# DISCRETE PHYSICS and the DERIVATION of ELECTROMAGNETISM from the FORMALISM of QUANTUM MECHANICS 

Louis H. Kauffman *<br>Department of Mathematics, Statistics and Computer Science, University<br>of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607-7045

and

H. Pierre Noyes ${ }^{\dagger}$<br>Stanford Linear Accelerator Center<br>Stanford University, Stanford, California 94309


#### Abstract

Freeman Dyson has recently focused attention on a remarkable derivation of electromagnetism from apparently quantum-mechanical assumptions about the motion of a single particle. We present a new version of the Feynman-Dyson derivation in a discrete context. In the course of our derivation we have uncovered a useful and elegant reformulation of the calculus of finite differences.


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## 1. INTRODUCTION

In unpublished work circa 1948 Richard Feynman discovered that a quantum mechanical particle whose coordinates and momenta obey the simplest nonrelativistic commutation relations will admit a description of acceleration that is compatible with Newton's second law and with the action of a classical electromagnetic field. This remarkable derivation was recently brought to the attention of the scientific community by the elegant paper of Freeman Dyson (Dyson 1990). In his editorial comment on the reconstructed proof Dyson remarks, "... here we find Galilean mechanics and Maxwell equations coexisting peacefully. Perhaps it was lucky that Einstein had not seen Feynman's proof when he started to think about relativity." The proof has been generalized by Tanimura (Tanimura 1992) in a paper that embeds the Feynman argument into the contexts of gravity and gauge theories.

There are many themes to consider in the project of understanding the FeynmanDyson derivation. In this paper we concentrate on the following consideration: Feynman and Dyson assume commuting spatial coordinates $X_{1}(t), X_{2}(t), X_{3}(t)$, each a differentiable function of the time $t$. This occurs in the context of commutation relations of the form $\left[X_{i}, \dot{X}_{j}\right]=\kappa \delta_{i j}$ ( $\kappa$ a constant) giving the formalism the outward appearance of quantum mechanics. In the usual approaches to quantum mechanics one has the corresponding equation $\left[q_{i}, p_{j}\right]=i \hbar \delta_{i j}$ where $q_{i}$ is the position operator and $p_{j}$ is the momentum operator. These operators are not themselves functions of time in the Schroedinger representation of quantum mechanics, but they are functions of time in the Heisenberg formulation. As a consequence the Feynman-Dyson derivation does apply directly to quantum mechanics in the Heisenberg formulation.

The derivation is not classicaL mechanics with the commutator interpreted as a Poisson Bracket. As noted by Tanimura (Tanimura 1992), the Leibnitz rule needed in the proof holds for Poisson brackets only if the dynamical variables are derived from a Hamiltonian or a Lagrangian. One major reason for being interested in the
proof stems from the fact that this assumption is not made. We wish to point out that in a context of discrete physics the derivation can still be carried out, and that in this context there need not be any demand for simultaneous values of position and momentum operators. In fact, this idea is simply meaningless in our discrete context.

Because the variables and fields in the Feynman-Dyson derivation are noncommutative, the question of Lorentz invariance requires a special analysis that we shall not attempt in this paper, but comment on briefly in the Appendix. There is nothing paradoxical about the Feynman-Dyson derivation as it stands. It is a piece of mathematical physics asking for a good interpretation.

The purpose of this paper is to analyze the Feynman-Dyson derivation in a context of discrete physics. In this context a spatial variable $X_{i}$ has values $X_{i}, X_{i}{ }^{\prime}$, $X_{i}{ }^{\prime \prime}, \ldots$ at successive values of discrete time. A measurement of velocity depends upon the difference of position values at two different (neighboring) values of discrete time. Thus we may (by convention) identify the value of $\dot{X}_{i}$ with $X_{i}^{\prime}-X_{i}$ and write $\dot{X}_{i}:=X_{i}^{\prime}-X_{i}$. Since velocity depends upon two times and position on only one time, the idea of simultaneous determination of position and velocity is meaningless in the discrete context.

In order to achieve our aims we have had to go to the roots of the calculus of discrete differences and discover an ordered version of this calculus that just fits the desired application. In this discrete ordered calculus (described in $\S 2$ and $\S 3$ of this paper) the operation of differentiation acts also to shift a product to its left by one time step. Thus $X \dot{X}:=X^{\prime}\left(X^{\prime}-X\right)$ while $\dot{X} X:=\left(X^{\prime}-X\right) X$. In the discrete ordered calculus, $\dot{X}$ and $X$ do not commute and a specific commutation relation such as $\dot{X} X-X \dot{X}=\kappa$ is regarded as a hypothesis about the structure of their noncommutativity.

Furthermore, the discrete ordered calculus (DOC) obeys the rule for the differentiation of the product, $(A B)^{\dot{C}}=A \dot{B}+\dot{A} B$, precisely without any time-shifting (see $\S 2$ ). This makes DOC an appropriate vehicle to support the calculus and
noncommutative algebra that we need for our work.
In $\S 4$ we work out the derivation of electromagnetism in this discrete context. We begin with the assumption of the commutation relations for $X_{i}(i=1,2,3)$ :

1. $\left[X_{i}, X_{j}\right]=0$
2. $\left[X_{i}, \dot{X}_{j}\right]=\kappa \delta_{i j}$.

Here the $\operatorname{dot}(\cdot)$ is the discrete derivative and $\kappa$ is a commuting scalar in DOC. We discuss reformulations of these equations in $\S 4$.

With $F_{i}=\ddot{X}_{i}$ and

$$
\begin{aligned}
H_{\ell} & =\frac{1}{2 \kappa} \epsilon_{j k \ell}\left[\dot{X}_{j}, \dot{X}_{k}\right] \\
F_{j} & =E_{j}+\epsilon_{j k \ell} \dot{X}_{k} H_{\ell} \\
(F & =E+v \times H)
\end{aligned}
$$

we show that
(i) div $H=0$
(ii) $\frac{\partial H}{\partial t}+\nabla \times E=0$.

This is the desired result. Note that $\epsilon_{i j k}$ is the alternating symbol, and that $F=E+v \times H$ defines $E$.

In order to interpret these equations as electromagnetism we need the other two Maxwell equations:

$$
\begin{gathered}
\operatorname{div} E=4 \pi \rho \\
\frac{\partial E}{\partial t}-\nabla \times H=4 \pi j
\end{gathered}
$$

In our context, following Dyson, we take these equaions as definitions of $\rho$ and $j$. With these conventions we have a non-commutative electromagnetic formalism. It remains to be understood how this formalism is related to standard electromagnetism, and how the considerations of special relativity enter into this noncommutative context.. It is our purpose, in this first paper, to put the derivation
on a firm footing in order to provide a platform for consideration of these problems in subsequent work.

