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ON THE LORENTZ INVARIANCE OF BIT-STRING GEOMETRY*

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Abstract

We construct the class of integer-sided triangles and tetrahedra that respectively correspond to two or three discriminately independent bit-strings. In order to specify integer coordinates in this space, we take one vertex of a regular tetrahedron whose common edge length is an even integer as the origin of a line of integer length to the "point" and three integer distances to this "point" from the three remaining vertices of the reference tetrahedron. This — usually chiral — integer coordinate description of bit-string geometry is possible because three discriminately independent bit-strings generate four more; the Hamming measures of these seven strings *always* allow this geometrical interpretation. On another occasion we intend to prove the rotational invariance of this coordinate description. By identifying the corners of these figures with the positions of recording counters whose clocks are synchronized using the Einstein convention, we define velocities in this space. This suggests that it may be possible to define boosts and discrete Lorentz transformations in a space of integer coordinates. We relate this description to our previous work on measurement accuracy and the discrete ordered calculus of Etter and Kauffman (DOC).

1. INTRODUCTION

The successful derivation of the Maxwell equations[1] from a discrete ordered calculus[2] has prompted a re-examination of the somewhat heuristic discussion of the Lorentz invariance of bit-string physics presented at this meeting three years ago.[3] We discuss here some of the problems which arise, but do not resolve all of them.

2. BIT-STRING GEOMETRY

2.1 BIT-STRING BASICS

Specify a *bit-string* $\mathbf{a}(S)$ by its S ordered elements a_s:

$$a_s \in 0, 1; \quad s \in 1, 2, 3, ...S \tag{2.1}$$

If we interpret the symbols "0" and "1" in the strings as *integers*, we can calculate the norm, or Hamming measure, $a(S)$ by the formula

$$a \equiv \Sigma_{s=1}^{S} a_s = |\mathbf{a}(S)| = a(S) \tag{2.2}$$

Because we interpret the symbols "0" and "1" as *integers* rather than bits, we can define the operator XOR, which combines two strings to form a third and is symbolized by "$\oplus$", in terms of the elements of the resulting string:

$$(\mathbf{a} \oplus \mathbf{b})_s \equiv (a_s - b_s)^2 \tag{2.3}$$

This is isomorphic to the usual meaning of XOR, addition *mod 2* or boolean symmetric difference in the sense that the element is 1 if a_s and b_s differ, and 0 if they are the same.

We introduce the *null string* $\mathbf{0}(S)$ with elements $0_s = 0$, the *anti-null string* $\mathbf{I}(S)$ with elements $I_s = 1$, and the *complement* to $\mathbf{a}$ defined by $\bar{\mathbf{a}} \equiv \mathbf{I} \oplus \mathbf{a}$; clearly this string is $\mathbf{a}$ with the "0" 's and "1" 's interchanged. We note that

$$\mathbf{a} \oplus \mathbf{a} = \mathbf{0}; \qquad \mathbf{a} \oplus \bar{\mathbf{a}} \oplus \mathbf{I} = \mathbf{0} \tag{2.4}$$

When n bit-strings do not combine under XOR to yield the null string when taken two, three, four up to and including n at a time, we say that they are *discriminately independent*, or *d.i.*

We introduce a second bit-string operation called *concatenation*, symbolized by "||" and defined by

$$e_k^{a\|b} \equiv a_k,\ k \in 1,2,...,S_a;\ e_k^{a\|b} \equiv b_j,\ j \in 1,2,...,S_b, k = S_a + j \tag{2.5}$$

2.2 TRIANGLES

If we now consider three non-null strings which are *not* independent under XOR, but any pair of which *are* independent:

$$\mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} = \mathbf{0}(S); \quad \mathbf{a} \neq \mathbf{b} \neq \mathbf{c} \neq \mathbf{a} \tag{2.6}$$

and note that, necessarily

$$|\mathbf{a} \oplus \mathbf{b}| = c \in |a-b|, |a-b|+2, ..., a+b-2, a+b \tag{2.7}$$

it follows that the Hamming measures a, b, c satisfy the triangle inequalities

$$|a-b| \leq c \leq a+b;\ \textit{cyclic and anticyclic on } a, b, c \tag{2.8}$$

If we now take any three arbitrary integers a, b, c which satisfy the triangle inequalities, we can define three numbers n_i, $i \in 1,2,3$ indirectly by the equations

$$a = n_1 + n_2; \quad b = n_2 + n_3; \quad c = n_3 + n_1 \tag{2.9}$$

This definition automatically insures that the triangle inequalities are satisfied because

$$|a-b| = |n_1 - n_3| \leq n_1 + n_3 = c \leq n_1 + n_3 + 2n_2 = a + b \quad Q.E.D. \tag{2.10}$$

Unfortunately for simplicity (but fortunately when it comes to modeling quantized

angular momenta) the inverse equations

$$n_1 = \frac{1}{2}(a - b + c)$$

$$n_2 = \frac{1}{2}(b - c + a) \tag{2.11}$$

$$n_3 = \frac{1}{2}(c - a + b)$$

show that the three numbers can be either integer or half-integer even when a, b, c are integer. In particular we see that if the three Hamming measures a, b, c contain one or three odd integers, the n_i are *half-integer.* Thus, although any three bit-strings which discriminate to the null string specify an integer sided triangle, not all integer sided triangles can be specified by three independent integers using Eq. 2.9; half-integers may be required.

We also note that the square of the area $A(a, b, c)$ is given by

$$A^2 = \frac{1}{16}(a + b + c)(a - b + c)(b - c + a)(c - a + b)$$

$$= \frac{1}{16}[(a + b)^2 - c^2][c^2 - (a - b)^2]$$

$$\textit{cyclic and anticyclic on } a, b, c \tag{2.12}$$

$$= (n_1 + n_2 + n_3)n_1 n_2 n_3$$

Although A^2 is an integer when all three n_i are integer, it is *not* necessarily the square of an integer. We will explore more fully elsewhere why "bit-string geometry" cannot simply be characterized as the construction of figures whose sides are, when embedded in a finite Euclidean 3-space, straight lines of integer length.

