# The Coherent Beam-Beam Interaction with Four Colliding Beams 

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#### Abstract

The coherent beam-beam interaction in the absence of Landau damping is studied with a computer simulation of four, space-charge compensated colliding beams. Results are presented for the modes, phase space structures, widths, and growth rates of coherent beam-beam resonances. These results are compared with solutions of the Vlasov equation, and with measurements made at the DCI storage ring which operated with space-charge compensated colliding beams.


## INTRODUCTION

The luminosity of storage ring colliders is limited by the effects of the electromagnetic fields of one beam on the particles of the other beam. This beam-beam interaction is parametrized by the beam-beam strength parameter,

$$
\begin{equation*}
\xi=\frac{r_{e}}{2 \pi} \frac{N \beta_{y}^{*}}{\gamma \sigma_{y}\left(\sigma_{x}+\sigma_{y}\right)} \tag{1}
\end{equation*}
$$

where $r_{e}$ is the classical electron radius, $N$ is the number of particles in the beam, $\beta_{y}^{*}$ is the vertical amplitude function at the interaction point, $\gamma$ is the beam energy in units of rest energy, and $\sigma_{x}$ and $\sigma_{y}$ are the rms horizontal and vertical beam sizes at the interaction point. The beam-beam interaction is not linear in displacement, and, in the usual case of two colliding beams, those nonlinearities introduce single particle nonlinear resonances and a spread in transverse oscillation tunes. The vertical tune spread is equal to $\xi$ which is sometimes denoted as $\xi_{\mathrm{y} \text {. The beam-beam }}$ luminosity limit could be due to the nonlinear resonances and the tune spread which are single particle, incoherent effects, or it could be due to coherent instabilities. ${ }^{1}$

Coherent beam-beam instabilities are expected based on solutions of the Vlasov equation. $2,3,4,5$ They are characterized by rapid, turn-by-turn, correlated variations of the beam distributions. They have been seen in a two-beam simulation that used particle-in-cell techniques to calculate electromagnetic fields. ${ }^{6}$ In this simulation there was qualitative agreement with Vlasov equation solutions for sixth and eighth order resonances, but higher order resonances were not seen, presumably because of Landau damping from the beam-beam tune spread. The instabilities that were observed occurred for $\xi \geq 0.05$ and could be avoided by the appropriate choice of operating point. This led to the conclusion that the coherent beam-beam effect was not likely to be important in operating or planned colliders. In contrast, turn-by-turn variations of beam sizes have been observed at LEP using a novel detector capable of imaging the beam on successive turns. ${ }^{7}$ More data are needed before drawing any conclusion about the relation between these variations and the beam-beam limit.

Incoherent beam-beam effects can be eliminated by colliding four beams in the field compensating configuration shown in Figure 1. Without incoherent effects there is a possibility of

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Figure 1: Four colliding beams in the field compensation configuration used in DCI. The four beams are collinear and arrive at the interaction point at the same time. (Derbenev numbers the beams differently than we do in this figure. This leads to some sign differences in the equations summarizing his work.)
substantially improved performance. However, there is no Landau damping from the beam-beam tune spread, and coherent instabilities could be much more important. The DCI storage ring at the Laboratoire de l'Accélérateur Linéaire (Orsay, France) had four colliding, space-charge compensated beams, and the beam-beam limit was not significantly different than with two beams. ${ }^{8}$ This is a strong indication of the importance of coherent beam-beam effects in this configuration, and the DCI performance limit was attributed to them.

This paper reports the results of a computer simulation of four colliding beams. Coherent beam-beam resonances are observed, and their mode structures, phase space structures, widths and growth rates are measured and compared with solutions of the Vlasov equation to study the underlying physics of the coherent beam-beam interaction Other results are compared with DCI measurements to understand the performance there.

## SIMULATION

The simulation is a modification of that of Krishnagopal and Siemann. ${ }^{6}$ Test particles were followed in transverse, four-dimensional phase-space for a large number of turns with each turn consisting of transport between the interaction points and beam-beam collisions. The initial phase space coordinates were chosen from Gaussian distributions with the nominal sizes.