We have taken great care to perform this derivation in the discrete ordered calculus. This involves taking the following definitions for partial derivatives of a function $f(X)$ :

$$
\begin{aligned}
\frac{\partial f}{\partial X_{i}} & =\frac{1}{\kappa}\left[f, \dot{X}_{i}\right] \\
\dot{f} & =\frac{\partial f}{\partial t}+\dot{X}_{j} \frac{\partial f}{\partial X_{j}}
\end{aligned}
$$

(The Einstein summation convention is in effect.) These definitions are discussed in $\S 4$.

We wish to close this introduction with a remark about the commutativity of $X$ and $X^{\prime} . X^{\prime}$ is regarded as the indicator of $X$ after one discrete time step. Formally, we can write both $X X^{\prime}$ and $X^{\prime} X$. However, in our conventions $X X^{\prime}$ means [measure $X^{\prime}$, then measure $X$ ] and this would require the observer to step backwards in time! For this reason we do not assume that $X X^{\prime}=X^{\prime} X$, and this gives us the formal freedom to postulate (in $\S 4$ ) a set of commutation relations among $\left\{X_{i}, X_{j}^{\prime}\right\}$ that can be regarded as the basis of our derivations. In a sequel to this paper we shall discuss actual numerical solutions to these relationships.

Obviously, much more work remains to be done in this domain. We shall discuss gravity/quantum formalism in a sequel to this paper.

## 2. Motivating a Discrete Calculus

In one-dimensional standard quantum mechanics in the Heisenberg formulation (Dirac 1947) the uncertainty principle takes the form of a commutation relation

$$
Q P-P Q=\hbar i
$$

where $Q$ and $P$ denote, respectively, the position and momentum operators for the quantum mechanical particle, and $\hbar$ is Planck's constant divided by $2 \pi\left(i^{2}=-1\right)$.

There are many interpretations of this formalism. In the Schrödinger picture of quantum mechanics the system is represented by a wave function, $\psi=\psi(x, t)$. where $x$ denotes the spatial coordinate and $t$ denotes the temporal coordinate. The operators $Q$ and $P$ are defined by the equations

$$
Q \psi=x \psi \quad P \psi=\frac{\hbar}{i} \frac{\partial \psi}{\partial x} .
$$

Thus

$$
(Q P-P Q) \psi=x \frac{\hbar}{i} \frac{\partial \psi}{\partial x}-\frac{\hbar}{i} \frac{\partial}{\partial x}(x \psi)=(\hbar i) \psi .
$$

Hence $Q P-P Q=\hbar i$.
The Heisenberg picture is not tied to this particular interpretation. It simply asserts that the order of application of the position and momentum operators matters-and that the difference of these orders is described by the commutation relations.

We can find an almost identical commutation relation by thinking about position and momentum in a classical but discrete context. In a discrete universe time goes forward in measured ticks, and space occurs only in discrete intervals. We can imagine position determined at an instant, but to find velocity or momentum the clock must advance one tick to allow computation of the ratio of change of position to change of time. In measuring position first and then momentum, we
advance the clock after determining position. If momentum is measured before position, the clock advances before the measurement of position and the position is determined at a later time. In this way $P Q$ and $Q P$ differ due to the intervening time step.

Let us quantify these last remarks by working with discrete position $X$ and discrete velocity $\dot{X}$. Let $X, X^{\prime}, X^{\prime \prime}, \ldots$ denote the sequence of values for $X$ at successive times $t_{0}, t_{1}, t_{2}, \ldots$. Define the value of $\dot{X}$ to be $X^{\prime}-X$ and write $\dot{X}:=X^{\prime}-X$ to indicate this evaluation. We regard $\dot{X}$ as a discrete velocity with the time-step normalized to 1 by convention.

Let $X \dot{X}$ denote the process-measure $\dot{X}$ then measure $X$. Thus, on evaluating, we find

$$
X \dot{X}:=X^{\prime}\left(X^{\prime}-X\right)
$$

since measuring $\dot{X}$ requires stepping forward in time to the position $X^{\prime}$.
On the other hand, $\dot{X} X$ denotes the process-measure $X$ then measure $\dot{X}$. Thus $\dot{X} X:=\left(X^{\prime}-X\right) X$.

We conclude that

$$
X \dot{X}-\dot{X} X:=X^{\prime}\left(X^{\prime}-X\right)-\left(X^{\prime}-X\right) X
$$

This difference is not zero, and if it turns out to be a constant $(\kappa)$ then we have the equation $X \dot{X}-\dot{X} X:=\kappa$, a discrete analog to the Heisenberg commutation relation.

In order to take the derivations of Dyson (Dyson 1990) and Tanimura (Tanimura 1992) and place them on a discrete foundation we shall develop a time ordered calculus that generalizes the ideas that have been presented in this section. We end this section with an informal discussion of some of the issues that are involved.

One issue that must be faced is the question of the commutativity of $X$ and $X^{\prime}$. We can formally write both $X^{\prime} X$ and $X X^{\prime}$. The first ( $X^{\prime} X$ ) means-measure $X$, take a time step, measure $X$ after the time step. However, $X X^{\prime}$ does not have operational meaning in this same sense, since $X^{\prime}$ demands a time step while $X$ asks for the value at a previous time. We therefore assume that $X^{\prime} X$ and $X X^{\prime}$ are distinct without yet making any explicit assumption about the value of their difference.

The second issue involves evaluation. We have been careful to write $\dot{X}:=$ $X^{\prime}-X$ rather than $\dot{X}=X^{\prime}-X$ since the dot in $\dot{X}$ is a special instruction to shift time to its left in the ordered calculus. The directed equals sign (:=) is used to indicate evaluation. Thus we can write $A \dot{B}:=A^{\prime}\left(B^{\prime}-B\right)$ and

$$
\begin{aligned}
A \dot{B} \dot{C} & :=(A \dot{B})^{\prime}\left(C^{\prime}-C\right) \\
& :=A^{\prime} \dot{B}^{\prime}\left(C^{\prime}-C\right) \\
& :=A^{\prime \prime}\left(B^{\prime}-B\right)^{\prime}\left(C^{\prime}-C\right) \\
& :=A^{\prime \prime}\left(B^{\prime \prime}-B^{\prime}\right)\left(C^{\prime}-C\right)
\end{aligned}
$$

(We assume that $(X Y)^{\prime}=X^{\prime} Y^{\prime}$.) Each step in evaluation must perform all the time shifts for any dot that is eliminated. We shall return to this issue in the next section.

