

# COUPLING IMPEDANCE OF A LONG SLOT AND AN ARRAY OF SLOTS IN A CIRCULAR VACUUM CHAMBER \*

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## Abstract

A method is presented for calculating the real part of the longitudinal impedance for both a small hole and a long slot in a beam vacuum chamber with a circular cross section. The length of the slot can be arbitrarily large, the only requirement on the dimensions of the slots is that its width be much smaller than  $c/\omega$ . Regular arrays of  $N$  slots periodically distributed along the pipe are also considered. The equation relating the real part of the impedance to the energy radiated by the slot into waveguide eigenmodes is the basis of this method. We show that in addition to broad band impedance, there are narrow peaks in  $\text{Re}Z$ . For a single slot, the peaks are located slightly above the cutoff frequencies of various waveguide modes. For a periodic array, additional peaks are generated at frequencies determined by the array period. These peaks can be partially suppressed by randomization of the positions of the slots.

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## I. INTRODUCTION

Existing theory for the impedance produced by small holes in the wall of a vacuum chamber of the accelerator has been developed in papers by Kurennoy [1] and Gluckstern [2]. Their theory applies Bethe's approach developed for study of diffraction of an electromagnetic wave on a perfectly conducting plane screen with a small hole [3] to the problem of radiation of the beam propagating in a circular pipe having a hole in its wall. It is based on utilization of small parameters  $\alpha_{el}/b^3$  and  $\alpha_{mg}/b^3$ , where  $\alpha_{el}$  is the electric and  $\alpha_{mg}$  is the magnetic polarizabilities of the hole, and  $b$  is the beam pipe radius. For circular holes,  $\alpha_{mg} \sim |\alpha_{el}| \sim w^3$ , where  $w$  is the radius of the hole, and these ratios are of the order of  $(w/b)^3$ . This theory also assumes that the wavelength of the electromagnetic waves radiated by the hole is much larger than the dimensions of the hole. In the first approximation of the perturbation theory, the impedance is expressed in terms of polarizabilities  $\alpha_{el}$  and  $\alpha_{mg}$  and turns out to be purely imaginary.

One of the practically important predictions of their theory refers to the optimal shape of the holes. To minimize the impedance for a given total area of the holes, they should have a shape of long narrow slots in the direction of the pipe axis; however, the theory becomes inapplicable if the length of the slot  $l$  exceeds either the pipe radius or the wavelength  $c/\omega$ .

In this paper, we show how to eliminate these constraints and how to calculate the impedance for an arbitrary  $l$ , assuming only that the width of the slot  $w$  is much smaller than  $b$  and  $c/\omega$ . Another important issue addressed in this paper is the real part of the impedance of holes which is responsible for the Robinson or multibunch instability in circular accelerators. As mentioned above, the first order theory predicts a purely imaginary impedance. To find  $\text{Re}Z$ , it is necessary to develop a second order of the perturbation theory. Fortunately, for our purpose it is not necessary to derive expressions for electromagnetic fields in the second order if use is made of the relation between  $\text{Re}Z$  and the energy radiated by the hole on a given frequency, which can be found using the first order expressions for the radiated field.

This approach also allows consideration of a periodic array of  $N$  slots. A previous paper [4] considers this problem, assuming an infinitely long periodic system,  $N \rightarrow \infty$ . It has been shown that in such a system the impedance exhibits sharp peaks at frequencies that allow the coherent buildup of waveguide modes due to positive interference of radiated field from different holes. For a finite number  $N$ , we find the same resonances as in infinite system; however, the width and height of the peaks both depend on the  $N$ . We discuss how the transition to the limit  $N \rightarrow \infty$  occurs.

To simplify our consideration, we calculate only the longitudinal impedance.

Section II reviews the approach of Ref. [1]. Section III discusses a method in which a long slot is considered to consist of infinitesimally small dipoles with given polarizabilities per unit length, showing that the imaginary part of the impedance for long slots does not depend on the length of the slot. In Sec. IV, we find the real part of the impedance of a short slot. Sections V and VI extend our consideration to long slots and arrays of slots. Section VII discusses randomization of the positions of the slots and how this effects the narrow band impedance peaks. Our conclusions are discussed in Sec. 8.

## II. SMALL HOLE

To calculate the longitudinal impedance of a circular beam pipe with a hole, it is convenient to consider an oscillating current traveling with the velocity of light along the axis of the pipe,

$$I(z, t) = I_0 \exp(-i\omega t + i\kappa z), \quad (1)$$

where  $\kappa = \omega/c$ . The pipe is assumed to have a small hole located at  $z = 0$  with characteristic dimensions much less than pipe radius  $b$ . Perturbation of the electromagnetic field caused by the hole can be represented as a superposition of the waveguide modes propagating away from the hole. We will consider in this section only axisymmetric  $E$  modes that contribute to the longitudinal impedance in the first order of the perturbation theory.