2.3 TETRAHEDRA

In order to extend our construction to a skeletal abstraction from a small number of the attributes of Euclidean 3-space, we next examine how to relate XOR to a tetrahedron whose faces will be integer sided triangles of the restricted class discussed in the last section, i.e. those which correspond to three bit-strings which combine under XOR to the null string. Consider four non-null strings which are not independent under XOR:

$$\mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \oplus \mathbf{d} = \mathbf{0}(S) \tag{2.13}$$

but which *are* pairwise independent:

$$\mathbf{a} \neq \mathbf{b};\ \mathbf{a} \neq \mathbf{c};\ \mathbf{a} \neq \mathbf{d};\ \mathbf{b} \neq \mathbf{c};\ \mathbf{b} \neq \mathbf{d};\ \mathbf{c} \neq \mathbf{d} \tag{2.14}$$

allowing us to define

$$\begin{aligned} \mathbf{h_{ab}} &\equiv \mathbf{a} \oplus \mathbf{b} = \mathbf{c} \oplus \mathbf{d} \neq \mathbf{0}(S) \\ \mathbf{h_{bc}} &\equiv \mathbf{b} \oplus \mathbf{c} = \mathbf{a} \oplus \mathbf{d} \neq \mathbf{0}(S) \\ \mathbf{h_{ca}} &\equiv \mathbf{c} \oplus \mathbf{a} = \mathbf{b} \oplus \mathbf{d} \neq \mathbf{0}(S) \end{aligned} \tag{2.15}$$

We also assume that all triplets are independent. For instance, if we consider the triplet $\mathbf{a}, \mathbf{b}, \mathbf{c}$, we can then generate from them the three strings just expressed and the fourth string

$$\mathbf{h_{abc}} \equiv \mathbf{d}(S) = \mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \neq \mathbf{0}(S) \tag{2.16}$$

Note that three strings independent under XOR generate four more, and hence seven strings which close under XOR. Because the operation XOR *discriminates* between two strings in the sense of saying that they are the same when their discriminant is zero or different when it is not zero, it was called *discrimination* by

Amson, Bastin and Kilmister. In 1965, John Amson[4] discovered that the sets of strings which generate the *combinatorial hierarchy* as eigenvectors of non-singular, square mapping matrices[5] are in fact *discriminately closed subsets* (DCsS).[6] Given two *discriminately independent* (or *d.i.*) strings of length 2, we generate $3 = 2^2 - 1$ DCsS; given 3 d.i. strings of length 4 we generate $7 = 2^3 - 1$ DCsS; given 7 d.i. strings of length 16 we generate $127 = 2^7 - 1$ DCsS; given 127 d.i. strings of length 256 we generate $2^{127} - 1$ DCsS. The Parker-Rhodes mapping construction cannot be continued beyond this point, because this large number of DCsS cannot be mapped by the 256^2 mapping matrices which are available.

Just as we were able to represent a triangle with integer sides generated by the discrimination of two d.i. strings using three integers or half-integers, it turns out (cf. Ref. 3) that the three discriminately independent strings just discussed can be shown to represent two chiral tetrahedra, whose opposite edges are equal, and all four of whose faces are the *same* triangle specified by

$$\mathbf{h_{ab}} \oplus \mathbf{h_{bc}} \oplus \mathbf{h_{ca}} = \mathbf{0}(S) \tag{2.17}$$

Note that this restriction is a necessary consequence of Eq. 2.13 and the definitions given in 2.15. Either one of this chiral pair of tetrahedra can be decomposed into four (usually distinct) tetrahedra each of which has this triangle as a base and the three remaining edges of length a, b, c; b, c, d; c, d, a or d, a, b. Further,

$$|\mathbf{b} \oplus \mathbf{c} \oplus \mathbf{d}| = h_{bcd} = a: \quad h_{cda} = b: \quad h_{dac} = c \tag{2.18}$$

This geometry is illustrated in figure 1. Please note that this is a corrected version of Fig. 4 in Ref. 3.

Since we are concerned in this paper with coordinates rather than the general geometry underlying the bit-string construction, we will not pursue the general case here further than to note that the seven integers $a, b, c, d = h_{abc}, h_{ab} = h_{cd}, h_{bc} =$

h_{ad} and $h_{ca} = h_{bd}$ can be represented uniquely by seven numbers n_i as follows:

$$a = n_1 + n_4 + n_5 + n_7$$

$$b = n_2 + n_5 + n_6 + n_7$$

$$c = n_1 + n_4 + n_5 + n_7$$

$$d = h_{abc} = n_1 + n_2 + n_3 + n_7 \tag{2.19}$$

$$h_{ab} = n_1 + n_2 + n_4 + n_6$$

$$h_{bc} = n_2 + n_3 + n_4 + n_5$$

$$h_{ca} = n_1 + n_3 + n_5 + n_6$$

The inversion brings in quarter integers as well as half-integers, making the space of bit-string tetrahedra compared to the space of tetrahedra with integer length edges correspondingly sparser than the space of bit-string triangles compared to the space of triangles with integer sides.

3. TWO POSSIBLE 3-SPACE "COORDINATE SYSTEMS"

N.B. The extension of these constructions to a *rigorous* discussion of rotations in 3-space using these coordinates has yet to be accomplished.

3.1 A SYMMETRIC BASIS

We pick as our basis three strings of length $4N$ with Hamming measure $2N$ (i.e. having an equal number of "0" 's and "1" 's) and which meet the condition

$$\mathbf{U}(4N) \oplus \mathbf{W}(4N) \oplus \mathbf{V}(4N) = \mathbf{I}(4N); \qquad U = V = W = 2N \tag{3.1}$$

from which it follows immediately that

$$\begin{aligned} \mathbf{U}(4N) \oplus \mathbf{W}(4N) &= \bar{\mathbf{V}}(4N); \qquad \bar{V} = 2N \\ \mathbf{W}(4N) \oplus \mathbf{V}(4N) &= \bar{\mathbf{U}}(4N); \qquad \bar{U} = 2N \\ \mathbf{V}(4N) \oplus \mathbf{U}(4N) &= \bar{\mathbf{W}}(4N); \qquad \bar{W} = 2N \end{aligned} \tag{3.2}$$

and hence that we have specified a regular tetrahedron with edges of length $2N$. Further, we can think of the edges U, W, V as meeting in a vertex on one side or the other the plane of the triangle specified by

$$\bar{\mathbf{U}}(4N) \oplus \bar{\mathbf{W}}(4N) \oplus \bar{\mathbf{V}}(4N) = \mathbf{0}(4N) \tag{3.3}$$

If we can label the corners of the tetrahedron with four distinct labels, we can distinguish two chiral possibilities. Otherwise we have an even more ambiguous situation in going from our *algebraic* model to the *geometrical* interpretation.