Returning to $X \dot{X}$ and $\dot{X} X$, we evaluate and find

$$
\begin{aligned}
X \dot{X}-\dot{X} X:= & X^{\prime}\left(X^{\prime}-X\right)-\left(X^{\prime}-X\right) X \\
= & X^{\prime}\left(X^{\prime}-X\right)-X\left(X^{\prime}-X\right) \\
& +X\left(X^{\prime}-X\right)-\left(X^{\prime}-X\right) X \\
= & \left(X^{\prime}-X\right)\left(X^{\prime}-X\right)+X X^{\prime}-X^{2}-X^{\prime} X+X^{2} \\
= & \left(X^{\prime}-X\right)^{2}+\left[X, X^{\prime}\right] .
\end{aligned}
$$

Thus

$$
X \dot{X}-\dot{X} X:=\left(X^{\prime}-X\right)^{2}+\left[X, X^{\prime}\right]
$$

where $[A, B]=A B-B A$. (In general, we will not assume that $\left[X, X^{\prime}\right]=0$.)

If $X$ and $X^{\prime}$ commute, then $[X, \dot{X}]:=\left(X^{\prime}-X\right)^{2}$. In this context one might assume that $\left(X^{\prime}-X\right)^{2}=\kappa$ is constant and declare that $X \dot{X}-\dot{X} X=\kappa$.

If $X$ and $X^{\prime}$ do not commute then the formula above shows how their commutator is related to $[X, \dot{X}]$.

To summarize: The dot in $\dot{X}$ is an instruction to take a time step. A product $A B$ means do $B$, then do $A$. Therefore $X \dot{X}:=X^{\prime}\left(X^{\prime}-X\right)$ since measuring $X$ after one time step yields $X^{\prime}$.

## 3. A Discrete Ordered Calculus-DOC

By a variable $X$ we mean a collection of algebraic entities $X, X^{\prime}, X^{\prime \prime}, X^{\prime \prime \prime}, \ldots$ called "the values of $X$ at successive steps of discrete time." No assumptions of commutativity are made for these variables, but we do assume that multiplication is associative and that multiplication distributes over addition and that there is a unit element, 1 , such that $1 X=X 1$ for all $X$. Furthermore we assume that $1^{\prime}=1$. Similarly, there is a 0 such that $0+X=X$ for all $X$ and $0^{\prime}=0$.

At this point the reader will see that we are assuming that a noncommutative ring $R$ has been given, and that $X, X^{\prime}, \ldots, Y, Y^{\prime}, \ldots$ belong to $R$. (Note that this means that we assme that $X^{\prime}+Y=Y+X^{\prime}$.) Thus we can speak concisely by saying that we assume as a given a (noncommutative) ring $R$ with unit (1) equipped with a unary operator ${ }^{\prime}: R \rightarrow R$ such that $1^{\prime}=1$ and $0^{\prime}=0$ and $(a+b)^{\prime}=a^{\prime}+b^{\prime}$ for all $a$ and $b$ in $R$, and $(a b)^{\prime}=a^{\prime} b^{\prime}$ for all $a, b$ in $R$. In the context of the ring $R$, we shall define a discrete ordered calculus by first adjoining to $R$ a special element $J$ whose sole purpose is to keep track of the time shifting. We assume that $J$ has the properties:

1. $J^{\prime}=J$
2. $A J=J A^{\prime}$ for all $A \in R$
(of course $J J^{\prime}=J J$ so this works for $J$ as well).

We let $\widehat{R}$ be the ring obtained from $R$ by formally adjoining $J$ to $R$ with these properties. Since $\widehat{R}$ is, by definition, a ring with unit this means that

$$
\begin{aligned}
(X+Y) J & =X J+Y J \\
J(X+Y) & =J X+J Y \\
J 0 & =0, \quad J 1=1, \quad \text { etc. }
\end{aligned}
$$

Now note that any expression in $\widehat{R}$ can be rewritten (using $A J=J A^{\prime}$ ) in the form of a sum of elements of the form $J^{k} Z$ where there is no appearance of $J$ in $Z$. We can define an evaluation map $E: \widehat{R} \rightarrow R$ by the equations
(i) $E(A+B)=E(A)+E(B)$ for any $A, B \in R$
(ii) $E\left(J^{k} Z\right)=Z$ whenever $Z \in R$.
$E$ is defined on $\widehat{R}$ by writing $A \in \widehat{R}$ as a sum of elements of the form $J^{k} Z$ and then applying (i) and (ii) above. For example

$$
E(A J+B J(C J))=E\left(J A^{\prime}+J^{2} B^{\prime \prime} C^{\prime}\right)=A^{\prime}+B^{\prime \prime} C^{\prime}
$$

(assuming that $A, B, C \in R$ ). It follows from our assumptions that $E: \widehat{R} \rightarrow R$ is well-defined, but we shall omit the proof of this fact. Note that, by definition, $E(E(X))=E(X)$ where we regard $R \subset \widehat{R}$ as the set of expressions in $\widehat{R}$ without any $J$ 's. In fact, we note that $\widehat{R} \cong \bigoplus_{n=0}^{\infty} J^{n} R$ where

$$
J^{n} R=\left\{J^{n} r \mid r \in R\right\} \quad \text { and } \quad\left(J^{n} r\right)\left(J^{m} s\right)=J^{n+m} r^{(m)} s
$$

where $r, s, \in R$ and $r^{(m)}=r^{\prime \prime \prime .}$, with $m$ "primes". With this reformulation, the evaluation map is obviously well defined. Now we are prepared to define differentiation in $\widehat{R}$ and therefore initiate the discrete ordered calculus (DOC).

Definition 3.1. Define $D: \widehat{R} \rightarrow \widehat{R}$ by the equation $D(X)=J\left(X^{\prime}-X\right)$. The presence of $J$ in $D X$ makes it a time-shifter for expressions on its left. [Compare this approach with (Etter and Kauffman 1994). That approach arose from discussions about an early version of the present paper.]

Proposition 3.1. Let $A, B \in \widehat{R}$. Then

$$
D(A B)=D(A) B+A D(B)
$$

Proof.

$$
\begin{aligned}
D(A B) & =J\left((A B)^{\prime}-A B\right) \\
& =J\left(A^{\prime} B^{\prime}-A^{\prime} B+A^{\prime} B-A B\right) \\
& =J\left(A^{\prime}\left(B^{\prime}-B\right)+\left(A^{\prime}-A\right) B\right) \\
& =J A^{\prime}\left(B^{\prime}-B\right)+J\left(A^{\prime}-A\right) B \\
& =A J\left(B^{\prime}-B\right)+J\left(A^{\prime}-A\right) B \\
& =A D(B)+D(A) B .
\end{aligned}
$$

We see from the proof of this proposition how the ordering convention in the discrete calculus has saved the product rule for differentiation.