The electromagnetic field of the  $m$ th axisymmetric  $E$  mode in a smooth wave guide is given by the following equations:

$$\begin{aligned} E_z^{(m)} &= \frac{\mu_m^2}{b^2} J_0\left(\mu_m \frac{r}{b}\right) \exp(\sigma i \kappa_m z), \\ E_r^{(m)} &= -\sigma \frac{i \mu_m \kappa_m}{b} J_1\left(\mu_m \frac{r}{b}\right) \exp(\sigma i \kappa_m z), \\ H_\theta^{(m)} &= -\frac{i \omega \mu_m}{c b} J_1\left(\mu_m \frac{r}{b}\right) \exp(\sigma i \kappa_m z), \end{aligned} \quad (2)$$

where  $J_0$  and  $J_1$  are the Bessel functions of the zeroth and first order,  $\mu_m$  is the  $m$ -th root of  $J_0$ ,  $b$  is the radius of the waveguide,  $\kappa_m = \sqrt{\omega^2 - \omega_m^2}/c$ , where  $\omega_m = c \mu_m / b$  is the cutoff frequency for the  $m$ th mode. The variable  $\sigma$  denotes the direction of the propagation of the wave;  $\sigma = +1$  corresponds to the waves propagating in the positive direction along the  $z$ -axes and  $\sigma = -1$  marks the waves traveling in the opposite direction.

The electromagnetic field scattered by the hole into the waveguide is characterized by the amplitudes  $a_m(\sigma)$  such that

$$F = h(z) \sum_{m=1}^{\infty} a_m(\sigma = 1) F^{(m)}(r, z, \sigma = 1) + h(-z) \sum_{m=1}^{\infty} a_m(\sigma = -1) F^{(m)}(r, z, \sigma = -1), \quad (3)$$

where  $h(z)$  is the step function and  $F$  denotes any of the components  $E_z$ ,  $E_r$ , or  $H_\theta$ . The factors  $a_m$  can be expressed in terms of the electric  $\alpha_{el}$  and magnetic  $\alpha_{mg}$  polarizabilities of the hole [1] \*

$$a_m = -\frac{2I_0}{c b^2 \kappa_m \mu_m J_1(\mu_m)} (\kappa \alpha_{mg} + \sigma \kappa_m \alpha_{el}). \quad (4)$$

Calculation using the following equation for the longitudinal impedance,

$$Z = -\frac{1}{I_0} \int_{-\infty}^{\infty} dz E_z(z, r=0) \exp(-i \omega z / c), \quad (5)$$

showed that  $Z$  is a purely imaginary linear function of the frequency  $\omega$  [1, 2],

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\*Our definition of  $\alpha_{el}$  and  $\alpha_{mg}$  agrees with the Bethe's paper [3]. They are two times larger than those used by Kurennoy [1, 5].

$$Z = -\frac{Z_0 i \omega}{2\pi c b^2} (\alpha_{el} + \alpha_{mg}) . \quad (6)$$

For a long slot of length  $l$  and width  $w$ ,  $l \gg w$ , parallel to the axis of the pipe,  $\alpha_{el}$  and  $\alpha_{mg}$  almost cancel each other:  $\alpha_{mg} \approx -\alpha_{el}$ . The degree of cancellation depends on the exact shape of the slot and the thickness of the wall  $t$ . For a slot with parallel edges and rounded ends in an infinitely thin wall [5],

$$\alpha_{mg} \approx -\alpha_{el} \approx \frac{1}{32} l w^2 , \quad \alpha_{el} + \alpha_{mg} \approx 2.1 \times 10^{-2} w^3 . \quad (7)$$

In the opposite limiting case of a thick wall,  $t \gg w$ ,

$$\alpha_{mg} \approx -\alpha_{el} \approx \frac{1}{4\pi^2} l w^2 , \quad (8)$$

and the sum  $\alpha_{el} + \alpha_{mg}$  is somewhat smaller than that given by Eq. (7).

The theory outlined above is based on the assumptions that the dimensions of the slot are much smaller than the pipe radius,  $w$ ,  $l \ll b$ , and that the wavelength corresponding to the frequency  $\omega$  is much larger than the dimensions of the slot,  $\kappa w$ ,  $\kappa l \ll 1$ . According to Eq. (6), in the first approximation of the perturbation theory, the real part of the impedance is equal to zero. In the next section we introduce a principle that allows us to extend the theory to the limit  $l > b$ ,  $\kappa l > 1$  (but  $w \ll b$ ).

### III. LONG SLOTS AND MULTIPLE HOLES

The applicability requirements of Eq. (6),  $l \ll b$  and  $\kappa l \ll 1$ , can actually be omitted because it is easily seen that the theory leading to Eq. (6) is linear.