We use a particular representation for these three basis strings:

$$\begin{aligned} \mathbf{U}(4N) &= \mathbf{I}(2N)\|\mathbf{0}(2N) \\ \mathbf{W}(4N) &= \mathbf{I}(N)\|\mathbf{0}(N)\|\mathbf{I}(N)\|\mathbf{0}(N) \\ \mathbf{V}(4N) &= \mathbf{I}(N)\|\mathbf{0}(2N)\|\mathbf{I}(N) \end{aligned} \tag{3.4}$$

We simplify our notation by omitting the concatenation symbol, i.e.

$$\mathbf{a}\|\mathbf{b}\|\mathbf{c}\|... \rightarrow \mathbf{abc}... \tag{3.5}$$

Note that this "product" can be non-commutative because it is often the case that

$$\mathbf{a}\|\mathbf{b} = \mathbf{ab} \neq \mathbf{ba} = \mathbf{b}\|\mathbf{a} \tag{3.6}$$

Using $\mathbf{U}, \mathbf{W}, \mathbf{V}$ as our reference system we can represent any *single* arbitrary string about which all we know is that its Hamming measure is a and its length $4N$ relative to this representation as

$$\mathbf{a}(4N; n_r, n_u, n_w, n_v)$$

$$= \mathbf{I}(n_r)\mathbf{0}(N - n_r)\mathbf{I}(n_u)\mathbf{0}(N - n_u)\mathbf{I}(n_w)\mathbf{0}(N - n_w)\mathbf{I}(n_v)\mathbf{0}(N - n_v) \tag{3.7}$$

The geometrical interpretation is sketched in figure 2. Here the "coordinates" n_i^a, $i \in r, u, w, v$, must obviously meet the two restrictions

$$0 \leq n_i \leq N; \quad a = \Sigma_{i=r,u,w,v}\, n_i^a \tag{3.8}$$

but, subject to these constraints, can be chosen arbitrarily. We will discuss elsewhere how to make counting the number of strings which can be brought into this form by permutation of their ordered elements while keeping the length and Hamming measure fixed into a well posed problem.

The algebraic consequences of this particular representation are that

$$\mathbf{h}_{au}(4N; n_r, n_u, n_w, n_v) \equiv \mathbf{a} \oplus \mathbf{U} =$$

$$= \mathbf{0}(n_r)\mathbf{I}(N - n_r)...\mathbf{0}(n_u)\mathbf{I}(N - n_u)\mathbf{I}(n_w)\mathbf{0}(N - n_w)\mathbf{I}(n_v)\mathbf{0}(N - n_v) \tag{3.9}$$

$$\mathbf{h}_{aw}(4N; n_r, n_u, n_w, n_v) \equiv \mathbf{a} \oplus \mathbf{W} =$$

$$= \mathbf{0}(n_r)\mathbf{I}(N-n_r)\mathbf{I}(n_u)\mathbf{0}(N-n_u)\mathbf{0}(n_w)\mathbf{I}(N-n_w)\mathbf{I}(n_v)\mathbf{0}(N-n_v) \tag{3.10}$$

$$\mathbf{h}_{av}(4N; n_r, n_u, n_w, n_v) \equiv \mathbf{a} \oplus \mathbf{V} =$$

$$= \mathbf{0}(n_j)\mathbf{I}(N-n_r)\mathbf{I}(n_u)\mathbf{0}(N-n_u)\mathbf{I}(n_w)\mathbf{0}(N-n_w)\mathbf{0}(n_v)\mathbf{I}(N-n_v) \tag{3.11}$$

The scalar integer equations which follow immediately are

$$a = n_r + n_u + n_w + n_v$$

$$\begin{aligned} h_{au} &= 2N - n_r - n_u + n_w + n_v = 2N - a + 2n_w + 2n_v \\ h_{aw} &= 2N - n_r + n_u - n_w + n_v = 2N - a + 2n_u + 2n_v \\ h_{av} &= 2N - n_r + n_u + n_w - n_v = 2N - a + 2n_u - 2n_w \end{aligned} \tag{3.12}$$

Noting that

$$h_{au} + h_{aw} + h_{av} = 6N - 3n_r + n_u + n_w + n_v = 6N - 3a + 4(n_u + n_v + n_w) \tag{3.13}$$

these equations can be immediately inverted to give the coordinates (n_r, n_u, n_w, n_v) in terms of the Hamming measures $a, h_{au}, h_{aw}, h_{av}$. The unique inversion of these equations is

$$\begin{aligned} n_r^a &= \frac{3}{2}N + \frac{1}{4}a - \frac{1}{4}(h_{au} + h_{aw} + h_{av}) \\ n_u^a &= -\frac{1}{2}N + \frac{1}{4}a + \frac{1}{4}(-h_{au} + h_{aw} + h_{av}) \\ n_w^a &= -\frac{1}{2}N + \frac{1}{4}a + \frac{1}{4}(h_{au} - h_{aw} + h_{av}) \\ n_v^a &= -\frac{1}{2}N + \frac{1}{4}a + \frac{1}{4}(h_{au} + h_{aw} - h_{av}) \end{aligned} \tag{3.14}$$

3.2 Coordinate description of a triangle, a line segment and bit-string polyhedra

It follows from the above that a string $\mathbf{b}(4N; n_r^b, n_u^b, n_w^b, n_v^b)$ whose discriminant with $\mathbf{a}$ is called $\mathbf{h}_{ab} = \mathbf{a} \oplus \mathbf{b}$ has Hamming measure

$$h_{ab} = h_{ab}(4N; |n_r^a - n_r^b|, |n_u^a - n_u^b|, |n_w^a - n_w^b|, |n_v^a - n_v^b|) \tag{3.15}$$

To proceed further, we must use care. If we follow the convention used in writing $\mathbf{a}$ and $\mathbf{b}$ given in Eq. 3.7 *and* the geometrical interpretation given in Fig. 2 in which they are *vectors* from the origin of length a and b respectively, we would have the string

$$\vec{h}_{ab} = \mathbf{I}(|n_r^a - n_r^b|)\mathbf{0}(N - |n_r^a - n_r^b|)\mathbf{I}(|n_u^a - n_u^b|)\mathbf{0}(N - |n_u^a - n_u^b|)$$

$$\mathbf{I}(|n_w^a - n_w^b|)\mathbf{0}(N - |n_w^a - n_w^b|)\mathbf{I}(|n_v^a - n_v^b|)\mathbf{0}(N - |n_v^a - n_v^b|) \tag{3.16}$$

rather than the string with the same Hamming measure obtained by discriminating $\mathbf{a}$ and $\mathbf{b}$, namely

$$\mathbf{h_{ab}} = \mathbf{a} \oplus \mathbf{b} =$$

$$\mathbf{0}(le(n_r^a, n_r^b))\mathbf{I}(|n_r^a - n_r^b|)\mathbf{0}(N - ge(n_r^a, n_r^b))$$

$$\mathbf{0}(le(n_u^a, n_u^b))\mathbf{I}(|n_u^a - n_u^b|)\mathbf{0}(N - ge(n_u^a, n_u^b))$$

$$\mathbf{0}(le(n_w^a, n_w^b))\mathbf{I}(|n_w^a - n_w^b|)\mathbf{0}(N - ge(n_w^b, n_w^b)) \tag{3.17}$$

$$\mathbf{0}(le(n_v^a, n_v^b))\mathbf{I}(|n_v^a - n_v^b|))\mathbf{0}(N - ge(n_v^a, n_v^b))$$

where $le(a, b)$ means the lesser of (a, b) and $ge(a, b)$ means the greater of (a, b). Consequently, we find it necessary from now on to distinguish *vector bit strings*

defined and interpreted by Eq. 3.7, Fig. 2 and Eq. 3.16 from the string obtained by discriminating two such strings, given by Eq. 3.17 and interpreted in figure 3. We then must, retrodictively, call the strings we have been discussing $\vec{a}, \vec{b}, \vec{U}, \vec{W}, \vec{V}$ and interpret $\bar{\mathbf{U}}, \bar{\mathbf{V}}, \bar{\mathbf{W}}$ and $\mathbf{h}_{ab}$ as line segments neither of whose ends are at the origin.

