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RADIATION AND RADIATION REACTION IN CONTINUOUS FOCUSING CHANNELS*

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Abstract

We show that the radiation damping rate of the transverse action of a particle in a straight, continuous focusing system is independent of the particle energy, and that no quantum excitation is induced. This absolute damping effect leads to the existence of a transverse ground state which the particle inevitably decays to, and yields the minimum beam emittance that one can ever attain, $\gamma\epsilon_{min} = \hbar/2mc$, limited only by the uncertainty principle. Due to adiabatic invariance, the particle can be accelerated along the focusing channel in its ground state without any radiation energy loss. These findings may apply to bent systems provided that the focusing field dominates over the bending field.

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1 INTRODUCTION

In an electron or positron storage ring the amplitude of transverse oscillations damps towards a stable closed trajectory. This radiation damping is caused by the emission of synchrotron radiation due to the uniform bending fields and by the replacement of the energy in the longitudinal direction only. The damping time is approximately equal to the time it takes to radiate away the initial energy of the particle. This radiation damping is counteracted by random fluctuations generated by the discrete photons emitted by each electron, which leads to an equilibrium beam emittance when the damping and excitation rates cancel [1, 2].

Radiation damping and excitation are, in principle, present in a straight magnetic or electric focusing system because particles with finite amplitude are bent back towards the straight line trajectory. However, these effects may be modified because the fields are not uniform in such a focusing system. Motivated by these considerations and also by proposals for accelerating charged particles in crystals [3, 4], in a recent paper [5] we study the radiation reaction effect on a charged particle undulating in a straight, continuous focusing system. In this paper we present more detailed radiation calculation and discuss extension of the study to a focusing-dominated bent system.

2 CHANNELING RADIATION

2.1 Stationary States

Consider an electrostatic focusing channel that provides a transverse continuous potential $V(x) = Kx^2/2$ for a charged particle, say a positron, where K is the focusing strength. The parabolic potential could be, for example, an approximation of the Lindhard potential in the case of planar crystal channeling [6, 7]. For simplicity, we take x as the single transverse dimension of the particle, which has relativistic energy $E = \gamma m$ and which moves freely (without acceleration) in the longitudinal z-direction with a constant momentum $p_z = \gamma m \beta_z$ in the absence of radiation. We set $e = \hbar = c = 1$ in most equations, but reinsert them when suitable. The effect of the additional transverse dimension will be discussed later. We consider the case in which the peak transverse momentum in one oscillation $p_{x,max}$ is much smaller than p_z. Defining $E_z = \sqrt{m^2 + p_z^2}$, we can approximate the total energy, $E = \sqrt{m^2 + p_z^2 + p_x^2} + V(x)$, as $E_z + E_x$, where $E_x = p_x^2/2E_z + V(x)$ is the so-called transverse energy. The motion of the particle is now decoupled into two parts: a free relativistic longitudinal motion and a transverse harmonic oscillation with an effective mass E_z.

We now move straight to quantum mechanical analysis of the system because we want to calculate the full radiation reaction including damping and excitation due to discrete photon emissions. Work on relativistic crystal channeling has shown that the spin degree of freedom plays a negligible role [8]. Therefore, we use the Klein-Gordon

equation to describe the general wavefunction $\Psi(x,z,t)$ of the channeled particle,

$$\left[(-i\vec{\nabla}-\vec{A})^2+m^2\right]\Psi=(i\partial_t-V)^2\Psi \quad . \tag{1}$$

In the absence of radiation, we let $\vec{A}=0$ and look for the energy levels E and the stationary states $\Psi(x,z,t)=e^{-iEt}|n,p_z\rangle$ of Eq. (1) by neglecting terms of the order $(E_x/E)^2$ [9]. We have

$$E\simeq E_z+E_x=\sqrt{m^2+p_z^2}+(n+1/2)\omega_z \quad , \tag{2}$$

$$|n,p_z\rangle=\frac{1}{\sqrt{L}}\exp\left(ip_z z\right)|n(p_z)\rangle \quad ,$$

$$|n(p_z)\rangle=(C_n/x_0)^{1/2}\exp\left(-x^2/2x_0^2\right)H_n(x/x_0) \quad , \tag{3}$$

where the transverse part of energy is $E_x=(n+1/2)\omega_z$, n is the transverse quantum number ($n=0,1,2...$), $\omega_z=\sqrt{K/E_z}$ is the transverse oscillation frequency, L is the length of the channel, $C_n=(2^n n!\sqrt{\pi})^{-1}$, $x_0=\sqrt{\omega_z/K}=1/\sqrt{\omega_z E_z}$ is the transverse ground state amplitude, and H_n is the nth order Hermite polynomial with its argument x scaled by x_0. It is clear that the transverse energy level E_x and the transverse wavefunction $|n(p_z)\rangle$ are controlled by both n and p_z.