First, let us show that in this theory the impedance of several holes located so that they do not cross talk to each other is equal to the sum of their individual impedances.<sup>†</sup> Indeed, the radiated electromagnetic field of an array of the holes will be the sum of the

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<sup>†</sup>For this to be true, the distance between the holes/slots should be large compared with the radius of the holes (width of the slots).

fields calculated separately for each hole. Since the impedance (5) is a linear functional of the electromagnetic field  $E_z$  on the axis, the impedance of the array is equal to the sum of the impedances of the individual holes.

For a single slot, if length  $l$  of a slot is comparable with  $b$  and/or  $\kappa^{-1}$ , we can consider its inner part (excluding the regions close to both ends, whose length  $l_1$  is such that  $w \ll l_1 \ll l$ ) to be a system of infinitesimally small magnetic and electric moments, uniformly distributed along the slot with the polarizabilities per unit length equal to  $\alpha_{el}/l$  and  $\alpha_{mg}/l$ , respectively. Since for an infinitely long slot  $\alpha_{el} = -\alpha_{mg}$ , the contributions  $dZ$  to the impedance from each infinitesimal element of length  $dz$  exactly cancel each other, because  $dZ$  is proportional to the sum  $dz\alpha_{el}/l$  and  $dz\alpha_{mg}/l$ . The remaining nonvanishing part of  $Z$  is due to the contribution of the ends of the slot, and will, as a matter of fact, give Eq. (6). This observation also explains why the impedance of a long slot does not depend on its length—only the end regions of the slot that have the length of several  $w$  contribute to  $Z$  [5].

We emphasize here that the conclusions that Eq. (6) remains valid for long slots and that the impedance of an array of slots is an additive quantity are true only in the first approximation of the perturbation theory. Section IV will show that additivity is not applicable for the real part of the impedance.

#### IV. REAL PART OF THE IMPEDANCE FOR A HOLE

The real part of the impedance of a hole arises in the second order of the perturbation theory based on the smallness of the parameters  $\alpha_{mg}/b^3$  and  $\alpha_{el}/b^3$ . At first glance, it seems necessary to do one more order of the perturbation theory and estimate the electromagnetic field in the second approximation in the small parameters of the theory. It turns out, however, that we can find the real part of the impedance without going to higher orders if use is made of the following relation between the  $\text{Re}Z$  and the energy  $P$  radiated per unit time by the hole (for example, see Ref. [6]):

$$P = \frac{1}{2} I_0^2 \text{Re}Z(\omega) . \quad (9)$$

The idea to use the energy radiated by the hole for estimating the loss factor due to the presence of the hole has been previously explored by M. Sands [7].

The energy flux  $P$  in Eq. (9) should include all the waves radiated by the hole, both inside and outside of the waveguide. The outside radiation will depend on the geometry and location of the conducting surfaces in that region and cannot be computed without knowing particular details of the specific design. Here we neglect its contribution, assuming that the thickness of the pipe wall is large enough so that the electromagnetic field does not penetrate through the hole.<sup>†</sup>

Inside the waveguide, we have to take into account the radiation going into all  $E$  and  $H$  modes, rather than axisymmetric  $E$  modes considered in the previous section. The amplitudes  $a_{n,m}$  ( $n$  is the azimuthal and  $m$  the radial number) of those waves have been calculated by Kurennoy [1]. For an  $E$  mode,

$$a_{n,m}^{(E)} = \frac{4I_0}{cb^2 \kappa_{n,m} \mu_{n,m} J'_n(\mu_{n,m}) (1 + \delta_{n,0})} (\kappa \alpha_{mg} + \sigma \kappa_{n,m} \alpha_{el}) , \quad (10)$$

and for an  $H$  mode,

$$a_{n,m}^{(H)} = -\frac{4nI_0}{cb^2 \kappa'_{n,m} (\mu_{n,m}^2 - n^2) J_n(\mu'_{n,m})} (\sigma \kappa'_{n,m} \alpha_{mg} + \kappa \alpha_{el}) , \quad (11)$$

where  $\omega_{n,m} = c\mu_{n,m}/b$ ,  $\omega'_{n,m} = c\mu'_{n,m}/b$ ,  $\kappa_{n,m} = \sqrt{\omega^2 - \omega_{n,m}^2}/c$ ,  $\kappa'_{n,m} = \sqrt{\omega^2 - \omega_{n,m}^2}/c$ ,  $\mu_{n,m}$  is the  $m$ th root of the Bessel function  $J_n$  of the  $n$ th order,  $\mu'_{n,m}$  is the  $m$ th root of the derivative  $J'_n$  and  $\delta_{0,n}$  is the Kronecker delta symbol. Note that for  $n = 0$ , Eq. (10) reduces to Eq. (4). The energy flow in the mode of unit amplitude is equal to

$$P_{n,m}^{(E)} = \frac{1 + \delta_{0,n}}{16} \omega \kappa_{n,m} \mu_{n,m}^2 J_n'^2(\mu_{n,m}) , \quad (12)$$

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<sup>†</sup>Note that though a thick wall strongly suppresses the radiation into the outer space through the slot, it does not eliminate it completely in the case of a long slot. Even if the wall is infinitely thick,  $H_{0,1}$  waves with the cutoff frequency  $\pi c/l$  can propagate from within the slot to the outside of the vacuum chamber. Depending on how effeciently those waves are excited by the beam, a part of the electromagnetic field of the beam will eventually channel through the wall.

and

$$P_{n,m}^{(H)} = \frac{1 + \delta_{n,0}}{16} \omega \kappa'_{n,m} (\mu_{n,m}'^2 - n^2) J_n^2(\mu_{n,m}') , \quad (13)$$

respectively.