I now realize that it is the failure to make the distinction between vector bit-strings and line segments connecting the tips of two vectors from a common origin which has rendered some of my previous work ambiguous or just plain wrong.

Once this distinction is made and followed consistently, any polyhedron in our bit-string space can be simply specified by giving the coordinates n_i, $i \in r, u, w, v$ of each vector bit-string to each of its corners. The lengths of its edges can then be *calculated* by discrimination. The more complex the polyhedron we represent, the smaller the number of polyhedra of that type with integer edges which will be *bit-string polyhedra.* It will be interesting to work out the connection between the five regular polyhedra of Euclidean Geometry, allowed regular bit-string polyhedra, and the combinatorial hierarchy. In particular, we know that the cube and the octhedron will be excluded because bit-string geometry cannot be used to describe a right angle between two line segments of equal length.

3.3 Positive and negative coordinates, rotations and rotational invariance

In order to treat rotations in a way compatible with the standard "vector model" for angular momenta in quantum mechanics, we consider a rotation of $\vec{a}$ whose origin coincides with $\vec{W}$ about $\vec{W}$ keeping this vector, the origin, and the length of $\vec{a}$ fixed. For simplicity here, we think of our bit-string vectors as embedded in a Euclidean 3-space. Then the point A describes an arc of a circle in a plane

perpendicular to $\vec{W}$. The intersection is a distance $|a_w|$ from the origin along $\vec{W}$, the radius of the circle is ρ_{aw}, and these two numbers as well as the scalars $2N, a$ and h_{aw} are kept fixed. This is illustrated in figure 4. Algebraically this implies that

$$a^2 - a_w^2 = \rho_{aw}^2 = h_{aw}^2 - (2N - a_w)^2 \tag{3.18}$$

or

$$2Na_w = h_{aw}^2 - a^2 - 4N^2 \tag{3.19}$$

But

$$h_{aw} + a = 2N + 2(n_u + n_v); \quad h_{aw} - a = 2N - 2(n_r + n_w) \tag{3.20}$$

Hence

$$Na_w = 2N(n_u + n_v - n_r - n_w) - 2(n_u + n_v)(n_r + n_w) \tag{3.21}$$

This allows us to specify a_w as a positive or negative number depending on whether a_w is above or below the plane through the origin perpendicular to $\vec{W}$, or equivalently as to whether the *geometrical* inner product defined by the sides of the triangle (OWA)

$$2\vec{a} \cdot \vec{w} = a^2 + 4N^2 - h_{aw}^2 \tag{3.22}$$

is greater than (above) or less than (below) zero. The geometrical interpretation is illustrated in figure 5.

3.4 A BASIS WITH ONE RIGHT ANGLE

Simplify notation still further:

$$\mathbf{I}(N) \rightarrow N; \quad \mathbf{0}(N) \rightarrow 0 \tag{3.23}$$

and for $0 \leq n \leq N$

$$\mathbf{I}(n) \rightarrow [n]; \quad \mathbf{0}(n) \rightarrow (n) \tag{3.24}$$

For our basis, specify

$$\mathbf{Z}(8N) = N0N0N0N0$$

$$\mathbf{X}(8N) = NN0N0000 \tag{3.25}$$

$$\mathbf{U}(8N) = NN00NN00$$

Then an arbitrary bit-string vector will have eight coordinates

$$\vec{a}(8N) = \Pi_{i=1}^{8}[n_i^a](N - n_i^a) \tag{3.26}$$

Define

$$l_i^{ab} = le(n_i^a, n_i^b); \quad g_i^{ab} = ge(n_i^a, n_i^b); \tag{3.27}$$

Then an arbitrary bit-string in this coordinate representation which connects the tip of $\vec{a}(8N)$ to the tip of $\vec{b}(8N)$ is given by

$$\mathbf{h}_{ab}(8N) = \Pi_{i=1}^{8}(l_i^{ab})[|n_i^a - n_i^b](g_i^{ab}) \tag{3.28}$$

We leave working out the further elaboration of this coordinate system for another occasion. For $N = 1$, it should be capable of describing the quantum numbers of one generation of quarks and leptons and the related baryons and bosons in the standard model, using the scheme we have already roughed out.[7]

4. VELOCITIES AND LORENTZ TRANSFORMATIONS

4.1 Boosts in one direction

In Ref. 3, Sec. 1.3, we started from two positive integers n_0 and n_1 and defined position and time coordinates for the interval between two events by $x_{01} = n_0 - n_1 = -x_{10}$ and $t_{01} = n_0 + n_1 = t_{10}$ where the units are such that $c = 1$. Then the velocity $v_{01} = -v_{10}$ and square of the invariant interval $\tau_{01} = \tau_{10}$ are given by

$$v_{01} = \frac{n_0 - n_1}{n_0 + n_1}; \qquad \tau_{01}^2 = t_{01}^2 - x_{01}^2 = 4n_0n_1 \tag{4.1}$$

and we find that we can define the usual Lorentz time dilation γ by

$$\gamma_{01}^2 \equiv \frac{t_{01}^2}{\tau_{01}^2} = \frac{(n_0 + n_1)^2}{4n_0n_1} = \frac{1}{1 - v_{01}^2} \tag{4.2}$$

If we now introduce a third integer n_2 and generalize our definitions, $x_{01} + x_{12} + x_{20} = 0$ and a little algebra gives us the familiar result that

$$v_{02} = \frac{v_{01} + v_{12}}{1 + v_{01}v_{12}} \tag{4.3}$$

With the interpretation that v_{12} is the velocity of the Lorentz boost which takes (x_{01}, t_{01}) to (x_{02}, t_{02}), a little more algebra suffices to show that

$$x_{02} = \gamma_{12}(x_{01} + v_{12}t_{01}); \qquad t_{02} = \gamma_{12}(t_{01} + v_{12}x_{01}) \tag{4.4}$$

The difficult with this treatment is that our geometrical interpretation in Ref. 3 only used positive integers, and this neat way of describing Lorentz boosts remained heuristic. Now that we have negative as well as positive coordinates when we *project* onto an axis of rotation, we can clearly adapt this derivation to our current context, so long as boosts are along a reference axis. The general Lorentz transformation can then be composed of a boost and a rotation, as in a more conventional context. We hope to have the formalism worked out soon.