2.2 Perturbation Approach

Coupling between the channeled particle and the radiation field, represented by the vector potential $\vec{A}$ in Eq. (1), leads to spontaneous emission of photons. By choosing Coulomb gauge, $\vec{\nabla}\cdot\vec{A}=0$, and ignoring the $\vec{A}^2$ term (double-photon emission), we arrive at

$$\left[-\nabla^2+m^2+i2\vec{A}\cdot\vec{\nabla}\right]\Psi(x,z,t)=(i\partial_t-V)^2\Psi(x,z,t) \quad . \tag{4}$$

Moving to the interaction representation we write $\Psi(x,z,t)=\exp(-i\mathcal{H}_0 t)\psi(x,z,t)$. Identifying $(\mathcal{H}_0-V)^2=(-\nabla^2+m^2)$, and neglecting $\ddot{\psi}(t)$ in the expansion of $(i\partial_t-V)^2\Psi(t)$ in Eq. (4), we obtain

$$\dot{\psi}(t)=e^{i\mathcal{H}_0 t}\left[(\mathcal{H}_0-V)^{-1}\vec{A}\cdot\vec{\nabla}\right]e^{-i\mathcal{H}_0 t}\psi(t) \quad . \tag{5}$$

Using first-order, time-dependent perturbation theory (Fermi's Golden Rule), we obtain the transition rate W_{fi} for the particle from an initial state $|n,p_z\rangle$ (with energy E) to a final state $|n',p_z'\rangle$ (with energy E'):

$$W_{fi}=2\pi|M_{fi}|^2\delta(E-E'-\omega_\gamma) \quad , \tag{6}$$

where the matrix element M_{fi} is defined by

$$|M_{fi}|^2=\left|\langle n',p_z';k_\gamma|(\mathcal{H}_0-V)^{-1}\vec{A}\cdot\vec{\nabla}|n,p_z;0\rangle\right|^2 \quad . \tag{7}$$

The vector potential $\vec{A}$ acting on the radiation field creates a photon of momentum $\vec{k}_\gamma$ and energy $\omega_\gamma(\omega_\gamma = |\vec{k}_\gamma|)$ with two possible polarizations $\hat{e}_1$ and $\hat{e}_2$ ($\hat{e}_1 \cdot \hat{e}_2 = 0$ and $\hat{e}_{1,2} \cdot \vec{k}_\gamma = 0$). The operator $(\mathcal{H}_0 - V)^{-1}$ can be approximated as $\mathcal{H}_0^{-1}$ by neglecting terms of the order (E_x/E). Therefore

$$|M_{fi}|^2 \simeq \frac{2\pi}{E'^2\omega_\gamma} \sum_{j=1}^{2} \left| \langle n', p'_z | e^{-i\vec{k}_\gamma \cdot \vec{x}} (\hat{e}_j \cdot \vec{\nabla}) | n, p_z \rangle \right|^2 \quad . \tag{8}$$

The matrix element $\langle n', p'_z | e^{-i\vec{k}_\gamma \cdot \vec{x}} (\hat{e}_j \cdot \vec{\nabla}) | n, p_z \rangle$ can be evaluated by integrating over the coordinate space. Let the radiated photon direction be (θ, ϕ), where θ is the emission angle relative to the longitudinal z axis and ϕ is the azimuthal angle in the x-y plane. Thus, the photon momentum is $\vec{k}_\gamma = k_\gamma(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and its polarization vectors can be chosen as $\hat{e}_1 = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$ and $\hat{e}_2 = (\sin\phi, -\cos\phi, 0)$. Because the transverse motion of the particle is restricted to be along the x direction, we can drop the y integral and take $\vec{\nabla} = (\partial_x, 0, \partial_z = ip_z)$. The integral over z simply gives rise to $\delta(p_z - p'_z - k_\gamma \cos\theta)$ as the channel length $L \gg \hbar/p_z$, which expresses the conservation of longitudinal momentum during the radiation process. We drop this δ function along with its normalization to simplify the relevant equations, then we find

$$\begin{aligned} \langle n', p'_z | e^{-i\vec{k}_\gamma \cdot \vec{x}} (\hat{e}_1 \cdot \vec{\nabla}) | n, p_z \rangle &= I^1_{n'n} \cos\theta\cos\phi - ip_z I^0_{n'n} \sin\theta \quad , \\ \langle n', p'_z | e^{-i\vec{k}_\gamma \cdot \vec{x}} (\hat{e}_2 \cdot \vec{\nabla}) | n, p_z \rangle &= I^1_{n'n} \sin\phi \quad , \end{aligned} \tag{9}$$

with

$$\begin{aligned} I^0_{n'n} &\equiv \langle n'(p'_z) | e^{-ixk_\gamma \sin\theta\cos\phi} | n(p_z) \rangle \quad , \\ I^1_{n'n} &\equiv \langle n'(p'_z) | e^{-ixk_\gamma \sin\theta\cos\phi} \partial_x | n(p_z) \rangle \quad . \end{aligned} \tag{10}$$

These matrix elements are integrals over x involving Hermite polynomials with different scaling parameters x_0 and x'_0 because the initial and the final transverse states have *different* effective masses labeled by p_z and p'_z.