In a standard commutative time-discrete calculus, one of the terms in the expansion of the derivative of a product must be time-shifted. The same phenomenon occurs in the infinitesimal calculus, but there an infinitesimal shift is neglected in the limit:

$$
\begin{aligned}
\frac{d}{d t}(f g) & =\lim _{h \rightarrow 0} \frac{f(t+h) g(t+h)-f(t) g(t)}{h} \\
& =\lim _{h \rightarrow 0} \frac{f(t+h) g(t+h)-f(t+h) g(t)+f(t+h) g(t)-f(t) g(t)}{h} \\
& =\lim _{h \rightarrow 0} f(t+h)\left(\frac{g(t+h)-g(t)}{h}\right)+\left(\frac{f(t+h)-f(t)}{h}\right) g(t) \\
& =\left[\lim _{h \rightarrow 0} f(t+h)\right] \frac{d g}{d t}+\left(\frac{d f}{d t}\right) g(t) .
\end{aligned}
$$

It is interesting to see how the evaluations work in specific examples. In writing
examples it is convenient to write $\dot{A}$ for $D(A)$. Thus $\dot{A}=J\left(A^{\prime}-A\right)$. For example:

$$
\begin{aligned}
(X \dot{Y}) & \left.=J\left((X \dot{Y})^{\prime}-X \dot{Y}\right)\right) \\
& =J\left(\left(X J\left(Y^{\prime}-Y\right)\right)^{\prime}-X J\left(Y^{\prime}-Y\right)\right) \\
& =J\left(X^{\prime} J\left(Y^{\prime \prime}-Y^{\prime}\right)-X J\left(Y^{\prime}-Y\right)\right) \\
& =J\left(J X^{\prime \prime}\left(Y^{\prime \prime}-Y^{\prime}\right)-J X^{\prime}\left(Y^{\prime}-Y\right)\right) \\
& =J^{2}\left(X^{\prime \prime} Y^{\prime \prime}-X^{\prime \prime} Y^{\prime}-X^{\prime} Y^{\prime}+X^{\prime} Y\right)
\end{aligned}
$$

Thus $E((X \dot{Y}))=X^{\prime \prime} Y^{\prime \prime}-X^{\prime \prime} Y^{\prime}-X^{\prime} Y^{\prime}+X^{\prime} Y$. On the other hand,

$$
\begin{aligned}
\dot{X} \dot{Y} & =J\left(X^{\prime}-X\right) J\left(Y^{\prime}-Y\right) \\
& =J^{2}\left(X^{\prime \prime}-X^{\prime}\right)\left(Y^{\prime}-Y\right) \\
X \ddot{Y} & =X J\left(\dot{Y}^{\prime}-\dot{Y}\right) \\
& =X J\left(J\left(Y^{\prime}-Y\right)^{\prime}-J\left(Y^{\prime}-Y\right)\right) \\
& \left.=X J^{2}\left(Y^{\prime \prime}-Y^{\prime}-Y^{\prime}+Y\right)\right) \\
& =J^{2} X^{\prime \prime}\left(Y^{\prime \prime}-2 Y^{\prime}+Y\right) . \\
\dot{X} \dot{Y}+X \ddot{Y} & =J^{2}\left[\left(X^{\prime \prime}-X^{\prime}\right)\left(Y^{\prime}-Y\right)+X^{\prime \prime}\left(Y^{\prime \prime}-2 Y^{\prime}+Y\right)\right] \\
& =J^{2}\left[X^{\prime \prime} Y^{\prime}-X^{\prime \prime} Y-X^{\prime} Y^{\prime}+X^{\prime} Y+X^{\prime \prime} Y^{\prime \prime}-2 X^{\prime \prime} Y^{\prime}+X^{\prime \prime} Y\right] \\
& =J^{2}\left[-X^{\prime \prime} Y^{\prime}-X^{\prime} Y^{\prime}+X^{\prime} Y+X^{\prime \prime} Y^{\prime \prime}\right] \\
& =(X \dot{Y}) .
\end{aligned}
$$

This is a working instance of our formula $D(A B)=D(A) B+A D(B)$. In the remainder of the paper it will be useful to write $A:=B$ to mean that $E(A)=E(B)$. In particular, we will often use this to mean that $B$ has been obtained from $A$ by expanding some derivatives and throwing away some or all of the left-most $J$ 's. This means that while it is true that $E(A)=E(B), A$ and $B$ cannot be substituted for one another in larger expressions, since they contain different time-shifting instructions.

An example of this usage is:

$$
\dot{X} \dot{Y}:=\dot{X}^{\prime}\left(Y^{\prime}-Y\right) .
$$

Note that

$$
\dot{X} \dot{Y}=\dot{X} J\left(Y^{\prime}-Y\right)=J \dot{X}^{\prime}\left(Y^{\prime}-Y\right)
$$

Thus

$$
E(\dot{X} \dot{Y})=E\left(\dot{X}^{\prime}\left(Y^{\prime}-Y\right)\right)
$$

In calculating, the $:=$ notation allows us to "do the J's in our heads."

## Discussion

With the DOC formalized we can return to the structure of the commutator $[X, \dot{Y}]=X \dot{Y}-\dot{Y} X$. We have

$$
\begin{aligned}
{[X, \dot{Y}] } & :=X^{\prime}\left(Y^{\prime}-Y\right)-\left(Y^{\prime}-Y\right) X \\
& :=X^{\prime}\left(Y^{\prime}-Y\right)-X\left(Y^{\prime}-Y\right)+X\left(Y^{\prime}-Y\right)-\left(Y^{\prime}-Y\right) X \\
& :=\left(X^{\prime}-X\right)\left(Y^{\prime}-Y\right)+X Y^{\prime}-X Y-Y^{\prime} X+Y X \\
& :=\left(X^{\prime}-X\right)\left(Y^{\prime}-Y\right)+\left(X Y^{\prime}-Y^{\prime} X\right)-(X Y-Y X) \\
{[X, \dot{Y}] } & :=\left(X^{\prime}-X\right)\left(Y^{\prime}-Y\right)+\left[X, Y^{\prime}\right]-[X, Y] .
\end{aligned}
$$

This formula will be of use to us in the next section.
Note how, in this formalism, we cannot arbitrarily substitute $\dot{X}$ for $X^{\prime}-X$ since the definition of the dot (".") as a time-shifter can change the value of an expression. Thus $\left(X^{\prime}-X\right)\left(Y^{\prime}-Y\right) \neq \dot{X} \dot{Y}$. It may be useful to write $X^{\prime}-X=\|\dot{X}\|$ where $\|\dot{X}\|$ is by definition the difference, stripped of its time-shifting properties. Then $\left(X^{\prime}-X\right)\left(Y^{\prime}-Y\right)=\|\dot{X}\|\|\dot{Y}\|$ and we can write

$$
[X, \dot{Y}]:=\|\dot{X}\|\|\dot{Y}\|+\left[X, Y^{\prime}\right]-[X, Y] .
$$

Since $[A, B]=A B-B A$ these commutators satisfy the Jacobi identity. That
is, we have

$$
[[A, B], C]+[[C, A], B]+[[B, C], A]=0 .
$$

The proof is by direct calculation.