Once this is worked out, we hope to show that *any* Lorentz transformation is equivalent to a single bit-string discrimination, but details still elude us.

5. RELATIONSHIP TO NON-COMMUTATIVITY AND THE DISCRETE ORDERED CALCULUS (DOC)

[This chapter and the Appendix are excerpted from my contribution to ANPA 16. We hope to relate them more closely to our tetrahedral coordinates on another occasion.]

5.1 Particles, NO-YES events and Measurement Accuracy

My conceptual foundations for reconstructing relativistic quantum mechanics and physical cosmology start from three inextricably entwined technical terms: *particle*, *event* and *conserved quantum number*. I join them together in the following way:

A **particle** *is a conceptual carrier of conserved quantum numbers between events.*

An **event** *is a finite spacial region which particles enter and leave during a finite time interval. Both the spacial dimensions and the time interval are fixed in the context of a particular application of the definition.*

The algebraic sum of the numerical values for each type of quantum number carried into the region by the entering particles is individually equal to the algebraic sum of the numerical values for that type of quantum number carried out of the region by the leaving particles. This statement defines a (set of) **conserved quantum number(s)**. *Note that the number of particles entering the region need not equal the number of particles leaving the region; in other words, particle number is not necessarily conserved.*

N.B. **In this paper we will usually consider the restricted case in which only one particle enters and the "same" particle** (*i.e., carrying the same quantum numbers*) **leaves the event.** This allows us to talk about a "single particle trajectory", — a luxury usually denied to us in discussing relativistic quantum mechanics.

The paradigm for an event we have in mind is a *counter firing* in which a counter of relevant spacial size Δx at a specified location in the laboratory does *not* fire during a time interval Δt, which we call a *NO-event*, or *does* fire during that time interval, which we call a *YES-event.*

We further assume that these NO-YES events can be *recorded*, using a clock at the counter (or calibrated in such a way that it can be thought of as "in" the counter) which has been synchronized to the laboratory clock using the Einstein *convention* in relation to spacial coordinates of the counter position *fixed* relative to the position of the laboratory clock ("origin") and three *fixed*, independent (in particular, non-coplanar) *directions.*

This allows us to represent the record made by a single counter as an ordered sequence of two distinct symbols such as "0" and "1". When we have specified how two such ordered sequences of symbols of the same length combine, we will call them *bit-stringsc* A general representation of our bat-strings and the operations used to concatenate them is given in Appendix I.

We take as our paradigm for *measurement accuracy* the smallest counter size Δx and time resolution Δt which we can either construct, or infer from the theory we are in the process of constructing.

This is a very powerful and restrictive definition, because it prohibits us from considering *fractional* space and time intervals. Once we have developed the theory far enough to give meaning to *interference*, as in optical interferometry, this assumption of a minimum distance *also* implies a maximum distance and time, which we can call the *event horizon.*

Up to this point we have treated length and time measurement as distinct. But the *System International*, employed universally by physicists in reporting the results of measurement and establishing the meaning of "fundamental constants",[8] takes time measurement to be primary and defines the unit of length:

"The meter is defined to be the length of path traveled by light in vacuum in 1/299 792 458 s[econds]. See B.W.Petley, *Nature*, **303**, 373 (1983)."

Thus, following current practice, we are no longer allowed to define Δx and Δt separately when specifying our lowest bound on measurement accuracy. In fact, we must make the *scale invariant* statement that

$$\frac{\Delta x}{c\Delta t} = 1 \tag{5.1}$$

in *any* system of units which allows us to talk about NO-YES events in a precise way.

5.2 Constant Velocity Commutation Relations

Consider first the case when the velocity is the same whenever measured. Consistent with our finite measurement accuracy postulate, and taking Δx and Δt to be, respectively, the smallest space and time intervals we can measure between events, either directly or indirectly, any distance will be an integral multiple of Δx and any time an integral multiple of Δt. Then any velocity will be a rational fraction.

Confining ourselves to velocities that can always be interpreted as *particulate* velocities, these must then always be rational fractions less that unity. They must also always be greater than zero because zero is *not measurable.* With this understood, in the current context we can define $r = n_r^\beta \Delta x$, $d_r^\beta > n_r^\beta$ s where n_r^β and $d_r^\beta > n_r^\beta$ are integers with no common factor other than unity, and $\Delta x = c\Delta t$. Then the desired representation of a constant velocity in units of c can be taken to be

$$\beta \equiv \frac{n_r^\beta}{d_r^\beta} \tag{5.2}$$

Note that in this context the minimum time interval between measurements which can yield the velocity must be $d_r^\beta \Delta t$ and the minimum space interval between two such measurements — attributed to the firings of two counters produced by a particle of this velocity — must be $n_r^\beta \Delta x$.

In order to make it possible to define *larger* space and time intervals we now introduce the representation

$$r(N_r^\beta; n_r^\beta, d_r^\beta) \equiv N_r^\beta n_r^\beta \Delta x \tag{5.3}$$

This allows us to introduce negative as well as positive distances from the implied origin simply by keeping d_r^β positive and allowing the n_r^β to include negative integers with $|n_r^\beta| < d_r^\beta$. Clearly this also extends our velocity space to negative rational fractions greater than -1. We can now displace our origin by a *phase* defined by

$$\phi(n, \delta n) \equiv \frac{\delta n}{n}, \quad \delta n \in -n+1, -n+2, ..., n-2, n-1; \quad N_r \to N_r + \delta n \tag{5.4}$$

In keeping with our finite and discrete measurement restriction, we must also assign an *event horizon* for our counter array given by $R_{max} = N_{max}\Delta x$ and the requirement $N_r^\beta \pm \delta n < N_{max}$. Note that we will always keep N_r^β, the number of spacial periods of the counters which can be used to measure β, a positive integer. We also assume that the largest time interval we can measure is $T_{max} = 2\pi N_{max}\Delta t$, where "$\pi$" is for us a rational fraction known only to an accuracy consistent with the measurement context considered.[9] Ref. 21 includes a preliminary discussion of how Stillman Drake's analysis[10] of the experiment by which Galileo arrived at the "times squared law" can be viewed as a *dynamical* measurement of $\pi/2\sqrt{2}$.