Different transport models were used based on the issue under study. For resonance studies and for comparisons with the Vlasov theory i) the horizontal and vertical dimensions were independent, ii) centroid feedback set the mean coordinates to zero before the beam-beam interaction, and iii) there was no radiation damping or quantum excitation. The DCI simulations were intended for comparison with experiments, and the transport had coupling between horizontal and vertical motions, radiation damping, quantum excitation, and no centroid feedback. In addition, either one or two interaction regions was possible, and the two interaction region model had phase advance errors between the interaction regions. These errors were consistent with estimates based on the DCI magnetic lattice with quadrupole gradient errors. ${ }^{9}$

The electromagnetic fields at the collision point were calculated by Lorentz transforming to the rest frame of a pair of beams and solving for the electrostatic fields there. First, a circular mesh was constructed for each pair of beams. The meshes had sixteen azimuthal bins and one hundred radial bins each with a size $\Delta \mathrm{r}=\left(\sigma_{\mathrm{x} 0}+\sigma_{\mathrm{y} 0}\right) / 20$ where $\sigma_{\mathrm{x} 0}$ and $\sigma_{\mathrm{y} 0}$ are the nominal horizontal and vertical rms beam sizes. Each mesh was centered; for example, the origin of the mesh for beams 1 and 3 was centered at $\bar{x}=\left(\bar{x}_{1}+\bar{x}_{3}\right) / 2$ and $\bar{y}=\left(\bar{y}_{1}+\bar{y}_{3}\right) / 2$ where $\bar{x}_{1}$ is the horizontal centroid of beam 1 , etc.

Particles were placed on this mesh by apportioning them to adjacent mesh sites with area weighting fractions ${ }^{10}$ and taking their charge into account. The resultant array was Fourier
analyzed in azimuth, and the real and imaginary parts of the Fourier coefficients were smoothed to reduce the effects of statistical fluctuations in the number of test particles at individual mesh sites. The smoothing was performed with the IMSL routine CSSCV11 that is based on a smoothing spline to approximate noisy data with the smoothing parameter found by cross-validation. ${ }^{12}$ The smoothed charge distributions together with the Green's function for Poisson's equation in polar coordinates gave the electrostatic fields.

## VLASOV THEORY FOR COHERENT INSTABILITIES OF FOUR COLLIDING BEAMS

## Introduction

There are several Vlasov equation solutions for the coherent beam-beam interaction. The initial work was by Derbenev and was devoted to four beams with transverse motion in two spatial dimensions. ${ }^{2}$ Dikansky and Pestrikov considered two beams, two transverse spatial dimensions, and synchrotron motion. ${ }^{3}$ Chao and Ruth studied two beams, only one transverse dimension, and no synchrotron motion. ${ }^{4}$ Zenkevich and Yokoya calculated the growth rates for two beams and one-dimensional oscillations including Landau damping. ${ }^{5}$ They found that the growth rates of low order resonances were diminished significantly by Landau damping.

There are qualitative disagreements between Chao and Ruth and Dikansky and Pestrikov when that calculation is restricted to one dimension. These disagreements arise from: i) different treatments of the focusing of the equilibrium distribution when making action-angle transformations (Chao and Ruth account for it while Dikansky and Pestrikov do not), and ii) an implicit assumption by Dikansky and Pestrikov that perturbations cannot be defocusing. These disagreements do not affect the four beam calculation.

## The Vlasov Equation Solution of Derbenev ${ }^{2}$

This section is a summary of the Derbenev paper. Neglecting radiation damping and synchrotron oscillations, the linearized Vlasov equations may be written in the form

$$
\begin{equation*}
\frac{\partial \mathrm{f}_{\mathrm{q}}}{\partial \mathrm{t}}+\frac{2 \pi}{\mathrm{~T}} \mathrm{Q}_{\alpha} \frac{\partial \mathrm{f}_{\mathrm{q}}}{\partial \varphi_{\alpha}}=-\frac{\partial \tilde{\mathrm{L}}_{\mathrm{q}}}{\partial \varphi_{\alpha}} \frac{\partial \mathrm{F}}{\partial \mathrm{I}_{\alpha}} \tag{2}
\end{equation*}
$$

where $\left\{\varphi_{\alpha}, \mathrm{I}_{\alpha}\right\}$ äre action-angle variables, T and $\mathrm{Q}_{\alpha}$ are the revolution period and betatron tune, respectively, and there is a summation over index $\alpha$. All four beams are assumed to have the same equilibrium distribution, $F$, and $f_{q}$ is the perturbation of the density distribution of beam $q$. The Lagrangian for interaction of a particle with fields excited by collective oscillations of other beams is $\tilde{L}_{q}$. For example, the Lagrangian of interaction of particle in beam 1 with the field of beams 2 and 4 can be represented in the form:

$$
\begin{equation*}
\tilde{\mathrm{L}}_{1}\left(\mathbf{r}_{\perp}, \mathrm{t}\right)=-2 \mathrm{e}^{2} \delta_{\mathrm{T}} \int \mathrm{~d} \Gamma_{\perp}^{\prime}\left(\mathrm{f}_{2}\left(\mathbf{p}_{\perp}^{\prime}, \mathbf{r}_{\perp}^{\prime}, \mathrm{t}\right)-\mathbf{f}_{4}\left(\mathbf{p}_{\perp}^{\prime}, \mathbf{r}_{\perp}^{\prime}, \mathrm{t}\right)\right) \ln \left(\left|\mathbf{r}_{\perp}-\mathbf{r}_{\perp}^{\prime}\right|\right) \tag{3}
\end{equation*}
$$

where $\delta_{\mathrm{T}}$ is a periodic $\delta$-function with period equal to the revolution period, $\mathbf{p}_{\perp}^{\prime}$ and $\mathbf{r}_{\perp}^{\prime}$ are transverse deviations from the equilibrium orbit, and $d \Gamma_{\perp}^{\prime}$ is a phase space volume element.

The four beam system can have four modes

$$
\begin{equation*}
\mathrm{f}_{+}^{ \pm}=\left[\mathrm{f}_{1}+\mathrm{f}_{3}\right] \pm\left[\mathrm{f}_{2}+\mathrm{f}_{4}\right] \text { and } \mathrm{f}_{-}^{ \pm}=\left[\mathrm{f}_{1}-\mathrm{f}_{3}\right] \pm\left[\mathrm{f}_{2}-\mathrm{f}_{4}\right] \tag{4}
\end{equation*}
$$

Instability develops from differences in the densities of the two beams moving in the same direction, so only two of these modes, the $f_{-}$modes, can be unstable. The "-" subscript is dropped in the equations that follow. The equations for the individual beams can be added and subtracted leading to a set of uncoupled equations for $f^{+}$and $f^{-}$. Stationary solutions of the form

$$
\begin{equation*}
\mathrm{f}^{ \pm}(\mathrm{I}, \varphi, \mathrm{t}+\mathrm{T})=\lambda \mathrm{f}^{\ddagger}(\mathrm{I}, \bar{\varphi}, \mathrm{t}) \tag{5}
\end{equation*}
$$

are sought. They are unstable if $|\lambda|>1$. The phase space distribution is Fourier transformed in angle

$$
\begin{equation*}
\mathrm{f}(\mathbf{I})=\sum_{\mathbf{m}} \mathrm{f}_{\mathbf{m}}(\mathbf{I}) \exp \{\mathbf{i m} \cdot \varphi\} \tag{6}
\end{equation*}
$$

The charge density is Fourier transformed

$$
\begin{equation*}
\sigma(\mathbf{k})=\int \mathrm{dx} \int d y \mathrm{e}^{\mathrm{ik} \cdot \mathbf{r}}\left[\int \mathrm{dp}_{\mathrm{x}} \int d p_{\mathbf{y}} \mathrm{f}(\mathbf{I}, \varphi)\right] \tag{7}
\end{equation*}
$$

and the equilibrium distribution is assumed to be a Gaussian. The result is an integral equation for $\mathrm{f}(\mathrm{k})=\sigma(\mathbf{k}) / \mathrm{k}$

$$
\begin{equation*}
\mathrm{f}^{ \pm}(\mathrm{k})=\mp 2 \mathrm{i} \sum_{\mathrm{m}} \frac{\mathrm{~m} \cdot \xi}{1-\lambda \exp (2 \pi \mathrm{im} \cdot \mathbf{Q})} \int \mathrm{g}_{\mathrm{m}}\left(\mathbf{k}, \mathrm{k}^{\prime}\right) \mathrm{f}^{ \pm}\left(\mathbf{k}^{\prime}\right) \mathrm{d}^{2} \mathrm{k}^{\prime} \tag{8}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{m}\left(k, k^{\prime}\right)=\frac{4}{{k k^{\prime}}^{\prime}} I_{m_{x}}\left(k_{x} k_{x}^{\prime}\right) I_{m_{y}}\left(k_{y} k_{y}^{\prime}\right) \exp \left(-\frac{k^{2}+k^{\prime 2}}{2}\right) \tag{9}
\end{equation*}
$$