The energy flux in each mode radiated by the slot is given by  $|a_{n,m}(\sigma = 1)|^2 P_{n,m}$  and  $|a_{n,m}(\sigma = -1)|^2 P_{n,m}$  in the forward and backward directions, respectively. It is evident that this radiation occurs only if the frequency  $\omega$  is larger than the cutoff frequency  $\omega_{n,m}$  (or  $\omega'_{n,m}$ ). It is interesting to notice that since  $\alpha_{mg}$  and  $\alpha_{el}$  have opposite signs, radiation in the backward direction in each mode is larger than in the forward direction.

The total energy flux  $P$  is

$$P = \sum_{E,H} \sum_{n,m} \sum_{\sigma=\pm 1} P_{n,m} |a_{n,m}|^2 , \quad (14)$$

where the summation is carried out over both directions of propagation,  $\sigma = \pm 1$ , all possible values of  $n$  and  $m$ , and also over  $E$  and  $H$  modes. Combining Eqs. (9)–(13) yields the following equation for the contribution of  $E$  and  $H$  modes into the real part of the impedance:

$$\text{Re}Z^{(E)} = \frac{Z_0}{\pi} \frac{\omega^2}{c^2 b^4} \sum_{n,m} \frac{1}{(1 + \delta_{n,0})} F^{(E)}\left(\frac{\omega}{\omega_{n,m}}\right) , \quad (15)$$

where

$$F^{(E)}(x) = \frac{\alpha_{mg}^2 x^2 + \alpha_{el}^2 (x^2 - 1)}{x \sqrt{x^2 - 1}} \quad (16)$$

for  $x > 1$ , and  $F^{(E)}(x) = 0$  for  $x < 1$ . For the  $H$  modes

$$\text{Re}Z^{(H)} = \frac{Z_0}{\pi} \frac{\omega^2}{c^2 b^4} \sum_{n,m} \frac{n^2}{\mu_{n,m}'^2 - n^2} F^{(H)}\left(\frac{\omega}{\omega'_{n,m}}\right) , \quad (17)$$

where

$$F^{(H)}(x) = \frac{\alpha_{el}^2 x^2 + \alpha_{mg}^2 (x^2 - 1)}{x \sqrt{x^2 - 1}} \quad (18)$$

for  $x > 1$ , and  $F^{(H)}(x) = 0$  for  $x < 1$ .



For large aspect ratio slots,  $l \gg w$  (but  $l \ll b$  and  $l\kappa \ll 1$ ),  $\alpha_{mg} \approx -\alpha_{el}$ , and  $F^{(E)}(x) = F^{(H)}(x)$ . The plot of the  $\text{Re}(Z^{(E)} + Z^{(H)})$  measured in units  $\alpha_{mg}^2 Z_0 / \pi b^6$  as a function of  $\omega b/c$  is shown in Fig. 1.

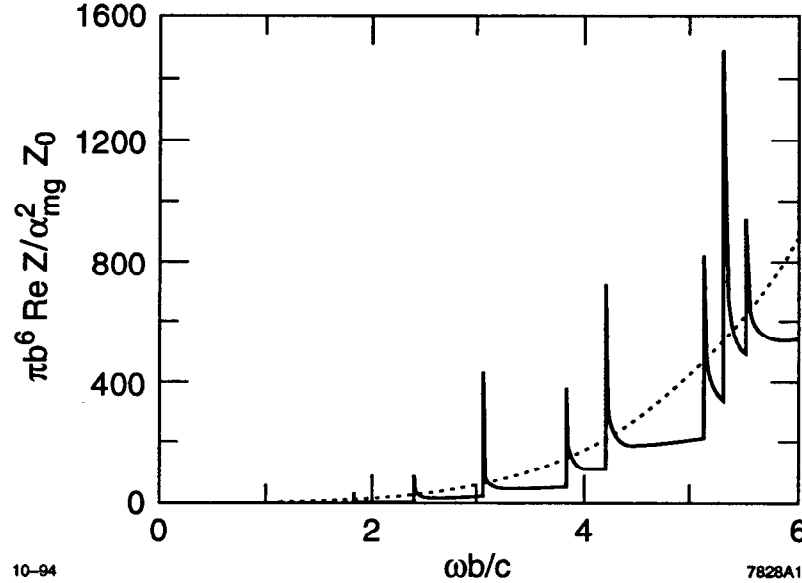


Figure 1. Real part of the impedance of a short large-aspect-ratio slot as a function of the frequency (solid curve), and a high-frequency approximation given by Eq. (20) (dotted curve).