We have been at some pains to make this representation consistent with the discussion of "measurement" in our fundamental reference.[11] For the constant velocity case at hand, we can envisage a "particle" traversing a counter array with counters spaced a fixed distance $n_r^\beta \Delta x$ apart which fire sequentially with a fixed time interval $d_r^\beta \Delta t$ between firings. We symbolize this sequence of firings by the sequence $R, R', R'', R''', ...$ For definiteness we consider positive velocity and take the first firing to be

$$R := r(N_r^\beta + \delta n; n_r^\beta, d_r^\beta) = (N_r^{beta} + \delta n) n_r^\beta \Delta x$$

Here, consistent with Kauffman's notation, we take the symbol "R" to stand for the instruction *measure* R, and the symbol ":=" to indicate that the value on the

right is the numerical value obtained by the measurement. Then

$$R' := r(N_r^\beta + \delta n + 1)n_r^\beta \Delta x$$

$$R'' := r(N_r^\beta + \delta n + 2)n_r^\beta \Delta x$$

$$R''' := r(N_r^\beta + \delta n + 3)n_r^\beta \Delta x \tag{5.5}$$

$$.....etc.$$

To measure velocity requires us to select *two* counter firings, measure the space and time intervals between them, and calculate the ratio; velocity measurement is intrinsically a more complicated process than the measurement of the spacial interval from an origin to the position R of some identified counter. To conform to Kauffman's usage, we assume that the " $'$ " is an *operator* which shifts us forward in the time sequence $R, R', R'', R'''...$ by one fixed time interval $d_r^\beta \Delta t$. The symbol $\dot{R}$ is to be interpreted as the evaluation of the interval between R' and R by *first* measuring R, *then* measuring R' and finally by dividing the difference between the two intervals by the (fixed) time interval for a single shift, namely $d_r^\beta \Delta t$. Clearly

$$\dot{R} := \frac{n_r^\beta \Delta x}{d_r^\beta \Delta t} = \beta c; \quad \Delta x = c\Delta t \tag{5.6}$$

In contrast to the notation in the Feynman-Dyson-Tanimura papers, which would seem to imply that position and velocity are measured at the *same* time (in violation of the uncertainty principle), we trust that our notation and explicit model make it clear that the two measurements are made at *different* times, and hence can yield different results when made in the opposite order. Now let $d_r^\beta \Delta t = 1$; interpret $R\dot{R}$ as the *process* — measure $\dot{R}$, then measure R. We now have

established in our context the fundamental relationship

$$R\dot{R} = R'(R' - R) := r'(r' - r) \tag{5.7}$$

However, if we first measure R and then measure $\dot{R}$, we obtain

$$\dot{R}R = (R' - R) := (r' - r)r \tag{5.8}$$

In any theory with finite space and time shifts these two values are necessarily *not* the same. In fact, by forming $R\dot{R} - \dot{R}R$ and adding and subtracting $-R(R' - R)$ we have that

$$R\dot{R} - \dot{R}R = (R' - R)^2 + [R, R']; \qquad [R, R'] \equiv RR' - R'R \tag{5.9}$$

Since for us, once we have removed the velocity operation symbol $\dot{R}$, measuring R and R' corresponds to ascertaining the positions of two counters which are fixed in the laboratory and can be measured as many times as we wish in any order without changing the result, we can take $[R, R'] \equiv RR' - R'R = 0$ and we have that

$$R\dot{R} - \dot{R}R \equiv [R, \dot{R}] = (R' - R)^2 := (r' - r)^2 = \kappa \tag{5.10}$$

where *kappa* is an arbitrary *constant scalar* fixed by the measurement accuracy context. This establishes the desired commutation relation for a single fixed velocity and a single spacial direction.

5.3 Two Different Velocities: Acceleration

Consider now two constant velocities which share a common counter firing at R'. The first velocity is $\beta_a = n_a/d_a$ and the second velocity is $\beta_b = n_b/d_b$. The first velocity can be measured using either counter firings at R_a and R' *or* at R' and R''_a *or* at R_a and at R''_a. The same is true for the second velocity using R_b, R' and R''_b. Even if all five counters are colinear, we cannot immediately apply our previous analysis because we have no guarantee that $d_a = d_b$ and hence cannot use our prescription $d\Delta t = 1$.

To deal with the most difficult case, we assume further that n_a, n_b, d_a, d_b have no common factor other than unity. In the current context we will assume that $R_a R'$, $R' R''_a$, $R_b R'$, $R' R''_b$ are colinear, but that we need more measured information to settle the questions about colinearity with respect to the spacial intervals $\Lambda_{ab''} = R''_b R_a$, etc. This information can be formalized by the definition of the *inner product*, a scalar given by

$$2\Lambda_{aa''} \cdot \Lambda_{bb''} := \lambda^2_{aa''}\lambda^2_{bb''} - \lambda^2_{ab''}; \qquad \lambda^2_{ab''} = \lambda^2_{ba''} \tag{5.11}$$

[The precise relationship to change in velocity due to a Lorentz transformation and to 3-space trajectories and accelerations remains to be worked out.]

6. CONCLUSION

The direct approach to bit-string coordinates attempted here remains frustratingly incomplete, but we freeze this effort here to keep the record clear, even though it does not go as far as we could wish toward meeting the promise made in the preliminary version of this paper, published in the *Proceedings of ANPA WEST 11.* We expect much of the algebra and geometry presented here will prove useful in a more satisfactory treatment. We believe that a treatment that follows a new approach, related more closely to the raising and lowering operators used in conventional treatments of quantum mechanical angular momentum, holds much promise. We have made significant progress along those lines and hope to have something useful to present in a few months.

7. APPENDIX. Formal derivation of finite and discrete Maxwell Equations

When I recently showed Ref. 1 to my colleague, M.Peskin, he noted that the "shift operator J" defined by Kauffman is, in our context of a single particle, isomorphic to the operator $U = exp(-iHT)$ representing a finite time shift in the Heisenberg representation. Then the formal steps in Kauffman's rigorous version of Feynman-Dyson-Tanimura "proof" go through easily. The difficulty with adopting Peskin's approach is that what *operational* context the Heisenberg formalism fits into is by no means obvious. So, for mathematical and physical clarity, one needs to invoke the DOC and discuss the relationship between measurement accuracy and the DOC. I am indebted to Peskin[12] for allowing me to quote his shortened version of the Kauffman proof below.

Define

$$\dot{X} = XU - UX = [X, U] \tag{7.1}$$

where U is the time shift operator from X to X' in time Δt (eg $U = e^{-iH\Delta t}$).

Notice that

$$(AB)^{\cdot} = [AB, U] = [A, U]B + A[B, U] = \dot{A}B + A\dot{B} \tag{7.2}$$

as required.

Postulate:

$$1.[X_i, X_j] = 0; \quad 2.\ [X_i, \dot{X}_j] = \kappa\delta_{ij}$$

Rewrite 2 as

$$[X_i, [X_j, U]] = -[X_j, [U, X_i]] - [U, [X_i, X_j]] \tag{7.3}$$

and noting that $[U, [X_i, X_j]] = [U, 0] = 0$ we find that

$$\kappa\delta_{ij} = [X_i, [X_j, U]] \ symmetric\ in\ i, j \tag{7.4}$$

Now *define*

$$H_l = \frac{1}{2\kappa}\epsilon_{jkl}[\dot{X}_j, \dot{X}_k] \tag{7.5}$$

Then

$$\nabla_l H_l = \frac{1}{2\kappa}\epsilon_{jkl}[[\dot{X}_j, \dot{X}_k], \dot{X}_l] \tag{7.6}$$

But this cyclic sum vanishes by the Jacobi identity. Thus

$$\nabla_l H_l = 0 \tag{7.7}$$

which is one of the two Maxwell equations we set out to derive.