## 4. Electromagnetism

In this section we give a discrete version of the Feynman-Dyson (Dyson 1990) derivation of the Maxwell equations from basic quantum mechanical formalism. We shall work in the discrete ordered calculus (DOC) of $\S 3$. We assume time-series variables $X_{1}, X_{2}$ and $X_{3}$ and the commutation relations
(i) $\left[X_{i}, X_{j}\right]=0 \quad \forall_{i j}$
(ii) $\left[X_{i}, \dot{X}_{j}\right]=\kappa \delta_{i j}$
where $\kappa$ is a constant and $\kappa$ commutes with all expressions in DOC.
We further assume that there are functions $F_{i}(X \dot{X})(i=1,2,3)$ such that $\ddot{X}_{i}=$ $F_{i}(X, \dot{X})$. (Here writing $F(X)$ means that $F$ is a function of $X_{1}, X_{2}, X_{3}$.) It is the purpose of this section to show that $F_{i}$ takes on the pattern of the electromagnetic field in vacuum. Our first task will be to rewrite the above relations in terms of the discrete ordered calculus.

Proposition 4.1. Given that $\left[X_{i}, X_{j}\right]=0$ for all $i, j=1,2,3$, and letting $\Delta_{i}=$ $X_{i}^{\prime}-X_{i}$, the equations $\left[X_{i}, \dot{X}_{j}\right]=\kappa \delta_{i j}$ imply the equations

$$
E\left(\left[X_{i}, X_{j}^{\prime}\right]+\Delta_{i} \Delta_{j}\right)=E\left(\kappa \delta_{i j}\right)
$$

Proof. First assume $\left[X_{i}, \dot{X}_{j}\right]=\kappa \delta_{i j}$. Then $\left[X_{i}, \dot{X}_{j}\right]:=\left\|\dot{X}_{i}\right\|\left\|\dot{X}_{j}\right\|+\left[X_{i}, X_{j}^{\prime}\right]-$ [ $\left.X_{i}, X_{j}\right]$ by the calculation at the end of $\S 2$. Here $\left\|\dot{X}_{i}\right\|=X_{i}^{\prime}-X_{i}=\Delta_{i}$ and $\left[X_{i}, X_{j}\right]=0$. Thus

$$
\left[X_{i}, \dot{X}_{j}\right]=\Delta_{i} \Delta_{j}+\left[X_{i}, X_{j}^{\prime}\right]
$$

This completes the proof.

Discussion. This proposition shows how the Heisenberg type relations $X_{i} \dot{X}_{j}-$ $\dot{X}_{j} X_{i}=\kappa \delta_{i j}$ translate into the time-series commutation relations $X_{i} X_{j}^{\prime}-X_{j}^{\prime} X_{i}:=$ $\kappa \delta_{i j}-\Delta_{i} \Delta_{j}$. From the point of view of discrete physics it is these relations that will implicate electromagnetism. Since there is no a priori reason for the elements of time series to commute with one another, we can regard the equations

$$
\begin{aligned}
X_{i} X_{j} & =X_{j} X_{i} \\
X_{i} X_{j}^{\prime}-X_{j}^{\prime} X_{i} & =\kappa \delta_{i j}-\Delta_{i} \Delta_{j} \\
\left(\Delta_{i}\right. & \left.=X_{i}^{\prime}-X_{i}\right)
\end{aligned}
$$

as setting the context for the discussion of the physics of a discrete particle. It is in this context that the patterns of electromagnetism will appear.

## Derivatives

We shall need to interpret certain derivatives in terms of our discrete formalism. First of all, we have

$$
\frac{\partial X_{i}}{\partial X_{j}}=\delta_{i j}
$$

Therefore

$$
\left[X_{i}, \dot{X}_{j}\right]=\kappa \frac{\partial X_{i}}{\partial X_{j}}
$$

Consequently, we make the following
Definition. Let $G$ be a function of $X$, then we define $\partial G / \partial X_{i}$ by the equation

$$
\frac{\partial G}{\partial X_{i}}=\kappa^{-1}\left[G, \dot{X}_{i}\right]
$$

We also wish to define $\partial G / \partial t$. This time derivative is distinct from $\dot{G}$. It should satisfy the usual relationship for multi-variable calculus:

$$
\dot{G}=\frac{\partial G}{\partial t}+\dot{X}_{j} \frac{\partial G}{\partial X_{j}} \quad(\text { summed over } j=1,2,3) .
$$

Therefore we define $\partial G / \partial t$ by the equation

$$
\frac{\partial G}{\partial t}=\dot{G}-\dot{X}_{j} \kappa^{-1}\left[G, \dot{X}_{j}\right] \quad(\text { summed over } j=1,2,3)
$$

when $\left[X_{m}, G\right]=0$ for $m=1,2,3$.
The condition $\left[X_{i}, G\right]=0$ for $i=1,2,3$ implies that $G$ has no dependence on $\dot{X}_{j}(j=1,2,3)$ under mild hypotheses on $G$. For if we assume that $G$ is either a polynomial or a (non-commutative) power series in $X_{j}$ and $\dot{X}_{j}$ then the equations $\left[X_{i}, \dot{X}_{j}\right]=\kappa \delta_{i j}$ and $\left[X_{i}, X_{j}\right]=0$ show that (under these assumptions) that $G$ has no occurance of $\dot{X}_{j}$. It is necessary to define $\frac{\partial}{\partial t}$ since our discrete theory does not carry the conventional time variable $t$.

With these definitions in hand, we can proceed to the consequences of the commutation relations (i) and (ii).

Lemma 4. 2.

$$
\left[\dot{X}_{i}, X_{j}\right]=\left[\dot{X}_{j}, X_{i}\right]
$$

Proof.

$$
\begin{aligned}
X_{i} X_{j} & =X_{j} X_{i} \\
\Rightarrow\left(X_{i} X_{j}\right) & =\left(X_{j} X_{i}\right) \\
\Rightarrow \dot{X}_{i} X_{j}+X_{i} \dot{X}_{j} & =\dot{X}_{j} X_{i}+X_{j} \dot{X}_{i} \\
\Rightarrow \dot{X}_{i} X_{j}-X_{j} \dot{X}_{i} & =\dot{X}_{j} X_{i}-X_{i} \dot{X}_{j} \\
\Rightarrow\left[\dot{X}_{i}, X_{j}\right] & =\left[\dot{X}_{j}, X_{i}\right]
\end{aligned}
$$

Lemma 4.3.

$$
\left[\dot{X}_{j}, \dot{X}_{k}\right]+\left[X_{j}, \ddot{X}_{k}\right]=0
$$

Proof.

$$
\begin{aligned}
& X_{j} \dot{X}_{k}-\dot{X}_{k} X_{j}=\kappa \delta_{j k} \\
\Rightarrow & \dot{X}_{j} \dot{X}_{k}+X_{j} \ddot{X}_{k}-\ddot{X}_{k} X_{j}-\dot{X}_{k} \dot{X}_{j}=0 \\
\Rightarrow & {\left[\dot{X}_{j}, \dot{X}_{k}\right]+\left[X_{j}, \ddot{X}_{k}\right]=0 }
\end{aligned}
$$