Because the functions  $F^{(E)}(x)$  and  $F^{(H)}(x)$  go to infinity when  $x \rightarrow 1$ ,  $\text{Re}Z$  has singularities at the cutoff frequencies  $\omega_{n,m}$  and  $\omega'_{n,m}$ . Formally, this happens because the amplitude of the radiated waves given by Eqs. (10) and (11) scales as  $\kappa_{n,m}^{-1}$  when  $\omega$  approaches a cutoff frequency. The physics of this effect is very simple—the group velocity of the waveguide eigenmodes goes to zero at the cutoff frequency so that the radiated waves almost stall when propagating away from the slot. As a result, the energy radiated by the slot will be accumulated, amplifying the amplitude of the waves. The situation becomes similar to the case when a resonant cavity is driven by an external source. As is well known,

the amplitude of the eigenmode goes to infinity when the driving frequency approaches the resonant frequency of the cavity.

The actual height of the cutoff peaks is determined by two effects which lie beyond the scope of the present theory. First, in addition to the beam field, the radiation fields must be taken into account when calculating the electric and magnetic dipole moments of the slot. This results in a finite height for the peaks, even for perfectly conducting walls. Assuming finite conductivity of the walls will further reduce the height of the peaks.

Since the real part of the impedance is of the second order in the ratio  $\alpha_{mg}/b^3$  and  $\alpha_{el}/b^3$ , it should be small compared with the imaginary part given by Eq. (6). To compare them by an order of magnitude, we assume that  $\omega \sim c/b$ ,

$$\frac{\text{Re}Z(\omega)}{\text{Im}Z(\omega)} \sim \frac{l^2 \omega}{b^3}, \quad (19)$$

which is indeed a small quantity when the dimensions of the slot are much less than  $b$ .

In the limit  $\omega \gg c/b$ , a large number of harmonics is involved in the sums (15) and (16). By considering them to be continuous variables, it is possible to integrate over  $n$  and  $m$  instead of summing. This integration is carried out analytically in the Appendix, yielding

$$\text{Re}Z = \frac{2}{3\pi} Z_0 \frac{\omega^4 \alpha_{mg}^2}{c^4 b^2}. \quad (20)$$

This function is also plotted in Fig. 1; it gives a good approximation of the averaged dependence of the  $\text{Re}Z$ , even for small frequencies.

Equation (20) can be derived in a much simpler way by considering radiation of the electric and magnetic dipoles representing the hole in a free space, and estimating the energy flow in a half space corresponding to the inner part of the waveguide [4]. This approach has been realized in Ref. [1] with a result which is four times smaller than ours. The discrepancy is due to the fact that the author of paper [4] used polarizabilities that are two times smaller than the correct ones. found, for example, in Ref [3].

## V. REAL PART OF THE IMPEDANCE FOR A LONG SLOT

As pointed out in Sec. II, the imaginary part of the impedance for a long slot does not depend on length  $l$ , and is given by Eq. (6). To find the real part of the impedance in the case when  $l$  is comparable or larger than  $b$  and/or  $\kappa^{-1}$ , we follow the logic of Sec. II, and consider the long slot as a distributed system of magnetic and electric dipoles. The field radiated by the slot consists of the waves coming from different elements of the slots with a relative phase advance between them. For two infinitesimal elements located at distance  $z$ , the phase advance is composed of two parts. The first part is due to the change of phase of the driving field of the beam, and is equal to  $\kappa z$ . The second part is caused by the relative phase shift of the two radiated waves, and is equal to  $-\sigma\kappa_{n,m}z$ , where  $\sigma = \pm 1$  for the forward and backward propagating waves. The total phase exponent,  $\exp(i\kappa z - i\sigma\kappa_{n,m}z)$  should be integrated over the length of the slot, yielding the factor

$$f_{n,m}(\sigma) = \frac{1}{l} \int_0^l \exp(i(\kappa - \sigma\kappa_{n,m})z) dz = \frac{1}{il(\kappa - \sigma\kappa_{n,m})} [\exp(i(\kappa - \sigma\kappa_{n,m})l) - 1] \quad (21)$$

for the  $E$  modes and a similar factor  $f'_{n,m}(\sigma)$ , for which  $\kappa_{n,m} \rightarrow \kappa'_{n,m}$  in Eq. (21), for the  $H$  modes. These factors multiply the amplitudes  $a_{n,m}^{(E)}$  and  $a_{n,m}^{(H)}$  in Eqs. (10) and (11). Combining all these changes, and taking into account that for a long slot,  $\alpha_{el} = -\alpha_{mg}$ , results in the following modifications of the functions  $F^{(E)}$  and  $F^{(H)}$  in Eqs. (15) and (17):