Finally, *define*

$$E_i = \ddot{X}_i - \epsilon_{ijk} H_k \tag{7.8}$$

We wish to prove that

$$\frac{\partial H_i}{\partial t} + \epsilon_{ijk}\nabla_j E_k = 0 \tag{7.9}$$

First we need to *define* $\partial/\partial t$ by

$$\dot{H} = \frac{d}{dt}H = \frac{\partial H}{\partial t} + (\dot{X}\cdot\nabla)H \tag{7.10}$$

Then

$$\frac{\partial H_i}{\partial t} = \dot{H}_i - \dot{X}_j\nabla_j H_i$$

$$= \frac{1}{2\kappa}\epsilon_{ikl}([\dot{X}_k, \dot{X}_l])^{\cdot} - \dot{X}_j\frac{1}{\kappa}[\frac{\epsilon_{ikl}}{2\kappa}[\dot{X}_k, \dot{X}_l], \dot{X}_j]$$

$$= \frac{1}{\kappa}\epsilon_{ikl}[\dot{X}_k, \ddot{X}_l] - \frac{1}{2\kappa^2}\dot{X}_j\epsilon_{ikl}[[\dot{X}_k, \dot{X}_l], \dot{X}_j] \tag{7.11}$$

$$
\begin{aligned}
\epsilon_{ijk}\nabla_j E_k &= \epsilon_{ijk}\frac{1}{\kappa}\ \left[\left(\ddot{X}_k - \epsilon_{klm}\dot{X}_l H_m\right), \dot{X}_j\right] \\
&= \frac{1}{\kappa}\,\epsilon_{ijk}\left[\dot{X}_j \ddot{X}_k\right]\cdot(-1) - \epsilon_{ijk}\epsilon_{klm}\epsilon_{mab}\frac{1}{2\kappa^2}\left[\dot{X}_l\left[\dot{X}_a, \dot{X}_b\right], \dot{X}_j\right] \\
&= -\frac{1}{\kappa}\,\epsilon_{ijk}\left[\dot{X}_j, \ddot{X}_k\right] \\
&\quad - \left(\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}\right)\epsilon_{mab}\frac{1}{2\kappa^2}\left(\left[\dot{X}_l, \dot{X}_j\right]\ \left[\dot{X}_a, \dot{X}_b\right] + \dot{X}_l\left[\left[\dot{X}_a, \dot{X}_b\right]\dot{X}_j\right]\right) \\
&= -\frac{1}{\kappa}\,\epsilon_{ijk}\left[\dot{X}_j \ddot{X}_k\right] + \frac{1}{2\kappa^2}\,\epsilon_{iab}X_j\left[\left[\dot{X}_a \dot{X}_b\right], \dot{X}_j\right] \\
&\quad - \epsilon_{jab}\frac{1}{2\kappa^2}\ \left[\dot{X}_i, \dot{X}_j\right]\ \left[\dot{X}_a, \dot{X}_b\right]
\end{aligned}
\tag{7.12}
$$

now

$$
\begin{aligned}
\epsilon_{jab}\left[\dot{X}_i, \dot{X}_j\right]\ \left[\dot{X}_a, \dot{X}_b\right] &= \left[\dot{X}_i, X_1\right]\ [X_2, X_3] \\
&\quad + \left[\dot{X}_i, \dot{X}_2\right]\ \left[\dot{X}_3, \dot{X}_1\right] + \left[\dot{X}_i, \dot{X}_3\right]\ \left[\dot{X}_1, \dot{X}_2\right]
\end{aligned}
\tag{7.13}
$$

for $i = 1$, eg

$$
= \left[\dot{X}_1, \dot{X}_2\right]\ \left[\dot{X}_3, \dot{X}_1\right] + \left[\dot{X}_1, \dot{X}_3\right]\ \left[\dot{X}_1, \dot{X}_2\right] = 0 \tag{7.14}
$$

so

$$
\begin{aligned}
\epsilon_{ijk}\nabla_j E_k &= -\frac{1}{\kappa}\,\epsilon_{ijk}\left[\dot{X}_j, \ddot{X}_k\right] + \frac{1}{2\kappa^2}\,\epsilon_{iab}\,X_j\left[\left[\dot{X}_a, \dot{X}_b\right]\dot{X}_j\right] \\
&= -\frac{\partial H}{\partial t} \qquad QED\ .
\end{aligned}
\tag{7.15}
$$

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FIGURE CAPTIONS

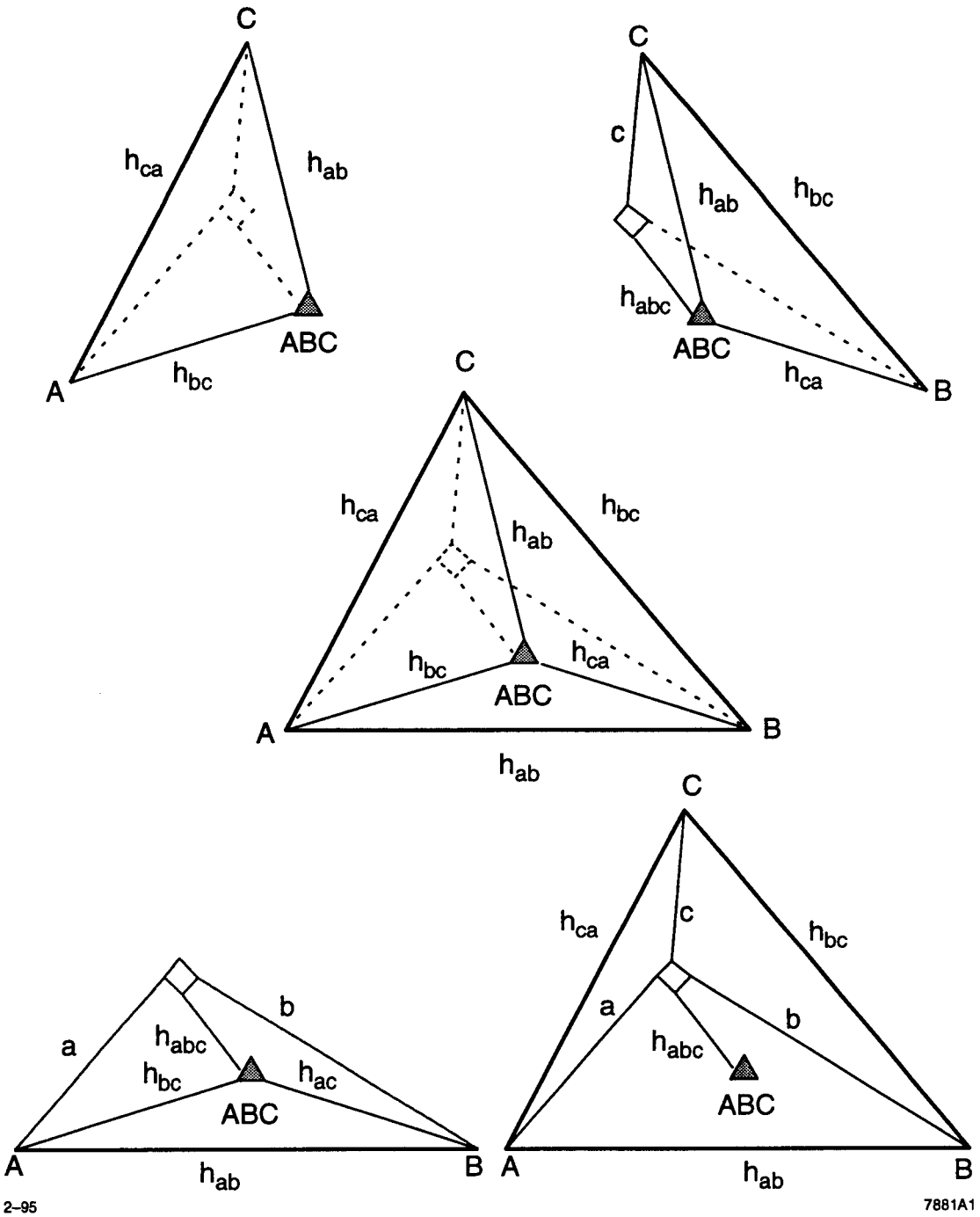
1) The tetrahedron generated by three discriminately independent bit-strings and its decomposition into four tetrahedra with a common vertex from which the lines of length a, b, c, d lead to the corners labeled A, B, C, ABC.