We will first expand the final transverse wavefunction as a superposition of the initial ones [10]:

$$\begin{aligned} \langle n'(p'_z) | &= \sum_{n''} \langle n'(p'_z) | n''(p_z) \rangle \langle n''(p_z) | \\ &= \sum_{n''} \left(\frac{n'!}{n''!} \cos\psi \right)^{1/2} P^{\alpha_-}_{\alpha_+}(\cos\psi) \langle n''(p_z) | \quad , \end{aligned} \tag{11}$$

where $\cos\psi = 2r/(1+r^2)$, $r = (E'_z/E_z)^{1/4}$, $\alpha_- = (n'' - n')/2$, $\alpha_+ = (n'' + n')/2$ and $P^{\alpha_-}_{\alpha_+}$ is the associated Legendre polynomial. Then Eq. (10) becomes

$$\begin{aligned} I^0_{n'n} &= \sum_{n''} \left(\frac{n'!}{n''!} \cos\psi \right)^{1/2} P^{\alpha_-}_{\alpha_+}(\cos\psi) \langle n''(p_z) | e^{-ixk_\gamma \sin\theta\cos\phi} | n(p_z) \rangle \quad , \\ I^1_{n'n} &= \sum_{n''} \left(\frac{n'!}{n''!} \cos\psi \right)^{1/2} P^{\alpha_-}_{\alpha_+}(\cos\psi) \langle n''(p_z) | e^{-ixk_\gamma \sin\theta\cos\phi} \partial_x | n(p_z) \rangle \quad . \end{aligned} \tag{12}$$

Since the states $|n''\rangle$ and $|n\rangle$ have the same effective mass labeled by p_z, we can obtain the following integrals from Ref. [11]:

$$\begin{aligned}
\langle n''|e^{-ixk_\gamma \sin\theta\cos\phi}|n\rangle &= [(-i)\mathrm{sgn}(\sin\theta\cos\phi)]^{\mu}\left(\frac{n''!}{n!}\right)^{1/2} e^{-\xi/2} L_n^{\mu}(\xi)\xi^{\mu/2} \quad , \\
\langle n''|e^{-ixk_\gamma \sin\theta\cos\phi}\partial_x|n\rangle &= [(-i)\mathrm{sgn}(\sin\theta\cos\phi)]^{\mu-1}\left(\frac{n''!}{2n!}\right)^{1/2} e^{-\xi/2} \\
&\quad \times\left[nL_{n-1}^{\mu-1}(\xi)\xi^{(\mu-1)/2} + L_{n+1}^{\mu+1}(\xi)\xi^{(\mu+1)/2}\right] \quad , \qquad (13)
\end{aligned}$$

where $\mathrm{sgn}(\sin\theta\cos\phi) = +1$ if $\sin\theta\cos\phi > 0$ or -1 if $\sin\theta\cos\phi < 0$, $\xi = (x_0 k_\gamma \sin\theta\cos\phi)^2/2$, $\mu = n - n''$, and $L_{n''}^{\mu}$ is the associated Laguerre polynomial. Therefore, with $I_{n'n}^{0,1}$ given by Eq. (12) and (13), the transition rate becomes

$$W_{fi} = \frac{(2\pi)^2}{E'^2\omega_\gamma}\left[\left|I_{n'n}^1\cos\theta\cos\phi - ip_z I_{n'n}^0\sin\theta\right|^2 + \left|I_{n'n}^1\sin\phi\right|^2\right]\delta(E - E' - \omega_\gamma) \quad . \qquad (14)$$

Note that we have not made any approximations in the derivation of Eq. (12) to (14) except for $E_x \ll E_z$. Thus, this set of equations describes an *arbitrary* radiation process under the channeling condition. We can further simplify these equations by additional approximations.