$$F^{(E)}(x) = \frac{2b^2}{\mu_{n,m}^2} \left( \frac{\alpha_{mg}}{l} \right)^2 \frac{1}{x\sqrt{x^2 - 1}} \times \left\{ \sin^2 \left[ \frac{l\mu_{n,m}}{2b} (x - \sqrt{x^2 - 1}) \right] + \sin^2 \left[ \frac{l\mu_{n,m}}{2b} (x + \sqrt{x^2 - 1}) \right] \right\}, \quad (22)$$

and  $F^{(H)}$  given by the same expression with  $\mu_{n,m}$  substituted by  $\mu'_{n,m}$ . The factors  $f_{n,m}(\sigma)$  reduce to 1 in the limit  $l \ll |\kappa - \sigma\kappa_{n,m}|^{-1}$ , reproducing the result for a short slot. In the opposite limit,  $l \gg |\kappa - \sigma\kappa_{n,m}|^{-1}$ , the effective length of the slot that contributes to the real part of the impedance turns out to be equal to  $|\kappa - \sigma\kappa_{n,m}|^{-1}$ , which means that  $\text{Re}Z(\omega)$  also does not depend on  $l$  in the limit  $l \gg \kappa^{-1}$  (but  $\kappa^{-1} \gg w$ ).

## VI. REGULAR ARRAY OF SLOTS

Consider an array of  $N$  identical slots distributed along the beam pipe such that the distance between the slots is equal to  $d_1$ . The system has a period  $d = l + d_1$ ; see Fig. 2 (we do not assume here short slots that allow  $l$  to be comparable with  $b$  and  $\kappa^{-1}$ ). Notice that the  $z$ -coordinate of the left end of the  $n$ th slot is equal to  $z_n = nd$ . The electromagnetic field scattered by the array is the sum of the fields of individual slots. As discussed above, in the first approximation of the perturbation theory, the impedance is equal to  $NZ$ , where  $Z$  is given by Eq. (6). However, since the energy radiated by the array of slots is a quadratic function of the amplitude of the waves, it will be shown below that, at resonant frequencies, there is a strong amplification in  $\text{Re}Z$  which scales as  $N^2$ .

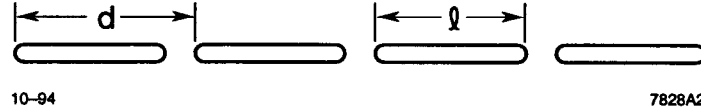


Figure 2. Periodic array of slots.

To find the radiation from  $N$  slots, it is necessary to sum their fields, taking into account the relative phase advance between the fields of deferent slots. As shown in the previous section, the phase advance between two adjacent slots is equal to  $\exp(i\kappa d - i\sigma\kappa_{n,m}d)$ . For  $N$  slots, the amplitude of  $(n, m)$   $E$  mode should be multiplied by the following factor:

$$g_{n,m}(\sigma) = \sum_{j=0}^{N-1} \exp[idj(\kappa - \sigma\kappa_{n,m})] = \frac{1 - \exp[idN(\kappa - \sigma\kappa_{n,m})]}{1 - \exp[id(\kappa - \sigma\kappa_{n,m})]}. \quad (23)$$

The square of the absolute value of  $g_{n,m}(\sigma)$ ,

$$|g_{n,m}(\sigma)|^2 = \frac{\sin^2[dN(\kappa - \sigma\kappa_{n,m})/2]}{\sin^2[d(\kappa - \sigma\kappa_{n,m})/2]}, \quad (24)$$

multiplies each sine term in Eq. (22) modifying the function  $F^{(E)}$  into the following expression:

$$\begin{aligned}
F^{(E)}(x) = & \frac{2b^2}{\mu_{n,m}^2} \left( \frac{\alpha_{mg}}{l} \right)^2 \frac{1}{x\sqrt{x^2-1}} \\
& \times \left\{ \sin^2 \left[ \frac{l\mu_{n,m}}{2b} (x - \sqrt{x^2-1}) \right] \frac{\sin^2 \left[ (dN\mu_{n,m}/2b) (x - \sqrt{x^2-1}) \right]}{\sin^2 \left[ (d\mu_{n,m}/2b) (x - \sqrt{x^2-1}) \right]} \right. \\
& \left. + \sin^2 \left[ \frac{l\mu_{n,m}}{2b} (x + \sqrt{x^2-1}) \right] \frac{\sin^2 \left[ (dN\mu_{n,m}/2b) (x + \sqrt{x^2-1}) \right]}{\sin^2 \left[ (d\mu_{n,m}/2b) (x + \sqrt{x^2-1}) \right]} \right\}, \quad (25)
\end{aligned}$$

For the  $H$  modes, the function  $F^{(H)}(x)$  contains  $\mu'_{n,m}$  instead of  $\mu_{n,m}$  in Eq. (25).