2) An arbitrary bit-string and the three fixed lengths h_{au}, h_{av}, h_{aw} which freeze its spatial position relative to the regular tetrahedron constructed from basis strings of length 2N.

3) The geometrical interpretation of the discriminant $\mathbf{h}_{ab}$ of two vector bit-strings **a** and **b** as the line segment connecting their tips of length h_{ab}.

4) A rotation about the W axis keeps h_{aw} constant.

5) The geometrical connection between the signed coordinate a_w, the integer lengths $a, W = 2N, h_{aw}$ and the radius ρ_{aw} illustrated in Fig. 4.


C
h_{ca}
h_{ab}
ABC
A
h_{bc}
C
c
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ABC
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Fig. 1

Fig. 2

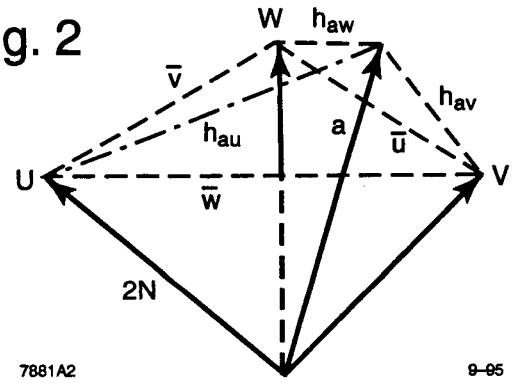


Fig. 3

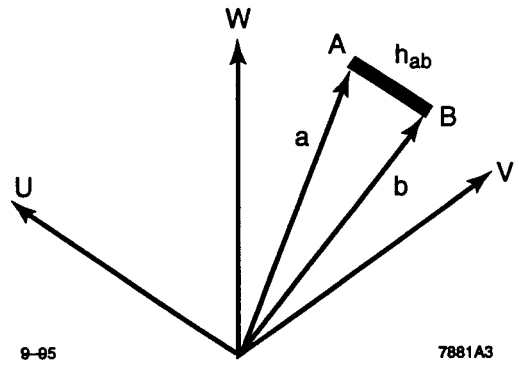


Fig. 4

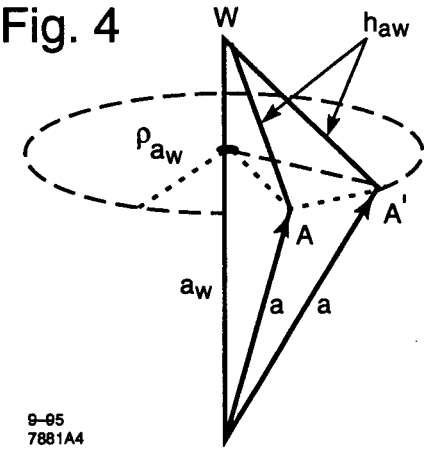
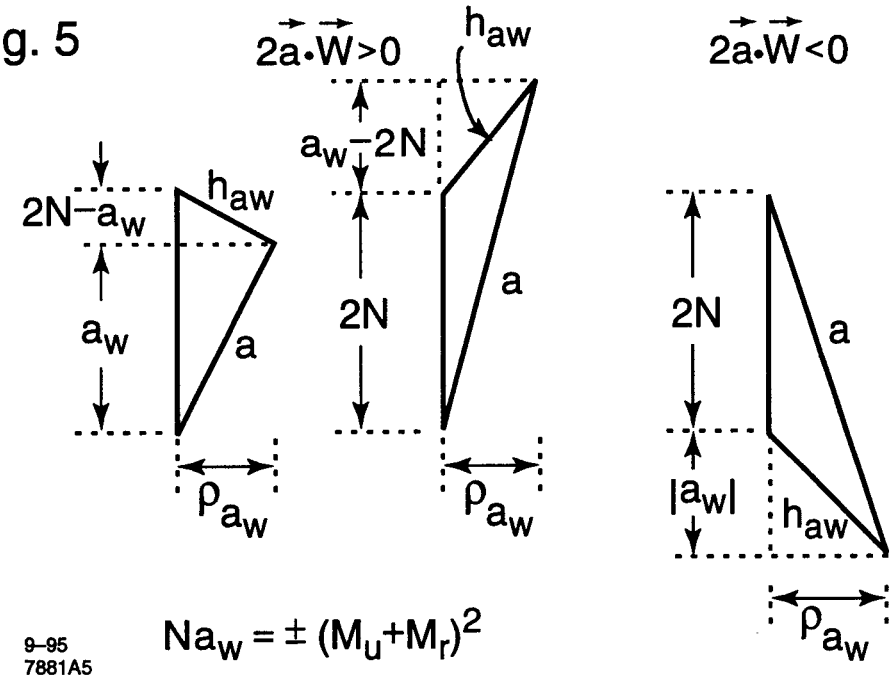


Fig. 5



$$Na_w = \pm (M_u+M_r)^2$$

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