There are resonances when $\mathbf{m} \cdot \mathbf{Q}$ is close to an integer. Defining $\boldsymbol{\varepsilon}=\mathbf{m} \cdot \mathbf{Q}-\mathrm{n}$ as the distance from resonance, and looking for eigenvalues of the form

$$
\begin{equation*}
\lambda=\exp (2 \pi \mathrm{i}(-\mathbf{m} \cdot \mathbf{Q}+\Delta)) \tag{10}
\end{equation*}
$$

with $\Delta \ll 1$ leads to

$$
\begin{equation*}
\Delta=-\varepsilon \pm \sqrt{\varepsilon^{2} \mp 2 \varepsilon \frac{m \cdot \xi}{\pi} C_{m}} \tag{11}
\end{equation*}
$$

where the minus sign inside the square root holds for $\mathrm{f}^{+}$, the plus sign holds for $\mathrm{f}^{-}$, and $\mathrm{C}_{\mathrm{m}}$ is the eigenvalue of

$$
\begin{equation*}
\ddot{C}_{m} f(\mathbf{k})=\int g_{m}\left(\mathbf{k}, \mathbf{k}^{\prime}\right) f\left(\mathbf{k}^{\prime}\right) \mathrm{d}^{2} \mathbf{k}^{\prime} \tag{12}
\end{equation*}
$$

These eigenvalues are positive and satisfy

$$
\begin{equation*}
\sum_{j} C_{m}^{j}=\int g_{m}(k, k) d^{2} k=\frac{4}{\pi} \frac{\ln \left(m_{x} / m_{y}\right)^{2}}{m_{x}^{2}-m_{y}^{2}} ;\left(\left|m_{x}\right|,\left|m_{y}\right| \gg 1\right) \tag{13}
\end{equation*}
$$

where the sum is over all eigenvalues.
Equation (11) contains the essential result. The $\mathrm{f}^{+}$mode is unstable for tunes above the resonance

$$
\begin{equation*}
0 \leq \varepsilon \leq 2 \frac{\mathbf{m} \cdot \xi}{\pi} C_{m} \tag{14}
\end{equation*}
$$

and the growth rate is

$$
\begin{equation*}
\tau^{-1}=2 \pi \sqrt{2|\varepsilon| \frac{\mathbf{m} \cdot \xi}{\pi} C_{m}-\varepsilon^{2}} \tag{15}
\end{equation*}
$$

The $\mathrm{f}^{-}$mode is an unstable for tunes below the resonance when

$$
\begin{equation*}
-2 \frac{\mathbf{m} \cdot \xi}{\pi} C_{\mathbf{m}} \leq \varepsilon \leq 0 \tag{16}
\end{equation*}
$$

with the, growth rate given by eq. (15).

## Round Beams

Our simulations have been done for the one dimensional case of nominally round beams, beams with equal sizes initially ( $\sigma_{x 0}=\sigma_{y 0} \equiv \sigma_{0}$ ), equal beam-beam strength parameters ( $\xi_{x}=\xi_{y}=\xi$ ), and equal betatron tunes $\left(\mathrm{Q}_{\mathrm{x}}=\mathrm{Q}_{\mathrm{y}} \equiv \mathrm{Q}\right)$. There is nothing to distinguish the two transverse directions, and, therefore, $\mathrm{m}_{\mathrm{x}}=\mathrm{m}_{\mathrm{y}} \equiv \mathrm{m}$. The resonance condition becomes

$$
\begin{equation*}
2 \mathrm{mQ}=\mathrm{n} \tag{17}
\end{equation*}
$$

Only even order resonances are allowed. The resonance widths expressed in units of tune are

$$
\begin{equation*}
|\Delta \mathrm{Q}| \leq 2 \frac{\xi}{\pi} \mathrm{C}_{\mathrm{m}} \tag{18}
\end{equation*}
$$

Taking the limit $\mathrm{m}_{\mathrm{x}}=\mathrm{m}_{\mathrm{y}}$ eq. (13) becomes

$$
\begin{equation*}
\sum_{\mathrm{j}} \mathrm{C}_{\mathrm{m}}^{\mathrm{j}}=\frac{4}{\pi \mathrm{~m}^{2}} \tag{19}
\end{equation*}
$$