The maximum value of  $|g_{n,m}|^2$  in Eq. (24) is equal to  $N^2$  and is attained when the following condition holds

$$d(\kappa - \sigma\kappa_{n,m}) = 2q\pi, \quad (26)$$

where  $q$  is an integer. For large  $N$ , Eq. (24) represents narrow peaks with a width at half height  $\Delta\omega/\omega \approx 1/(2qN)$  at the resonant frequencies. This implies that the  $Q$  factor for these resonances can be estimated as  $Q \approx qN$ . For the problems where only the integral strength of the impedances is important, and  $N \gg 1$ ,  $|g_{n,m}(\sigma)|^2$  can be approximated by a sum of delta functions,

$$|g_{n,m}(\sigma)|^2 = 2\pi N \sum_{q=-\infty}^{\infty} \delta[d(\kappa - \sigma\kappa_{n,m}) - 2\pi q]. \quad (27)$$

If  $d/b = 2\pi q/\mu_{n,m}$  (or  $d/b = 2\pi q/\mu'_{n,m}$ ), Eq. (26) is satisfied by the cutoff frequency  $\omega_{n,m}$  (or  $\omega'_{n,m}$ ). In this case, the height of the resonant peaks will be strongly amplified because of the superposition of the cutoff singularity for a single peak with a maximum of the  $|g_{n,m}|^2$  function.

In the limit of very large  $N$ ,  $N \rightarrow \infty$ , the width of the resonances becomes so narrow that it will actually be determined by the finite conductivity of the walls  $\sigma$ . The transition to this regime occurs when  $Q$  becomes comparable to  $b/\delta$ , where  $\delta$  is the skin depth at the resonant frequency. Previously, this regime has been studied in detail for an infinitely long periodic bellow in Ref. [8], where the resonance conditions (26) have also been found.

## VII. RANDOMIZATION OF THE SLOT POSITIONS

In cases when a large number of slots is involved, the resonant peaks in the impedance may create a real danger for the stability of the beam. One of the ways to improve the situation is to randomize the positions of the slots. As we will show below, randomization can help to suppress the resonances.

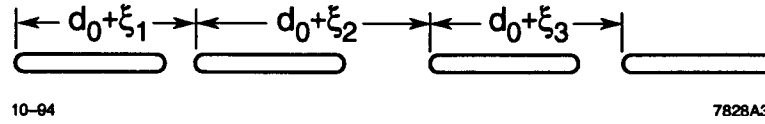


Figure 3. Random distribution of slots on the beam pipe.

Assume that the slots are randomly distributed along the pipe so that the location of the  $j$ th slot relative to the  $(j - 1)$ th slot is equal to  $d + \xi_j$ , where  $\xi_j$  are independent random Gaussian variables whose average value is equal to zero,  $\langle \xi_j \rangle = 0$ , and the rms value is equal to  $\sqrt{\langle \xi_j^2 \rangle} = \Delta d$ ; see Fig. 3. The  $z$ -coordinate of the  $j$ th slot is given by the following expression:

$$z_j = jd + \sum_{s=1}^j \xi_s .$$

In this case, Eq. (23) for the factor  $g_{n,m}(\sigma)$  takes the following form

$$g_{n,m}(\sigma) = \sum_{j=0}^{N-1} \exp \left[ i(\kappa - \sigma \kappa_{n,m}) \left( jd + \sum_{s=1}^j \xi_s \right) \right] . \quad (28)$$

Because the impedance functions contain the square of the absolute value of this factor, we will find the averaged value of  $|g_{n,m}(\sigma)|^2$ . We have

$$\begin{aligned} |g_{n,m}(\sigma)|^2 &= \left\langle \sum_{p,j} \exp \left[ iKd(p-j) + iK \sum_{s=p}^j \xi_s \right] \right\rangle \\ &= \sum_{p,j} \exp [iKd(p-j)] \left[ \int_{-\infty}^{\infty} d\xi \rho(\xi) \exp(iK\xi) \right]^{|p-j|} , \end{aligned} \quad (29)$$

where  $K = \kappa - \sigma\kappa_{n,m}$ ,  $\rho(\xi)$  is the probability distribution function for the random variable  $x$ . For the Gaussian distribution of the locations of the slots,  $\rho(\xi) = (1/\sqrt{2\pi}\Delta d) \exp(-\xi^2/2(\Delta d)^2)$ , and the integration in Eq. (27) can be easily performed yielding

$$|g_{n,m}(\sigma)|^2 = \sum_{p,j} \exp \left[ iKd(p-j) - K^2\Delta d^2 |p-j|/2 \right]. \quad (30)$$