Since $\ddot{X}_{j}=F_{j}(X, \dot{X})$ we have
Lemma 4.9'.

$$
\left[\dot{X}_{j}, \dot{X}_{k}\right]+\left[X_{j}, F_{k}\right]=0 .
$$

Thus

$$
\begin{aligned}
{\left[X_{\ell},\left[X_{j}, F_{k}\right]\right] } & =-\left[X_{\ell},\left[\dot{X}_{j}, \dot{X}_{k}\right]\right] \\
& =\left[\dot{X}_{j},\left[\dot{X}_{k}, X_{\ell}\right]\right]+\left[\dot{X}_{k},\left[X_{\ell}, \dot{X}_{j}\right]\right]
\end{aligned}
$$

(by the Jacobi identity)
$=\left[\dot{X}_{j},-\delta_{k \ell} \kappa\right]+\left[\dot{X}_{k}, \kappa \delta_{\ell j}\right]$

$$
=0+0=0 .
$$

Also, since

$$
\left[X_{j}, F_{k}\right]=-\left[\dot{X}_{j}, \dot{X}_{k}\right]=+\left[\dot{X}_{k}, \dot{X}_{j}\right]=-\left[X_{k}, F_{j}\right] .
$$

Thus

$$
\left[X_{j}, F_{k}\right]=-\left[X_{k}, F_{j}\right]
$$

We now define the field $H$ by the equation

$$
-\kappa \epsilon_{j k \ell} H_{\ell}=\left[X_{j}, F_{k}\right]
$$

where $\epsilon_{j k \ell}$ is the alternating symbol for 123 . That is, $\epsilon_{123}=+1$ and $\epsilon_{a b c}=\operatorname{sgn}(a b c)$ if $a b c$ is a permutation of 123 where $\operatorname{sgn}(a b c)$ is the sign of the permutation. Otherwise, $\epsilon_{a b c}=0$.

Note that $\left[X_{\ell},\left[X_{j}, F_{k}\right]\right]=0$ implies that $\left[X_{\ell},-\epsilon_{j k \ell} H_{\ell}\right]=0$ which in turn implies that $\left[X_{\ell}, H_{s}\right]=0$. This implies that $H_{s}$ has no dependence upon $\dot{X}$ since $\dot{X}$ has nontrivial commutator with $X$. Under these circumstances we will regard
$H$ as a function of $X$ and compute $\partial H / \partial t$ according to the formula

$$
\frac{\partial H}{\partial t}=\dot{H}-\dot{X}_{j} \kappa^{-1}\left[H, \dot{X}_{j}\right]
$$

as discussed above.

Definition.

$$
E_{j}=F_{j}-\epsilon_{j k \ell} \dot{X}_{k} H_{\ell}
$$

With this definition of $E$, we have $F=E+v \times H$ where $v=\dot{X}$.
Lemma 4.4.

$$
\left[X_{m}, E_{j}\right]=0
$$

Proof.

$$
\begin{aligned}
{\left[X_{m}, E_{j}\right] } & =\left[X_{m}, F_{j}-\epsilon_{j k \ell} \dot{X}_{k} H_{\ell}\right] \\
& =\left[X_{m}, F_{j}\right]-\left[X_{m}, \epsilon_{j k \ell} \dot{X}_{k} H_{\ell}\right] \\
& =-\kappa \epsilon_{m j \ell} H_{\ell}-\epsilon_{j k \ell} X_{m} \dot{X}_{k} H_{\ell}+\epsilon_{j k \ell} \dot{X}_{k} H_{\ell} X_{m} \\
& =-\kappa \epsilon_{m j \ell} H_{\ell}-\epsilon_{j k \ell} X_{m} \dot{X}_{k} H_{\ell}+\epsilon_{j k \ell} \dot{X}_{k} X_{m} H_{\ell} \\
& =-\kappa \epsilon_{m j \ell} H_{\ell}-\epsilon_{j k \ell}\left[X_{m}, \dot{X}_{k}\right] H_{\ell} \\
& =-\kappa \epsilon_{m j \ell} H_{\ell}-\epsilon_{j k \ell} \kappa \delta_{m k} H_{\ell} \\
& =\left(-\kappa \epsilon_{m j \ell}-\kappa \epsilon_{j m \ell}\right) H_{\ell} \\
& =0 .
\end{aligned}
$$

Thus $E$ also has no dependence upon $\dot{X}$.

Remark 4.5.

$$
H_{\ell}=\frac{\kappa^{-1}}{2} \epsilon_{j k \ell}\left[\dot{X}_{j}, \dot{X}_{k}\right] .
$$

Proof.

$$
-\kappa \epsilon_{j k \ell} H_{\ell}=\left[X_{j}, F_{k}\right]
$$

and by Lemma $4.3^{\prime}$,

$$
\left[X_{j}, F_{k}\right]=-\left[\dot{X}_{j}, \dot{X}_{k}\right]
$$

Thus

$$
\epsilon_{j k \ell} H_{\ell}=\kappa^{-1}\left[\dot{X}_{j}, \dot{X}_{k}\right] .
$$

From this it follows that

$$
H_{\ell}=\frac{\kappa^{-1}}{2} \epsilon_{j k \ell}\left[\dot{X}_{j}, \dot{X}_{k}\right]
$$

Lemma 4.6.

$$
\operatorname{div} H=\sum_{i=1}^{3} \frac{\partial H_{i}}{\partial X_{i}}=0
$$

Proof.

$$
\begin{aligned}
\sum_{\ell} \frac{\partial H_{\ell}}{\partial X_{\ell}} & =\kappa^{-1} \sum_{\ell}\left[H_{\ell}, \dot{X}_{\ell}\right] \\
& =\frac{\kappa^{-2}}{2} \epsilon_{j k \ell}\left[\left[\dot{X}_{j}, \dot{X}_{k}\right], \dot{X}_{\ell}\right] \\
& =0 \quad \text { (by the Jacobi identity) }
\end{aligned}
$$

Thus div $H=0$.