Assuming that the largest eigenvalue scales with the same power of $\mathrm{m}^{[2]}$

$$
\begin{equation*}
|\Delta \mathrm{Q}| \propto \frac{\xi}{\mathrm{m}^{2}} \tag{20}
\end{equation*}
$$

A straightforward extension of the Chao and Ruth calculation which is based on a waterbag model for the equilibrium distribution gives

$$
\begin{equation*}
|\Delta \mathrm{Q}| \leq \frac{32 \xi}{\pi\left(4 \mathrm{~m}^{2}-1\right)} \tag{21}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau^{-1} \approx 4 \pi \sqrt{8|\Delta Q| \xi / \pi-\mathrm{m}^{2} \Delta \mathrm{Q}^{2}} \tag{22}
\end{equation*}
$$

for the one-dimensional beam-beam interaction. We compare these equations to simulation results for widths and growth rates although there are reservations in doing so: $i$ ) the waterbag distribution used to obtain these equations is approximate, and $i i$ ) we measure the growth rate of $\sigma^{2}-\sigma_{0}^{2}$ which is beyond the scope of the linearized Vlasov equation, and in comparing our measured growth rates with eq. (22) we are primarily interested in the order of magnitude and dependence on $\Delta Q$.

Table 1: Parameters for Comparisons with Vlasov Equation

$=$| Betatron Tune, $\mathrm{Q}=\mathrm{Q}_{\mathrm{x}}=\mathrm{Q}_{\mathrm{y}}$ | $\sim 0.8$ |
| :--- | :--- |
| Coupling | Independent Horizontal and Vertical Motions |
| Radiation | No Radiation Damping Or Fluctuations |
| Feedback | Centroid Feedback, $\overline{\mathrm{x}}_{1}=\overline{\mathrm{y}}_{1}=\ldots=0$, Before Collisions |
| Number of Test Particles/Beam, $\mathrm{N}_{\mathrm{TP}}$ | 50,000 |

## COMPARISONS WITH SIMULATIONS

An extensive study of the tenth order resonance $Q=8 / 10$ was performed using the parameters given in Table 1. The nature of the coherent instability is illustrated in Figures 2 and 3. The beam sizes are stable and equal to the nominal sizes for roughly the first 200 turns. They increase rapidly after that eventually reaching a condition that repeats every fifth turn. The horizontal size of beam 1 is plotted in Figure 2. Vertical sizes behave the same when the tunes are equal. As shown in Figure 3 beams 1 and 2 behave identically as do beams 3 and 4. This is as expected since the tune is above the resonance value of $8 / 10$, and, therefore, the $f^{+}$mode should be unstable. Figure 4 shows the sizes on the other side of $Q=8 / 10$ where the $f^{-}$mode should be unstable. As expected in this figure beams 1 and 4 have the same behavior as do beams 2 and 3 .


Figure 2: The RMS horizontal size of beam 1 normalized to the nominal size.

This resonance corresponds to $\mathrm{m}=5$, and there should be five-fold structure in the horizontal and vertical phase spaces as the instability develops. This is seen clearly and is illustrated in Figure 5 for $Q=0.80075$. The difference $f_{2}-f_{4}$ starts out essentially uniform. Fivefold structure has developed by turn 300. It persists during the rapid growth of beam size, but phase space has become badly distorted by the time the beam has reached its limiting behavior. The vertical phase space also has five-fold structure during the growth of the instability, and it evolves in a similar manner through turn 1000. Other resonances have phase space structure determined by the resonance order. Figure 6 is an example for the 18 th order resonance $\mathrm{Q}=$ 14/18.


Figure 3: Horizontal beam sizes for turns $250-400$ where the instability appears.


Figure 4: Horizontal beams sizes on Turn Number


Figure 5: Horizontal phase space plots for $\mathrm{Q}=0.80075$ and $\xi=0.0173$ for beam $2\left(\mathrm{f}_{2}\right)$ on the left and for $\left(f_{2}-f_{4}\right)$, the difference between beams 2 and 4 , on the right. The plots cover $\pm 5$ times the nominal RMS sizes, and the dashed lines are contours of negative value.


## Position

Figure 6: Horizontal phase space contours of $\left(f_{2}-f_{4}\right)$ for $Q=0.777853$ and $\xi=0.0173$, a point within the $\mathrm{Q}=14 / 18$ resonance. The plots cover $\pm 5$ times the nominal RMS sizes, and the dashed lines are contours of negative value.