2.3 Wiggler Regime and Undulator Regime

The two δ functions in the previous subsection clearly indicate that the total energy and longitudinal momentum of the electron and the photon are conserved during the radiation process, just as one might expect. In order to conserve longitudinal momentum, we need $p_z' = p_z - \omega_\gamma\cos\theta$. Let us assume the photon energy $\omega_\gamma \ll E$ throughout the rest of the paper. Then the longitudinal energy, $E_z = \sqrt{m^2 + p_z^2}$, must accordingly decrease by an amount

$$\Delta E_z \simeq (p_z/E_z)\Delta p_z \simeq \omega_\gamma\beta\cos\theta < \omega_\gamma \quad . \qquad (15)$$

where $\beta \equiv p/(E - V) \simeq p_z/E_z$. Since the total energy of the particle is reduced by an amount ω_γ from energy conservation, its transverse energy $E_x = E - E_z$ must decrease by

$$\Delta E_x = \omega_\gamma(1 - \beta\cos\theta) > 0 \quad . \qquad (16)$$

We also know $E_x = (n + 1/2)\omega_z$, it follows that

$$(n + 1/2)\omega_z - (n' + 1/2)\omega_z' = \omega_\gamma(1 - \beta\cos\theta) > 0 \quad . \qquad (17)$$

For a small change in E_z, $\omega_z' = \sqrt{K/(E_z - \Delta E_z)} \simeq \omega_z(1 + \Delta E_z/2E_z)$. Substituting Eq. (15) for ΔE_z, we obtain an equation that relates the change of the transverse quantum number to the photon energy and its emission angle θ,

$$(n - n')\omega_z = (1 - \beta\cos\theta)\omega_\gamma + (\omega_\gamma\beta\cos\theta)(n' + 1/2)\omega_z/2E_z > 0 \quad , \qquad (18)$$

which is always positive definite. We therefore conclude that both the transverse energy and the transverse quantum number of the particle always decrease after a photon emission process for all possible photon angles. We will come back to this point in the next section.

Introducing the harmonic number $\nu = n - n'$ and the pitch angle of the particle $\theta_p = p_{x,max}/p_z \simeq \sqrt{2E_x/E_z}$, we find from Eq. (18) a condition for the photon energy

$$\omega_\gamma \simeq \frac{\nu\omega_z}{1-\beta\cos\theta+\theta_p^2/4} \simeq \frac{2\gamma^2\nu\omega_z}{1+\gamma^2\theta^2+\gamma^2\theta_p^2/2} \quad . \tag{19}$$

Note that $\gamma\theta_p$ in the above equation plays the same role as the undulator strength parameter in the conventional undulator radiation using alternating bending magnets [12]. In the "Wiggler" regime where $\gamma\theta_p \gtrsim 1$, or $p_x \gtrsim m$, the transverse motion of the particle is classical because its quantum level $n \gg 1$. Taking the classical limit for the transverse motion ($n \to \infty, \hbar \to 0$, but $n\hbar \to E_x/\omega_z$ remains fixed) and applying Eq. (19), we can write both the associated Legendre polynomial and the associated Laguerre polynomial in terms of the appropriate Bessel functions [11]:

$$\left(\frac{n'!}{n''!}\cos\psi\right)^{1/2} P_{\alpha_+}^{\alpha_-}(\cos\psi) \to J_{\alpha_-}(\nu a) \quad ; \tag{20}$$

$$\langle n''|e^{-ixk_\gamma\sin\theta\cos\phi}|n\rangle \to (-i)^\mu J_\mu(\nu b) \quad ,$$

$$\langle n''|e^{-ixk_\gamma\sin\theta\cos\phi}\partial_x|n\rangle \to \frac{(-i)^{\mu-1}}{x_0}\sqrt{\frac{n}{2}}[J_{\mu-1}(\nu b)+J_{\mu+1}(\nu b)] \quad , \tag{21}$$

where $a = \theta_p^2\cos\theta/8(1-\beta\cos\theta+\theta_p^2/4)$ and $b = \theta_p\sin\theta\cos\phi/(1-\beta\cos\theta+\theta_p^2/4)$. Note that in this limit, Eq. (20) does not yield $\delta_{n'n''}$ even for $\omega_\gamma \ll E$ because any small deviation in the scaling parameter x_0 in the Hermite polynomial can be amplified as $n \to \infty$. Putting Eq. (20) and (21) back into Eq. (12) and (14), we obtain

$$\begin{aligned} W_{fi} \simeq\ & \frac{\pi^2}{\omega_\gamma}\Big[(S_{\nu 3}\theta_p\cos\theta\cos\phi - 2S_{\nu 1}\beta\sin\theta)^2 + (S_{\nu 3}\theta_p\sin\phi)^2\Big] \\ & \times\delta[(1-\beta\cos\theta+\theta_p^2/4)\omega_\gamma - \nu\omega_z] \quad , \end{aligned} \tag{22}$$

where $S_{\nu 1} = \sum_l J_l(\nu a)J_{\nu-2l}(\nu b)$ and $S_{\nu 3} = \sum_l J_l(\nu a)[J_{\nu-2l-1}(\nu b)+J_{\nu-2l+1}(\nu b)]$. Compared with Eq.(57) and Eq.(58) in Ref. [12], the analogy between channeling radiation and the conventional undulator radiation is obvious.

