{5.4}$$

Both equations (5.3) and (5.4) are in agreement with the prediction (2.13).

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