Lemma 4.7.

$$
\frac{\partial H_{\ell}}{\partial t}=\epsilon_{j k \ell} \frac{\partial E_{j}}{\partial X_{k}} .
$$

Proof.

$$
\begin{aligned}
H_{\ell} & =\frac{\kappa^{-1}}{2} \epsilon_{j k \ell}\left[\dot{X}_{j}, \dot{X}_{k}\right] \\
\frac{\partial H_{\ell}}{\partial t} & =\dot{H}_{\ell}-\dot{X}_{j} \kappa^{-1}\left[H_{\ell}, \dot{X}_{j}\right]
\end{aligned}
$$

(See the discussion of $\partial / \partial t$ given earlier in this section.)
Now

$$
\begin{aligned}
\dot{H}_{\ell} & =\frac{\kappa^{-1}}{2} \epsilon_{j k \ell}\left[\dot{X}_{j}, \dot{X}_{k}\right] \\
& =\frac{\kappa^{-1}}{2} \epsilon_{j k \ell}\left(\left[\ddot{X}_{j}, \dot{X}_{k}\right]+\left[\dot{X}_{j}, \ddot{X}_{k}\right]\right) \\
& =\kappa^{-1} \epsilon_{j k \ell}\left[\ddot{X}_{j}, \dot{X}_{k}\right] \\
& =\kappa^{-1} \epsilon_{j k \ell}\left[F_{j}, \dot{X}_{k}\right] \\
& =\kappa^{-1} \epsilon_{j k \ell}\left[E_{j}+\epsilon_{j r s} \dot{X}_{r} H_{s}, \dot{X}_{k}\right] \\
& =\kappa^{-1} \epsilon_{j k \ell}\left[E_{j}, \dot{X}_{k}\right]+\kappa^{-1}\left[\dot{X}_{k} H_{\ell}, \dot{X}_{k}\right]-\kappa^{-1}\left[\dot{X}_{\ell} H_{k}, \dot{X}_{k}\right]
\end{aligned}
$$

And

$$
\begin{aligned}
& {\left[\dot{X}_{k} H_{\ell}, \dot{X}_{k}\right]-\left[\dot{X}_{\ell} H_{k}, \dot{X}_{k}\right]} \\
& =\dot{X}_{k} H_{\ell} \dot{X}_{k}-\dot{X}_{k} \dot{X}_{k} H_{\ell}-\dot{X}_{\ell} H_{k} \dot{X}_{k}+\dot{X}_{k} \dot{X}_{\ell} H_{k} \\
& =\dot{X}_{k}\left[H_{\ell}, \dot{X}_{k}\right]-\dot{X}_{\ell}\left[H_{k}, \dot{X}_{k}\right]+\left[\dot{X}_{k}, \dot{X}_{\ell}\right] H_{k} \\
& =\kappa \dot{X}_{k} \frac{\partial H_{\ell}}{\partial X_{k}}-\kappa \dot{X}_{\ell} \frac{\partial H_{k}}{\partial X_{k}}+\left[\dot{X}_{k}, \dot{X}_{\ell}\right] H_{k}
\end{aligned}
$$

Thus

$$
\dot{H}_{\ell}=\kappa^{-1} \epsilon_{j k \ell}\left[E_{j}, \dot{X}_{k}\right]+\dot{X}_{k} \frac{\partial H_{\ell}}{\partial X_{k}}-\dot{X}_{\ell} \frac{\partial H_{k}}{\partial X_{k}}+\kappa^{-1}\left[\dot{X}_{k}, \dot{X}_{\ell}\right] H_{k}
$$

Hence

$$
\frac{\partial H_{\ell}}{\partial t}=\dot{H}_{\ell}-\dot{X}_{j} \frac{\partial H_{\ell}}{\partial X_{j}}=\epsilon_{j k \ell} \frac{\partial E_{j}}{\partial X_{k}}-\dot{X}_{\ell} \frac{\partial H_{k}}{\partial X_{k}}+\kappa^{-1}\left[\dot{X}_{k}, \dot{X}_{\ell}\right] H_{k}
$$

However the second term on the right vanishes because div $H=0$, and the third term vanishes by symmetry. To see this, note that

$$
\begin{aligned}
H_{1} & =\kappa^{-1}\left[\dot{X}_{2}, \dot{X}_{2}\right] \\
H_{2} & =-\kappa^{-1}\left[\dot{X}_{1}, \dot{X}_{3}\right] \\
H_{3} & =\kappa^{-1}\left[\dot{X}_{1}, \dot{X}_{2}\right] .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\kappa\left[\dot{X}_{k}, \dot{X}_{\ell}\right] H_{k}= & {\left[\dot{X}_{1}, X_{\ell}\right]\left[\dot{X}_{2}, \dot{X}_{3}\right]-\left[\dot{X}_{2}, \dot{X}_{\ell}\right]\left[\dot{X}_{1}, \dot{X}_{3}\right] } \\
& +\left[\dot{X}_{3}, \dot{X}_{\ell}\right]\left[\dot{X}_{2}, \dot{X}_{2}\right]
\end{aligned}
$$

This vanishes for $\ell=1,2,3$. Therefore

$$
\frac{\partial H_{\ell}}{\partial t}=\epsilon_{j k \ell} \frac{\partial E_{j}}{\partial X_{k}} .
$$

This Lemma completes the derivation of Maxwell's equations. We have shown that

$$
\operatorname{div} H=0
$$

and

$$
\frac{\partial H}{\partial t}+\nabla \times E=0
$$

As Dyson (Dyson 1990) remarks, the other two Maxwell equations

$$
\begin{array}{r}
\operatorname{div} E=4 \pi \rho \\
\frac{\partial E}{\partial t}-\nabla \times H=4 \pi \vec{j}
\end{array}
$$

can be taken to define the external charge and current densities $\rho$ and $j$. However it is important to realize that our entire theory has applied only to a single trajectory.

We can regard this trajectory (and its "particle") as defining an electromagnetic field, or we can regard this particle as moving in an external field with these properties. We cannot have it both ways. The analysis so far in no way takes into account the self-interaction of this particle or its interactions with other particles and fields. Of course, our talk at this stage about the "trajectory" of a particle is an analog of a physical trajectory. The trajectory we talk about is in the space of $A \times A \times A$ where $A$ denotes the non-commutative operator algebra that underlies the theory. An eventual interpretation of this theory in terms of trajectories in physical space is a possible consequence of further analysis of our formalism. It is beyond the scope of this preliminary paper.

We feel that the foregoing analysis of the Feynman-Dyson derivation in a discrete context lays bare much of the beautiful structure of the electromagnetic formalism and its relation to a condition of discrete time. We hope to probe this structure more deeply in subsequent papers.

## APPENDIX 1: Historical Remarks

One of us (HPN) has already claimed that the Feynman proof is not paradoxical (Noyes 1991) in the context of the finite and discrete reconciliation between quantum mechanics and relativity (Noyes 1987) achieved by a new fundamental theory (Noyes 1994a, Noyes 1992, McGoveran 1991, McGoveran 1989, Noyes 1989) Noting that the Feynman postulates,

$$
\mathcal{F}_{k}(x, \dot{x} ; t)=m \ddot{x}_{k} ;\left[x_{i}, x_{j}\right]=0 ; m\left[x_{i}, \dot{x}_{j}\right]=i \hbar \delta_{i j}
$$

are independent of or linear in $m$, we can replace them by the scale invariant postulates

$$
f_{k}(x, \dot{x} ; t)=\ddot{x}_{k} ;\left[x_{i}, x_{j}\right]=0 ;\left[x_{i}, \dot{x}_{j}\right]=\kappa \delta_{i j}
$$

where $\kappa$ is any fixed constant with dimensions of area over time $\left[L^{2} / T\right]$ and $f_{k}$ has the dimensions of acceleration $\left[L / T^{2}\right]$. This step is suggested by Mach's conclusion (Mach 1875) that it is Newton's third law which allows mass ratios to be
measured, while Newton's second law is simply a definition of force. Hence in a theory which contains only "mass points", the Newtonian scale invariance of classical MLT physics reduces to the Galilean scale invariance of a purely kinematical LT theory. Breaking scale invariance in such a theory requires not only some unique specification of a particulate mass standard, but also the requirement that this particle have some absolute significance.