In the "undulator" regime where $\gamma\theta_p \ll 1$, the transverse oscillation amplitude is so small that the associated quantum level n can be very close to 1, so the above classical limit may not be valid. However, since both n' and n'' are comparable to n, we have in this case: $(n'!\cos\psi/n''!)^{1/2}P_{\alpha_+}^{\alpha_-}(\cos\psi) \to \delta_{n'n''}$ as $E_z' \simeq E_z$ or $\cos\psi \simeq 1$. Moreover, we can directly evaluate Eq. (13) by the dipole approximation [8] where terms beyond the order x are neglected:

$$\langle n''|e^{-ixk_\gamma\sin\theta\cos\phi}|n\rangle \simeq \langle n''|-ixk_\gamma\sin\theta\cos\phi|n\rangle = \frac{-i\sin\theta\cos\phi}{1-\beta\cos\theta}\sqrt{\frac{n\omega_z}{2E_z}}\delta_{n'',n-1} \quad ,$$

$$\langle n''|e^{-ixk_\gamma\sin\theta\cos\phi}\partial_x|n\rangle \simeq \langle n''|\partial_x|n\rangle = \sqrt{\frac{nE_z\omega_z}{2}}\delta_{n'',n-1} \quad . \tag{23}$$

We have made use of $n'' = n' < n$ after each radiation and $k_\gamma = \omega_\gamma \simeq \omega_z/(1-\beta\cos\theta)$ for $\gamma\theta_p \ll 1$ in Eq. (23) above. Thus, for an *arbitrary* transverse level n in the undulator regime, the transition rate is nonzero only if $n' = n-1$ (the dipole selection rule) and is given by

$$W_{fi} \simeq \frac{2\pi^2 n\omega_z}{E_z\omega_\gamma}\left[\frac{\cos^2\phi(\cos\theta-\beta)^2}{(1-\beta\cos\theta)^2} + \sin^2\phi\right]\delta[(1-\beta\cos\theta)\omega_\gamma - \omega_z] \quad . \tag{24}$$

This result is consistent with Eq. (22) in the limit of large n but small $\gamma\theta_p$. Therefore, we can calculate the rate of change of the particle's total energy from the dipole transition rate in this regime:

$$\frac{dE}{dt} = \sum_f \int \frac{d^3\vec{k}_\gamma}{(2\pi)^3}(E'-E)W_{fi} = -\frac{2}{3}\frac{r_e K}{mc}\gamma^2 n\hbar\omega_z \quad , \tag{25}$$

where $r_e = e^2/mc^2$ is the classical electron radius. After identifying $n\hbar\omega_z$ with the *rms* amplitude of the oscillating particle in the large n limit ($n\hbar\omega_z \simeq E_x = K\langle x^2\rangle$), we see that dE/dt in the above expression is identical to the classical radiation power, which is proportional to $E^2F_\perp^2$ ($F_\perp$ being the transverse focusing field strength).

3 RADIATION REACTION IN A STRAIGHT CHANNEL

3.1 Absolute, Asymmetric Damping

We now turn to the radiation reaction of the channeled particle. the transverse quantum level n of the particle always decreases after a random photon emission. This conclusion is valid for all oscillation amplitudes, although we focus on the undulator regime where $\gamma\theta_p \ll 1$ to illustrate the unique feature of radiation reaction in a focusing channel. With the dipole transition rate given by Eq. (24), we can calculate the rate of change of the transverse quantum level

$$\frac{dn}{dt} = \sum_f \int \frac{d^3\vec{k}_\gamma}{(2\pi)^3}(n'-n)W_{fi} = -\frac{2}{3}\frac{r_e K}{mc}n \quad . \tag{26}$$

We see that n damps exponentially with an energy-independent damping constant, $\Gamma_c = 2r_eK/3mc$. Note that in the case of radiation in a bending magnet, there is an additional term of opposite sign independent of the quantum level in question that represents the excitation of transverse oscillations [2]. That term is absent in Eq. (26) and the radiation damping is absolute because no quantum excitation is induced by random photon emissions. Since the action of the transverse oscillation is $J_n = E_x/\omega_z = (n+1/2)\hbar$, the decrement of the transverse energy level n leads to the radiation damping of this action given by $dJ_n/dt = -\Gamma_c(J_n - \hbar/2)$.