The beams are nominally round, but there are no restrictions forcing them to stay round. However, they do remain round to a substantial degree although some variation with the azimuthal angle in physical space does occur. This is illustrated in Figure 7.


Figure 7: Contour plots in physical space for $f_{2}-f_{4}$. The figures cover $\pm 5 \sigma_{0}$, and the conditions are those of Figure 4: $\mathrm{Q}=0.79925$ and $\xi=0.0173$.

The instability growth rate was estimated by fitting the square of the beam size, corrected for the equilibrium size, $\sigma^{2}-\sigma_{0}^{2}$, with an exponential during the initial rise of the instability. This quantity was chosen for fitting since it is proportional to the emittance increase due to the instability. Figure 8 shows the results. The width is about three-quarters of that given by eq. (21),
and the growth rate about half that of eq. (22). This is reasonable agreement given the approximate waterbag distribution and the tenuous connection between the growth rate defined by eq. (10) and the growth rate of $\sigma^{2}-\sigma_{0}^{2}$.


Figure 8: The growth rate of $\sigma^{2}-\sigma_{0}^{2}$ from the simulation compared with eq. (22) (plotted as solid lines).

Note that the growth at $Q=8 / 10$ is not exactly zero as would be expected. We found that the growth rates at the centers of resonances depend on the number of test particles and tend to zero as $N_{T P}^{1 / 2}$ where $N_{T P}$ is the number of test particles. The growth rate at $\Delta \mathrm{Q}=-0.0005$, a point of maximum growth in Figure 8, changed by less than $25 \%$ when the number of test particles was varied from 25,000 to 150,000 .

Resonance widths were measured by tracking for different values of Q and $\xi$. The resultant dependence on resonance order is given in Figure 9. As expected the widths are linear in $\xi$, but they depend on m as $\mathrm{m}^{-2.6}$ to $\mathrm{m}^{-2.9}$ in contrast to the $\mathrm{m}^{-2}$ dependence expected from the waterbag model, eq. (21), and the assumption that the largest eigenvalue in the Derbenev solution scales with the same power of $m$ as the sum rule, eq. (20).

m
Figure 9: Resonance widths for different resonance orders. The first stable and last unstable points on each side of the resonance are indicated. The resonances are: $8 / 10(\mathrm{~m}=5), 11 / 14(\mathrm{~m}=$ 7 ), and $14 / 18(\mathrm{~m}=9)$. The resonance width for $\mathrm{m}=5, \xi=0.0173$ was doubled for inclusion in the figure.

## DCI PERFORMANCE

Four DCI operating points documented in LeDuff et al ${ }^{8}$ were chosen for comparison between measurements and simulations. Two models of DCI, ONE and TWO, named after the number of interaction regions, were used. Reference [9] has details of these models. Parameters - are given in Table 2.

Table 2: Parameters of DCI Models

| Parameter | ONE | TWO |
| :---: | :---: | :---: |
| Interaction regions | 1 | 2 |
| Q | $\sim 0.88$ | $\sim 1.76$ |
| $\beta^{*}(\mathrm{~m}) \quad \begin{aligned} & \text { Horizontal } \\ & \text { Vertical }\end{aligned}$ | $\begin{aligned} & 2.18 \\ & 2.18 \\ & \hline \end{aligned}$ | $\begin{aligned} & 2.18 \\ & 2.18 \end{aligned}$ |
| $\varepsilon(\mu \mathrm{m}) \quad$Horizontal <br> Vertical | $\begin{aligned} & 0.282 \\ & 0.015 \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.282 \\ & 0.015 \\ & \hline \end{aligned}$ |
| Coupling Resonance Width ${ }^{13}$ | 0.001 | 0.002 |
| Arc Errors \{Horizontal, Vertical\} | ---- | $\{.0005,-.002\},\{-.0005, .002\}$ |
| Energy ( $\gamma$ ) | $1.57 \times 10^{3}$ | $1.57 \times 10^{3}$ |
| Fractional Energy Loss per Turn | $7.1 \times 10^{-6}$ | $14.2 \times 10^{-6}$ |
| Feedback | No Feedback | No Feedback |
| Nominal Beam Sizes ( $\mu \mathrm{m}$ ) | 569 (fully coupled) | 569 (fully coupled) |
| Number of Test Particles/Beam, $\mathrm{N}_{\text {TP }}$ | 10,000 | 10,000 |
| Turns of Tracking | 20,000 | 10,000 |