As has been remarked recently (Noyes 1994b), this aspect of scale invariance had already been introduced into the subject by Bohr and Rosenfeld in 1933 (Bohr 1933). In their classic paper, they point out that because QED depends only on the universal constants $\hbar$ and $c$, the discussion of the measurability of the fields can to a large extent be separated from any discussion of the atomic structure of matter (involving the mass and charge of the electron). Consequently, they are able to derive from the non-relativistic uncertainty relations the same restrictions on measurability (over finite space-time volumes) of the electromagnetic fields that one obtains directly from the second-quantized commutation relations of the fields themselves. Hence, to the extent that one could "reverse engineer" their argument, one might be able to get back to the classical field equations and provide an alternative to the Feynman derivation based on the same physical ideas.

Turning to the commutation relations themselves, we note that a velocity measurement requires a knowledge of the space interval and the time interval between two events in two well separated space-time volumes. Further, to embed these two positions in laboratory space, we must (in a relativistic theory) know the time it takes a light signal to go to one of these two positions and back to the other via a third reference position with a standard clock. Thus we need three rather than two reference events to discuss the connection between position and velocity measurements. We can then distinguish a measurement of position followed by a measurement of velocity from a measurement of velocity followed by a measurement of position. The minimum value of the difference between the product of position and velocity for measurements performed in the two distinct orders then specifies the constant in the basic "commutation relation" needed in the Feynman
derivation. So long as this value is finite and fixed, we need not know its metric value. This specifies what we mean by discrete physics in the main body of the text.

Relativity need not change this situation. Specify $c$ in a scale invariant way as both the maximum speed at which information can be transferred (limiting group velocity) and the maximum distance for supraluminal correlation without information transfer (maximum coherence length). If the unit of length is $\Delta L$ and the unit of time is $\Delta T$, then the equation $(\Delta L / c \Delta T)=1$ has a scale invariant significance. Further, the interval $I$ specified by the equation $c^{2} \Delta T^{2}-\Delta L^{2}=I^{2}$ can be given a Lorentz invariant significance. We can extend this analysis to includes the scale invariant definition $\Delta E / c \Delta P=1$ and the Lorentz invariant interval in energy-momentum space $\left(\Delta E^{2} / c^{2}\right)-\Delta P^{2}=\Delta m^{2}$ provided we require that $\frac{\Delta P \Delta L}{\Delta m}=\frac{\Delta E \Delta T}{\Delta m}$. Then, given any arbitrary particulate mass standard $\Delta m$, mass ratios can be measured using a Lorentz invariant and scale invariant LT theory. We trust that this dimensional analysis of the postulates used in the Feynman proof already removes part of the mystery about why it works, and suggests how it can be made "Lorentz invariant" in a finite and discrete sense.

## APPENDIX 2. On The Form of the Derivative

In our discrete ordered calculus (see section 3) we have defined the derivative $\dot{X}$ by the formula $\dot{X}=J\left(X^{\prime}-X\right)$ where $J$ is a formal element satisfying $J^{\prime}=J$ and $X J=J X^{\prime}$ for all $X$. Thus we have the equation

$$
\dot{X}=J X^{\prime}-J X=X J-J X=[X, J]
$$

This suggests that the equation $\dot{X}=[X, J]$ is an analog of the corresponding equation in the Heisenberg formulation of quantum mechanics:

$$
\dot{X}=[X, U]
$$

where $U$ is the time-evolution operator.

Michael Peskin has pointed out to us (private conversation) that we could accomplish the discretization of time in our theory by taking $U=e^{-i H \Delta t}$ (formally) where $\Delta t$ denotes a discrete timestep. Then $\dot{X}=[X, U]$ and $X^{\prime}=U^{-1} X U$ serves to define the time-step from $X$ to $X^{\prime}$. Our approach and Peskin's meet if we identify $J$ and $U$ ! The physical interpretation of this identification deserves further investigation.

In any case it is interesting to note the differentiation formula $(X Y)^{\cdot}=\dot{X} Y+$ $X \dot{Y}$ follows directly from the formula $\dot{X}=[X, J]$ without the necessity of introducing the step $X^{\prime}$. This relationship between the discrete ordered calculus and the algebra of commutators will be used in the next installment of our work.

## REFERENCES

Bohr, N. and Rosenfeld, L. 1933 Det. Kgl. Danske Videnskabernes Selskab., Mat.fys. Med. XII, (8).

Dirac, P.A.M. 1947 The Principles of Quantum Mechanics, Third Edition, Oxford, 1947, p. 117 et. seq.

Dyson, F. J. 1990 Amer. J. Phys., 58, 209.
Etter, T. and Kauffman, L. H. 1994 "Discrete Ordered Calculus" (in preparation).
Mach, E. 1875 Science of Mechanics (4 ed., Chicago, Open Court Publishing Co. 1919).

McGoveran, D. O. and Noyes, H. P. 1989 "Foundations for a Discrete Physics", SLAC-PUB-4526.

McGoveran, D. O. and Noyes, H. P. 1991 Physics Essays, 4, 115-120.
Noyes, H. P. 1987 "On the Discrete Reconciliation between Quantum Mechanics and Relativity" in Schladming Proceedings, Recent Developments in Mathematical Physics, pp 287-290.

Noyes, H. P. and McGoveran, D. O. 1989 Physics Essays, 2, 76-100.
Noyes, H. P. 1991 "Comment on 'Feynman's Proof...'...", SLAC-PUB-5588, (unpublished).

Noyes, H. P. 1992 Physics Today, March, pp 99-100.
Noyes, H.P. 1994a "Bit String Physics: a Novel 'Theory of Everything' ", in Proc. Workshop on Physics and Computation (PhysComp '94), D. Matzke, ch., IEEE Comp.Soc. Press, Los Amitos, CA, 1994, pp. 88-94.

Noyes, H. P. 1994b "On the Measurability of Electromagnetic Fields: A New Approach", in Marshak Memorial Volume, E. G. Sudarshan, ed., World Scientific.

Tanimura, S. 1992 Annals of Physics 220, 229-247.


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