One can use classical radiation reaction to obtain a similar result for the radiation damping of the transverse oscillation amplitude that damps exponentially (the change of energy modifies the amplitude damping). It also clearly shows how to extend the results to the case where $\gamma\theta_p \gtrsim 1$. More importantly, the quantum mechanical

calculation above automatically takes into account the full radiation reaction and shows the absence of excitation in this system (a surprising result viewed from the standpoint of electron synchrotrons and storage rings). It is difficult if not impossible to model the radiation reaction effect of discrete photon emissions *classically* for $\gamma\theta_p \ll 1$, because the time during which a typical photon is emitted is much longer than the oscillation period in the undulator regime [2].

The excitation-free reaction of radiation comes from the fact that the transverse quantum level must *decrease* after each radiation process. In the longitudinal direction the particle recoils against the emitted photon in order to conserve the longitudinal momentum between the two particles. However in the transverse direction the existence of the focusing force destroys the momentum balance and suppresses the recoil effect. The external focusing environment absorbs the excess transverse momentum during the process of radiation. In this sense, the radiation reaction of a channeled particle in the transverse dimension is similar to that in the Mössbauer effect [14].

Another novel characteristic of this radiation reaction is that the relative damping rate of the transverse action can be much faster than the relative damping rate of the longitudinal momentum, i.e., the radiation reaction is asymmetric in these two dimensions. The rate of change of the longitudinal momentum can be obtained from the energy loss equation, Eq. (25), with the approximation $p_z \simeq E_z \simeq E$. We obtain

$$\left|\frac{1}{p_z}\frac{dp_z}{dt}\right| \simeq \frac{1}{E}\left|\frac{dE}{dt}\right| \simeq \frac{\Gamma_c}{2}\gamma^2\theta_p^2 \quad , \tag{27}$$

which is less than Γ_c for $\gamma^2\theta_p^2 < 2$. In the undulator regime we have the condition $\gamma\theta_p \ll 1$, thus

$$\left|\frac{1}{J_n}\frac{dJ_n}{dt}\right| \simeq \Gamma_c \gg \left|\frac{1}{p_z}\frac{dp_z}{dt}\right| \quad . \tag{28}$$

When the pitch angle of the particle is increased to the extent that the undulator condition is no longer satisfied, the transverse damping rate is much more complex and Eq. (28) should be modified. In this case, we expect that the relative damping rate of the transverse action approaches that of the longitudinal momentum just as the case of synchrotron radiation. However, some asymmetry between these two degrees of freedom always exists because the focusing force suppresses the radiation reaction in the transverse direction.

3.2 Transverse Ground State

Because of the lack of recoil and excitation in the transverse dimension, the particle damps exponentially to its transverse ground state ($n = 0$), and this ground state is stable against further radiation (energy and momentum conservation forbid further radiation). In the ground state the particle reaches the minimum value of the action $J_0 = \hbar/2$. Relating this minimum action to a normalized emittance, we find

$$\gamma\epsilon_{min} \equiv J_0/mc = \lambda\!\!\!^{-}_c/2 \quad , \tag{29}$$

where $\lambda\!\!\!^{-}_c = \hbar/mc$ is the Compton wavelength. This minimum is also the fundamental emittance limited by the uncertainty principle.

One can estimate the time needed for a particle to damp to its ground state. Suppose the particle enters the focusing channel with a transverse energy $(n_i+1/2)\omega_z$ satisfying the undulator condition, it reaches the ground state in a time $t_g \sim \ln(n_i)/\Gamma_c$. To illustrate the range of damping times, let us consider two extreme examples: crystal channels and conventional focusing devices for accelerators. The channeling strength for a typical crystal channel is $K \sim 10^{11}\mathrm{GeV/m}^2$, so $\Gamma_c \sim (10\mathrm{nsec})^{-1}$. When a 100MeV particle is initially barely captured by the crystal channel, the transverse energy of the particle is of the order of the maximum channeling potential energy 100eV, and the corresponding quantum number n_i is about 500. Thus, in the absence of any dechanneling effects such as multiple scattering [15], the time it takes to damp to the ground state is $t_g \sim 60$nsec. For a conventional focusing device, the focusing strength is about $K \sim 30\mathrm{GeV/m}^2$, so $\Gamma_c \sim (30\mathrm{sec})^{-1}$. The damping time to the ground state in this case depends upon the logarithm of the initial state n_i, but will usually be several e-folding times.

One can also calculate the total energy loss of a particle when it is damped to the transverse ground state. By replacing $n = n_i \exp(-\Gamma_c t)$ and $\omega_z \simeq \sqrt{K/E}$ in Eq. (25) and integrating over time, we find the final energy retained in the ground state $n_f = 0$ is

$$E_f = E_i/[1 + (\gamma\theta_p)_i^2/4]^2 \quad . \tag{30}$$

Since Eq. (30) is derived in the undulator regime where $\gamma\theta_p \ll 1$, we conclude that a particle can radiate to the ground state while losing only a negligible amount of total energy, provided that its initial pitch angle satisfies the undulator condition. Thus, particles that enter the focusing channel with the same initial energy but different initial pitch angles will all end up in the transverse ground state with a very small relative longitudinal energy spread of $(\gamma\theta_p)_i^2/2$.