Figure 10 shows the results for one point, $\mathrm{Q}=0.865$ and $\xi=0.022$. There are strong, low order resonances, $\mathrm{Q}=12 / 14$ and $\mathrm{Q}=7 / 8$, on the two sides of this operating point leading to a region of width $\Delta Q \approx 0.011$ where the beam size has increased by less than $10 \%$ after tracking for 20,000 turns. This is to be compared with a measured stable operating region of $\Delta \mathrm{Q} \approx 0.0020$ which was extracted from Figure 3 of reference [8] and reduced by a factor of two for comparison with the one interaction region model. The measured widths for stable operation at the four operating points are compared with the widths from simulation in Table 3. The simulation predicts stable regions 3-5 times wider than those measured.

The measurements themselves could be in error due to effects such as quadrupole power supply regulation narrowing the observed stable operating point, but this cannot be tested and must remain speculation at best. Synchrobetatron resonances could play a role, but the DCI bunch length was short compared to $\beta^{*}$, and the dispersion at the interaction region was small. ${ }^{14}$ There are indications that phase advance errors between the interaction regions affected the DCI tune shift limit with two beams. 9 A simulation of DCI with two interaction regions and phase advance errors was performed to see if these errors affect the four-beam performance. These results are shown in Figure 10. Phase advance errors between interaction points do not change the width of the stable region in this case.


Figure 10: RMS beam size in DCI normalized to the nominal size for operating point $1: Q \sim$ $0.865, \xi=0.0218$. Even order resonances up to 30th order are plotted with widths from eq. (21).

The resonances bounding each of the regions as well as the resonances up to 30th order within each region are listed in Table 3. In three of the regions (all except IV) one of the bounding resonances could be a lower, odd order resonance if the round beam symmetry were broken. In addition, two of the regions, I and IV, overlap resonances that could be lower odd order resonances if the round beam symmetry were broken. Closed orbit offsets at the collision point or unequal horizontal and vertical tunes could break this symmetry and could possibly be the cause of the disagreement between measurements and this simulation.

Table 3: Comparison of Measured and Simulated Widths of Stable Operating Regions

| Region | I | II | III | IV |
| :--- | :---: | :---: | :---: | :---: |
| $\mathrm{Q}, \xi$ | $0.865,0.022$ | $0.884,0.018$ | $0.894,0.014$ | $0.907,0.011$ |
| Measured Width | 0.0020 | 0.0027 | 0.0027 | 0.0034 |
| Simulated Width | 0.011 | 0.009 | 0.009 | 0.013 |
| Bounding Resonances | $12 / 14,7 / 8$ | $7 / 8,16 / 18$ | $16 / 18,9 / 10$ | $9 / 10,11 / 12$ |
| Overlapped Resonances <br> (up to 30th order) | $19 / 22,26 / 30$ | $23 / 26$ | $25 / 28$ | $20 / 22$ |

Measured widths are from LeDuff et al. ${ }^{8}$ The table entries are the stable operating regions from Figure 3 of that reference divided by two for comparison with simulations of ONE.

## SUMMARY AND CONCLUSIONS

The simulation results are in excellent agreement with the qualitative features of Derbenev's theory: $i$ ) the tune dependence of the stability of the $\mathrm{f}^{+}$and $\mathrm{f}^{-}$modes, and $i i$ ) the phase space structure of the unstable modes are as expected. The widths and growth rates are comparable to those calculated from an extension of the waterbag model of Chao and Ruth, but the widths observed in the simulation decrease more rapidly with resonance order.

The simulation agrees with the locations of stable operating points in DCI , but predicts operating regions three to five times wider than those measured. There are a number of possible explanations, but they are difficult to explore because DCI is no longer available for colliding beam experiments.

High order coherent beam-beam instabilities have been observed with modest beam-beam strength parameter, $\xi \sim 0.02-0.04$; the width of an 18th order resonance was measured for Figure 9, and the effects of 14 th and 22nd order resonances are shown in Figure 10. The appearance of these high order resonances is in contrast to the two-beam situation where resonances higher than eighth order were never observed in simulation. The absence of Landau damping makes the coherent beam-beam interaction the important, limiting phenomena for space charge compensated colliding beams.

We wish to thank Alex Chao for his interest and for numerous helpful comments. This work was supported by the Department of Energy, contract DE-AC03-76SF00515.

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[^0]:    * Work supported by the Department of Energy, contract DE-AC03-76SF00515. Submitted to Physical Review E