An estimate of the peak value of  $|g_{n,m}(\sigma)|^2$  attained at the  $K = 2\pi q/d$  is

$$\begin{aligned} \max(|g_{n,m}(\sigma)|^2) &= \sum_{p,j} \exp(-\lambda |p-j|) \\ &\approx \int_0^N dj \int_0^N dp \exp(-\lambda |p-j|) = \frac{2}{\lambda^2} (\lambda N - 1 + e^{-\lambda N}), \end{aligned} \quad (31)$$

where  $\lambda = 2\pi^2 q^2 \Delta d^2 / d^2$ . When  $\Delta d \ll d/\pi q \sqrt{2N}$ , we recover the previous result,  $\max(|g_{n,m}(\sigma)|^2) = N^2$ . However, in the opposite limit,  $\Delta d \gg d/\pi q \sqrt{2N}$ , we have

$$\max(|g_{n,m}(\sigma)|^2) = \frac{d^2 N}{\pi q^2 \Delta d^2}, \quad (32)$$

so that the maximum values of  $\text{Re}Z(\omega)$  will be now multiplied by the factor given by Eq. (31). Notice that  $\text{Re}Z(\omega)$  now scales  $\propto N$ , rather than  $\propto N^2$  [Eq. (24)] in the case of a regular structure. According to Eq. (31), higher  $q$  values will attenuate faster than the smaller ones.

## VIII. CONCLUSIONS

We have presented a formalism for calculating the real part of the impedance for slots and arrays of slots, suitable for both short and long slots. In addition to broad band impedance, this theory predicts the existence of narrow band peaks at the cutoff frequencies of different waveguide modes. In regular arrays of slots, additional peaks develop at the frequencies given by Eq. (26). At these frequencies, the radiation of different slots interfere coherently so that the amplitude of the field radiated by the slots is equal to the number of slots times the amplitude radiated by a single slot. Since the real part of the impedance is proportional

to the radiated energy,  $\text{Re}Z$  at these frequencies turns out to be proportional to the number of slots squared. The situation here is completely analogous to the diffraction of light on a periodic diffraction grating. As is well known, at some angles, the electromagnetic field transmitted through the individual slits of the grating is summed coherently, resulting in the intensity of light being proportional to the number of slits squared.

Notice that in addition to the resonant and cutoff peaks studied in this paper, other sources of narrow band impedance in an array of slots are trapped modes below the cutoff frequency, which are studied in Ref. [9].

Since high values of the shunt impedance can cause instability of the beams, developing methods for suppression or even elimination of the peaks is desirable. One such approach was considered in this paper. It consists of randomization of the positions of the slots to break the periodicity of the array and destroy the coherent buildup of the radiated waves. If the degree of randomization is high enough, as shown in Sec. 6, it results in a suppression of the height of the peaks, making them eventually proportional to the number of the slots. Of course, the price for such a way of decreasing the peaks will be a diminished total area of the slots per unit length of the pipe.

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## APPENDIX

Assume large aspect ratio slots for which  $\alpha_{mg} \approx -\alpha_{el}$ . Then Eqs. (15)–(18) can be combined to yield

$$\text{Re}Z = \frac{Z_0}{\pi} \frac{\omega^2 \alpha_{mg}^2}{c^2 b^4} (I_1 + I_2), \quad (33)$$

where

$$\begin{aligned} I_1 &= \sum_{n,m} \frac{1}{(1 + \delta_{n,0})} F\left(\frac{\omega}{\omega_{n,m}}\right), \\ I_2 &= \sum_{n,m} \frac{n^2}{\mu_{n,m}'^2 - n^2} F\left(\frac{\omega}{\omega_{n,m}}\right), \end{aligned} \quad (34)$$

and

$$F(x) = \frac{2x^2 - 1}{x\sqrt{x^2 - 1}}. \quad (35)$$

In the limit of high frequency,  $\omega \gg c/b$ , a large number of harmonics in Eqs. (34) will contribute to  $\text{Re}Z$ , and we can change the summation by the integration over  $n$  and  $m$ . Note that, for  $n, m \gg 1$ ,

$$\mu_{n,m} \approx \mu_{n,m}' \approx nf\left(\frac{m}{n}\right), \quad (36)$$

where  $f(x)$  is implicitly given by the following equation [10]:

$$\pi x = \sqrt{f^2 - 1} - \arccos \frac{1}{f}. \quad (37)$$

This reduces the sum  $I_1 + I_2$  to

$$I_1 + I_2 = \int_0^\infty \int_0^\infty dndm \frac{f^2}{f^2 - 1} F\left(\frac{\omega b}{cnf}\right). \quad (38)$$

The integration can be carried out if, instead of  $n$  and  $m$ , we use the integration variables  $\xi = nc/\omega b$  and  $\zeta = nc f(n/m)/\omega b$ ,

$$I_1 + I_2 = \frac{\omega^2 b^2}{\pi c^2} \int_0^1 d\xi \int_\xi^1 d\zeta \frac{\zeta}{\sqrt{\zeta^2 - \xi^2}} F\left(\frac{1}{\zeta}\right) = \frac{2\omega^2 b^2}{3\pi c^2}. \quad (39)$$

Together with Eq. (33), Eq. (39) gives Eq. (20).

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