3.3 Two Transverse Degrees of Freedom

We have left out the other transverse degree of freedom of the particle for the sake of simplicity. If the y direction is free of any force, the particle radiating a photon with a momentum component in the y direction must recoil by the same magnitude to conserve total momentum in this direction. In general, quantum excitations are present in a force-free dimension. However, if a continuous focusing force also exists in the y direction, and if both transverse oscillations satisfy the conditions $\gamma\theta_p^x \ll 1$ and $\gamma\theta_p^y \ll 1$, then it is straightforward to extend the discussion above to both transverse dimensions because radiation reaction effects in the x and the y directions are completely decoupled. Photons are emitted by changing either n_x or n_y by one, and all the previous results apply to both dimensions. In the case where the oscillation amplitude is large in the x or in the y direction, there is some coupling between the two transverse degrees of freedom. But if we define the total transverse energy

$$E_\perp = p_x^2/2E_z + K_1x^2/2 + p_y^2/2E_z + K_2y^2/2 \quad , \tag{31}$$

from the conservation of both energy and longitudinal momentum, it follows that $E_\perp$ always decreases after a random photon emission. Combining this with the existence

of a focusing axis in the continuous focusing system, we conclude that the particle must damp to a mutual transverse ground state ($n_x = 0$ and $n_y = 0$) that is stable against further radiation.

3.4 Adiabatic Acceleration

We note that all the results obtained here are not affected by adiabatic acceleration along the longitudinal direction, since both the action and the stationary states in our system are adiabatic invariants. The condition for adiabatic acceleration is given by

$$\frac{dE_{accel}}{dt} \ll \omega_z E \simeq \sqrt{KE} \quad . \tag{32}$$

Using the previous examples, we get $\omega_z E \sim 10^5$GeV/m for a crystal channel and 2GeV/m for a conventional focusing device when the energy of the particle is only 100MeV. Obviously, the above inequality is guaranteed by any foreseeable acceleration mechanism. We conclude that the particle, once damped to its transverse ground state in a continuous focusing channel, can be accelerated adiabatically along the channel without any further radiation loss. Therefore, the theoretical minimum transverse emittance can be retained at a much higher accelerated particle energy, and the relative longitudinal energy spread can be reduced through acceleration.

4 RADIATION REACTION IN BENT SYSTEMS

4.1 Bending Magnet and Storage Ring

We have shown that the radiation reaction in a straight, continuous focusing channel is fundamentally different from that in a bending magnet. In a uniform magnetic field, the radiating particle recoils against the emitted photon by both reducing its orbital quantum number and by shifting the center of its circular orbit [2]. This latter change is allowed due to the translational invariance of the system in the plane perpendicular to the magnetic field, i.e., the system is degenerate with regard to the orbiting centers. The center shift is even necessary in order that the tangent of the particle trajectory be continuous before and after the emission. Therefore, the photon emission yields a random recoil of the electron due to variations in both angle and magnitude of the photon's momentum. The resulting random shifts in the orbit center give rise to the random excitations of radial betatron oscillations.

On the other hand, the existence of a focusing axis in a straight, continuous focusing environment removes such a degeneracy and therefore eliminates any quantum excitation to the particle from random photon emissions. In a conventional storage ring, the stored particles are confined by both bending and focusing fields. However, the focusing field is typically so much weaker than the bending field that its radiation effect is negligible. On the average, radiation damping in a conventional storage ring shrinks the momentum vector of the particle proportionally [1, 16].

4.2 Focusing-dominated Bent System

Nevertheless, the above results of straight, focusing channels can be extended to bent systems provided that the focusing field is much stronger than the bending field. Let us consider a bent system with the radius of curvature ρ. A highly relativistic particle of energy E

being bent by a uniform magnetic field, $B = E/ec\rho$, radiates electromagnetic energy at the rate:

$$\frac{dE}{dt} = -\frac{2}{3}\frac{r_e c}{(mc^2)^3}\frac{E^4}{\rho^2} \quad , \tag{33}$$

and the characteristic damping (or anti-damping) rate in all three degrees of freedom due to the bending is

$$\Gamma_b \sim \frac{1}{E}\left|\frac{dE}{dt}\right| = \frac{2}{3}\frac{r_e c}{\rho^2}\gamma^3 \quad . \tag{34}$$

In addition, the particle radiates while executing rapid betatron oscillations around the ideal bent trajectory due to the focusing field. If the bending is adiabatic, the transverse damping rate due to betatron oscillations can then be approximated by $\Gamma_c = 2r_e K/3mc$, as discussed in the previous section. Taking the ratio of these two rates, we obtain:

$$\frac{\Gamma_b}{\Gamma_c} = \frac{\lambdabar_\beta{}^2}{(\rho/\gamma)^2} \quad , \tag{35}$$

where $\lambdabar_\beta = \sqrt{E/K} = 1/\omega_s$ represents the betatron wavelength. Since the radiation formation length due to the bending is of the order ρ/γ [1, 2], Eq. (??) shows that when this length is much longer than the betatron wavelength, the transverse damping due to local oscillations is much stronger than the radiation effects from the global bending of the trajectory. Therefore, in such a system, the radiation reaction is dominated by the focusing field. To illustrate the choice of parameters for such a system, we consider a numerical example: a focusing-dominated low energy electron ring. Let us assume that the radius of the ring is $\rho = 33$m and that $E = 0.1$GeV electrons circulate around the ring. A rather weak magnetic field $B = 0.01$T is required to confine the particles on the ideal circular trajectory. Suppose along the ideal trajectory, the electrons are continuously focused with the focusing strength $K = 30\text{GeV/m}^2$, so the betatron wavelength $\lambdabar_\beta$ is about 5.8cm and the radiation formation length ρ/γ is about 17cm. From Eq. (??), we see that the transverse damping rate due to the focusing field is about nine times as fast as the characteristic damping (or anti-damping) rate from the bending field.

In a straight system, quantum excitation is absent because the transverse energy level must decrease after each radiation process to satisfy the kinematic constraints. However, in a bent system, the transverse betatron oscillations can be coupled with the energy loss through the dispersion function η. The instantaneous emission of a typical photon with energy ω_γ results in a change δx_β in the betatron displacement and a change $\delta x'_\beta$ in the betatron slope given by [1]

$$\delta x_\beta = \eta\frac{\omega_\gamma}{E} \quad , \quad \delta x'_\beta = \eta'\frac{\omega_\gamma}{E} \quad . \tag{36}$$

We can estimate the transverse energy change induced by this effect to the first order in ω_γ/E.

$$(\Delta E_x)_{\rm dis} \simeq K x_\beta(\delta x_\beta) + E x'_\beta(\delta x'_\beta) = K(x_\beta \eta + \lambda\!\!\!^{-}_\beta{}^2 x'_\beta \eta')\frac{\omega_\gamma}{E} \quad . \tag{37}$$

Since the transverse energy is discrete with minimum level spacing ω_s, the transverse quantum level change is not allowed when the induced change is smaller than ω_s. If we make the simplifying assumption that $\eta \sim \lambda\!\!\!^{-}_\beta{}^2/\rho$ is a constant, so that $\eta' = 0$; and use the dipole radiation photon energy $\omega_\gamma \sim 2\gamma^2\omega_s$ as the typical photon energy in a focusing-dominated system, then the condition $(\Delta E_x)_{\rm dis} < \omega_s$ is equivalent to

$$\frac{\lambda\!\!\!^{-}_\beta}{(\rho/\gamma)} < \frac{1}{2\gamma\theta_p} \quad . \tag{38}$$

Note that Eq. (38) is derived in the undulator regime where $\gamma\theta_p \ll 1$. Thus the inequality is guaranteed in the focusing-dominated system with $\lambda\!\!\!^{-}_\beta < \rho/\gamma$. These considerations suggest that quantum excitation may be prohibited even in dispersive systems as long as certain conditions are satisfied. We should emphasize that the above discussion is not a proof; however, it points out a new regime in focusing-dominated systems where the discrete photon emissions actually occur over a length scale long compared with betatron oscillations. This radiation process may lead to adiabatic variations of both the particle trajectory and the off-energy closed orbit without any quantum excitations to betatron oscillations.

5 CONCLUSION

The basic results obtained here apply to straight or adiabatically bent, focusing-dominated systems. The excitation-free, asymmetric radiation reaction in such systems is the direct consequence of the kinematic requirements and does not depend on the various approximations used here. There may be interesting applications of this phenomenon in beam handling, cooling and acceleration. For example, in a sufficiently low-energy, focusing-dominated electron ring, the absolute transverse damping could perhaps be utilized to obtain ultra-cool beams in transverse phase space with negligible total energy loss. Proposals of miniature linacs powered by lasers focusing systems. The results of this paper provide a radiation damping mechanism to prevent emittance growth. The existence of a transverse ground state for the accelerated particles might also be relevant and important. However, when realistic systems are considered, some of the results shown here may be modified. For instance, if other sources of excitation (multiple Coulomb scattering, imperfections, etc.) are present, then the beam may not reach the minimum emittance. When these additional effects are included, the actual equilibrium beam emittance will depend upon the details of the application considered.

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